Alternating Projection Methods

Failure in the Absence of Convexity

Matthew Tam
AMSI Vacation Scholar with generous support from CSIRO
Supervisor: Prof. Jon Borwein

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The Problem

- The setting: A Hilbert space, $\mathcal{H}$.
- The players: $r \geq 2$ sets $S_1, \ldots, S_r$ with corresponding projections P_{S_i}.
- The problem: Given an initial point x_0 we seek a feasible point in $\cap_{i=1}^r S_i$.

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von Neumann (1933)

Suppose S_1, S_2 are closed subspaces, then $\forall x_0 \in \mathcal{H}$:

$$(P_{S_2} P_{S_1})^n x_0 \to P_{S_1 \cap S_2}(x_0)$$

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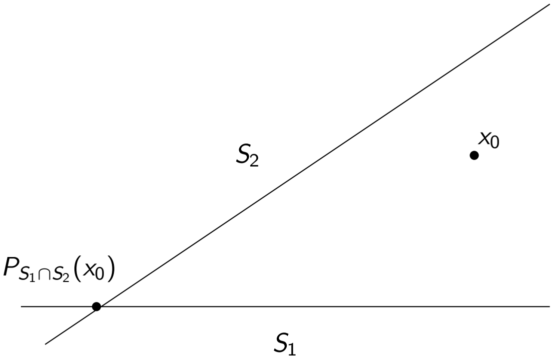
~~von Neumann (1933)~~ Bregman (1965)

Suppose S_1, S_2 are closed convex sets, then $\forall x_0 \in \mathcal{H}$:

$$(P_{S_2} P_{S_1})^n x_0 \overset{w.}{\rightharpoonup} x \text{ where } x \in S_1 \cap S_2$$

Why It Works?



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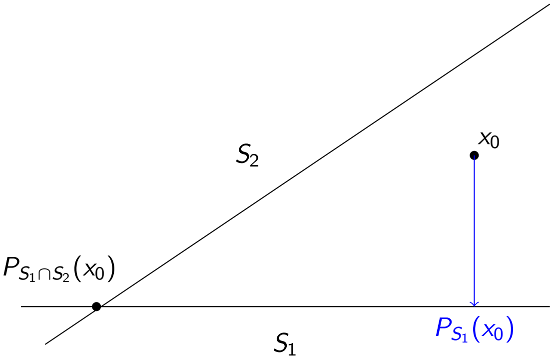

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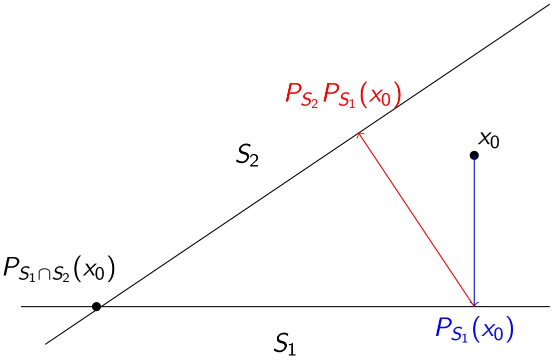

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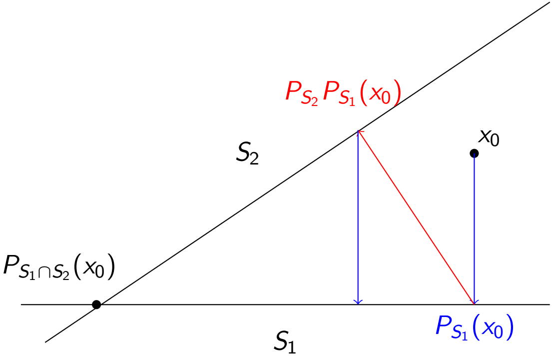

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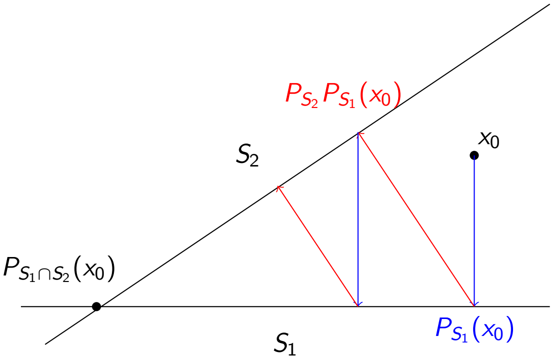

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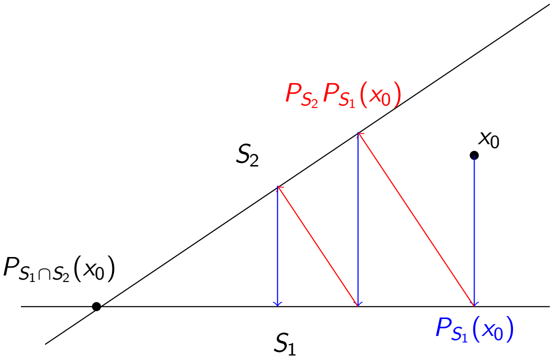

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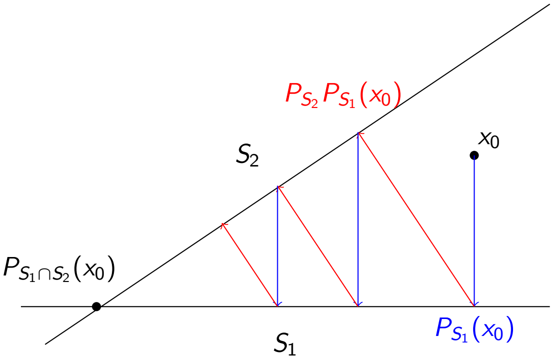

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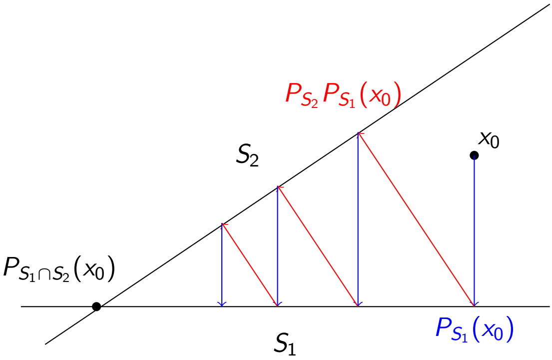

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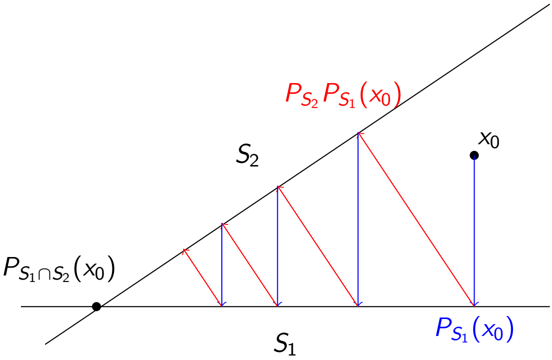

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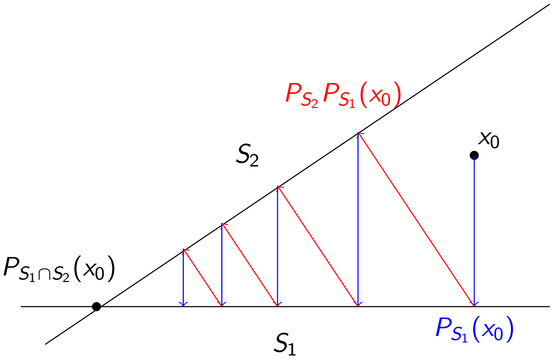

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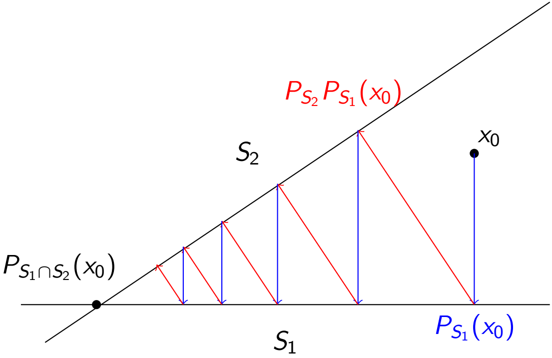

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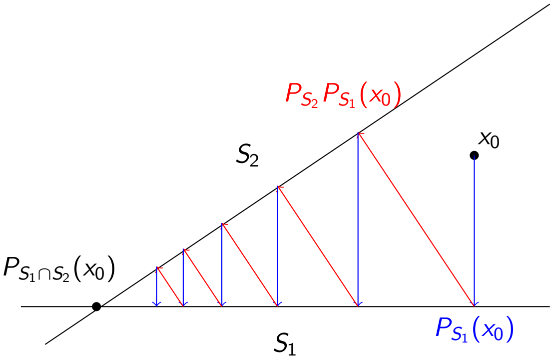

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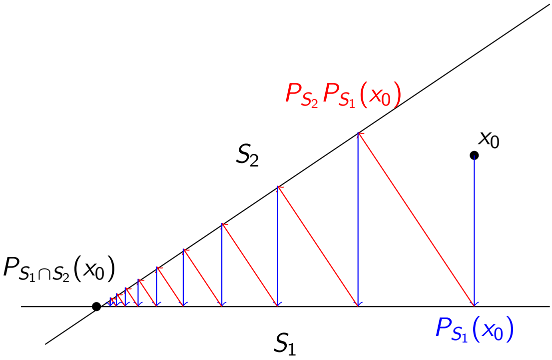

Douglas-Rachford (1959)

Suppose S_1, S_2 are closed convex sets, then $\forall x_0 \in \mathcal{H}$:

$$x_{n+1} := \frac{x_n + R_{S_2} R_{S_1}(x_n)}{2} \qquad \text{where } R_{S_i}(x) := 2P_{S_i}(x) - x$$

then $x_n \xrightarrow{w.} x$, a fixed point, with $P_{S_1}(x) \in S_1 \cap S_2$.

Dysktra (1986)

Suppose S_1, S_2 are closed convex sets, then $\forall x_0 \in \mathcal{H}$:

$$x_n^1 := x_{n-1}^2, \quad x_n^i := P_{S_i}(x_n^{i-1} - I_i^{n-1}), \quad I_n^i := x_n^i - (x_n^{i-1} - I_{n-1}^i)$$

with initial values $x_0^2 := x_0$, $I_0^i := 0$, then $x_n \to P_{S_1 \cap S_2}(x_0)$.

The Hubble Telescope

[1] http://spectrum.ieee.org/aerospace/astrophysics/software-for-optical-systems-spells-the-end-of-blur/0

The Hubble Telescope

Before correction

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The Hubble Telescope

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After correction

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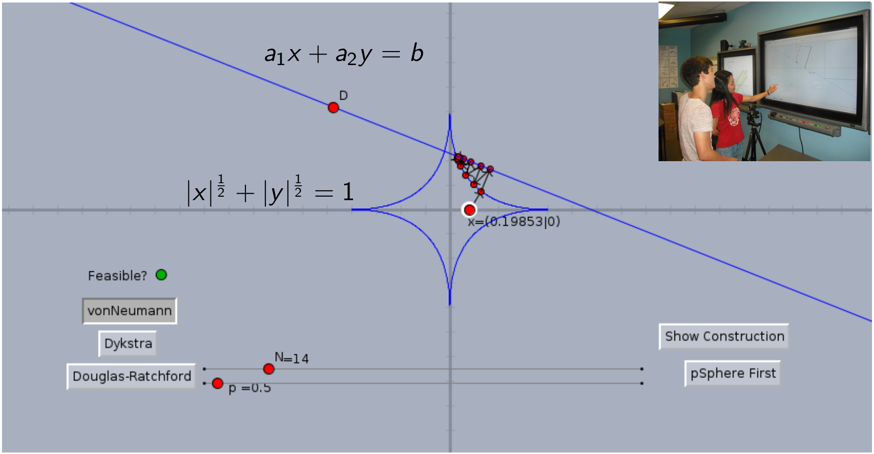
Project Aims

- Investigate behaviour of three alternating projection variants:
  - von Neumann
  - Dykstra
  - Douglas-Rachford
- Particularly, cases when the underlying subsets are non-convex.[1]
- Partially answer the question: "*When does convergence fail?*"
- Develop some tools to help better understand behaviour, which are:
  - Visual
  - Interactive
  - *Hands-on* (literally!)
- Even behaviour in $\mathbb{R}^2$ is poorly understood.

[2] *Douglas–Rachford Algorithm in the Absence of Convexity*, Borwein, JM & Sims, B, (2011).

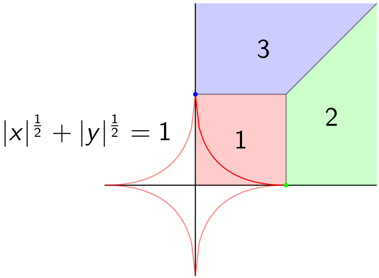
Cinderella in Action



A Line and 1/2-sphere

We consider the case when where the sets are the 1/2-sphere and a line.

- Projection onto the line is simply the orthogonal projection.
- The 1/2-sphere is more difficult:

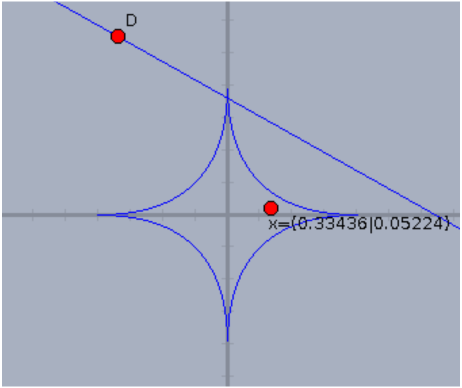


Results

- Sufficient conditions for failure of von Neumann and Dysktra's methods.
- Behaviour of Douglas-Rachford in particular cases.
- Some examples of failure.

Despite theoretical justification, these methods appear *fairly* robust.

Example: von Neumann



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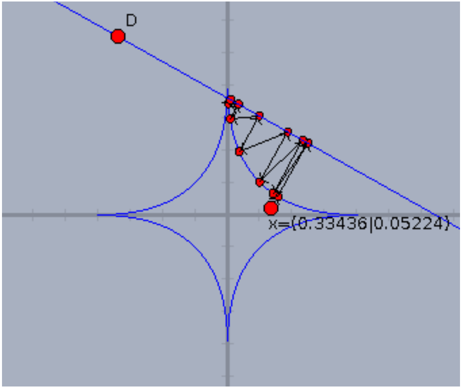

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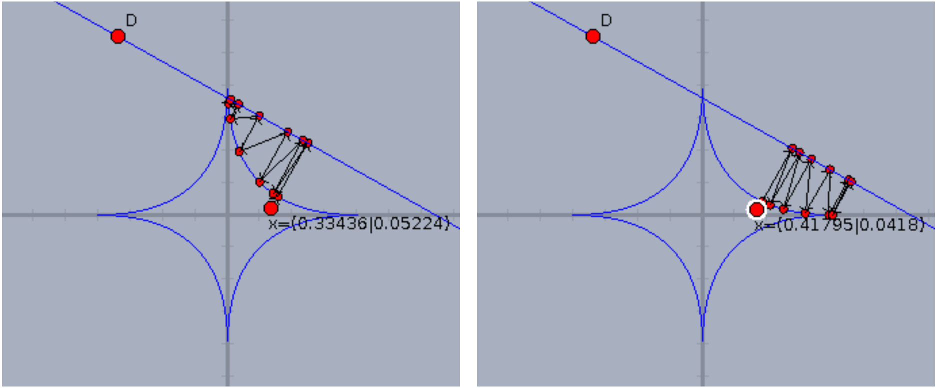
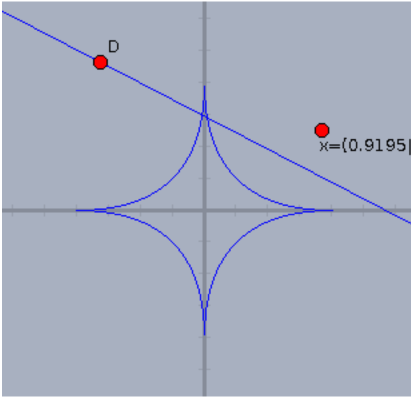


Example: Dykstra



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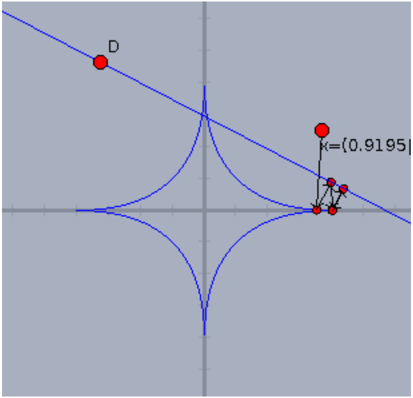

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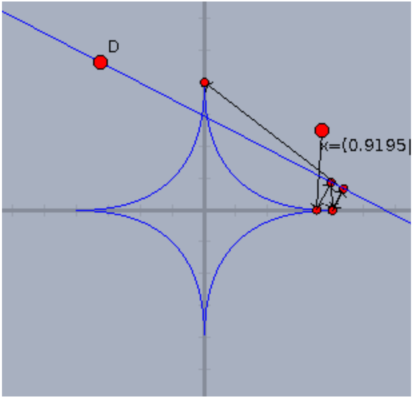

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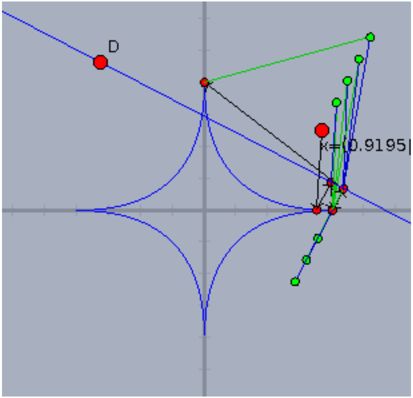

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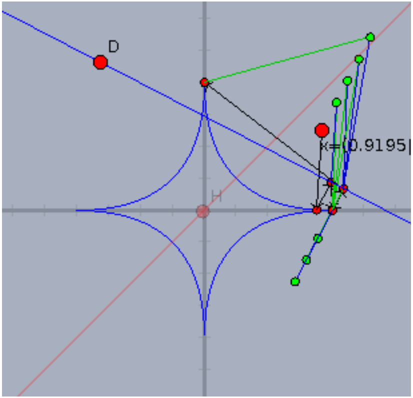

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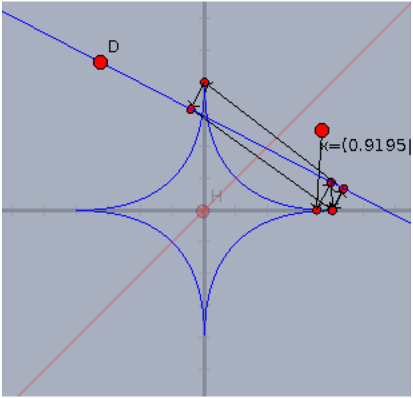

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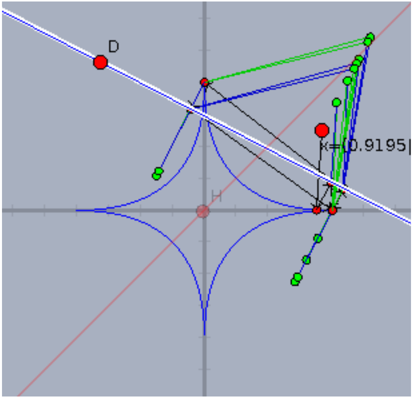
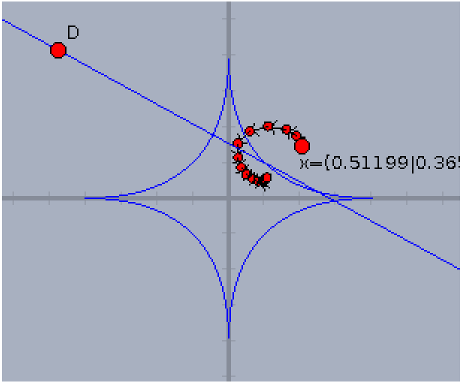


Example: Douglas-Rachford



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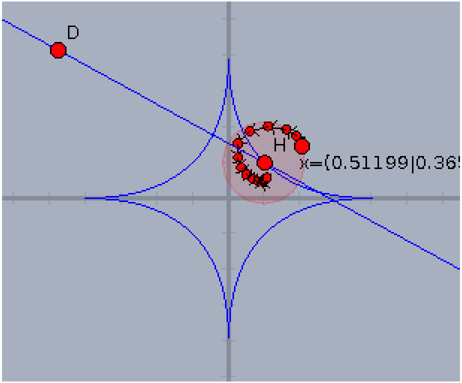

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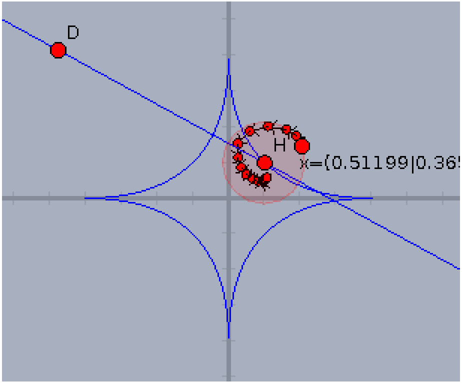

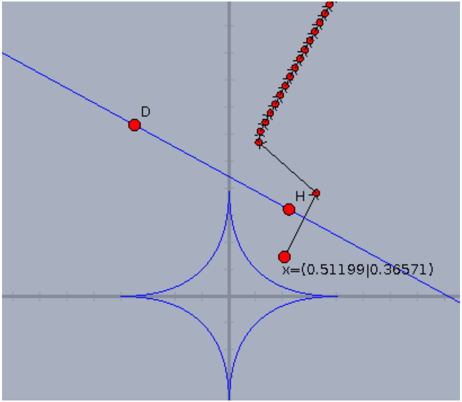

Where To Next?

Explain algorithms, in terms of:

- Local convergence results.
- Behaviour of iterations, in the case of divergence.
- Behaviour for infeasible problems. (Can infeasibility be *detected*?)
- Convergence Rate.
- Acceleration schemes.
- More examples.

Ultimately, a more complete theory for non-convex sets.

Questions?

A big thanks the Big Day In organisers, AMSI and CSIRO;
And of course to my supervisor Jon Borwein.