

Convex Feasibility Problems

Laureate Prof. Jonathan Borwein with Matthew Tam

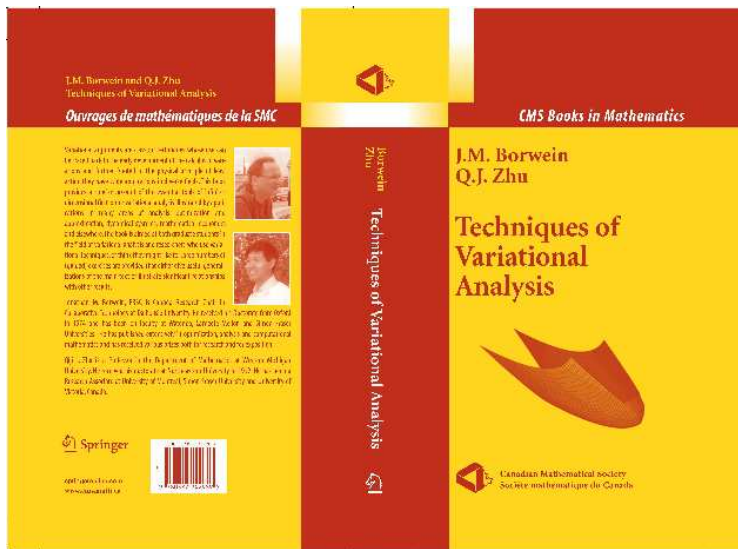
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Techniques of Variational Analysis



This lecture is based on Chapter 4.5: Convex Feasibility Problems

Let X be a Hilbert space and let $C_n, n = 1, \dots, N$ be convex closed subsets of X . The **convex feasibility problem** is to find some point

$$x \in \bigcap_{n=1}^N C_n,$$

when this intersection is non-empty.

In this talk we discuss **projection algorithms** for finding such a feasibility point. These algorithms have wide ranging applications including:

- solutions to convex inequalities,
- minimization of convex nonsmooth functions,
- medical imaging,
- computerized tomography, and
- electron microscopy

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We start by defining projection to a closed convex set and its basic properties. This is based on the following theorem.

Theorem 4.5.1 (Existence and Uniqueness of Nearest Point)

Let X be a Hilbert space and let C be a closed convex subset of X . Then for any $x \in X$, there exists a unique element $\bar{x} \in C$ such that

$$\|x - \bar{x}\| = d(C; x).$$

Proof. If $x \in C$ then $\bar{x} = x$ satisfies the conclusion. Suppose that $x \notin C$. Then there exists a sequence $x_i \in C$ such that $d(C; x) = \lim_{i \rightarrow \infty} \|x - x_i\|$. Clearly, x_i is bounded and therefore has a subsequence weakly converging to some $\bar{x} \in X$.

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Since a closed convex set is weakly closed (Mazur's Theorem), we have $\bar{x} \in C$ and $d(C; x) = \|x - \bar{x}\|$. We show such \bar{x} is unique. Suppose that $z \in C$ also has the property that $d(C; x) = \|x - z\|$. Then for any $t \in [0, 1]$ we have $t\bar{x} + (1 - t)z \in C$. It follows that

$$\begin{aligned} d(C; x) &\leq \|x - (t\bar{x} + (1 - t)z)\| = \|t(x - \bar{x}) + (1 - t)(x - z)\| \\ &\leq t\|x - \bar{x}\| + (1 - t)\|x - z\| = d(C; x). \end{aligned}$$

That is to say

$$t \rightarrow \|x - z - t(\bar{x} - z)\|^2 = \|x - z\|^2 - 2t\langle x - z, \bar{x} - z \rangle + t^2\|\bar{x} - z\|^2$$

is a constant mapping, which implies $\bar{x} = z$. ●

The nearest point can be characterized by the normal cone as follows.

Theorem 4.5.2 (Normal Cone Characterization of Nearest Point)

Let X be a Hilbert space and let C be a closed convex subset of X . Then for any $x \in X$, $\bar{x} \in C$ is a nearest point to x if and only if

$$x - \bar{x} \in N(C; \bar{x}).$$

Proof. Noting that the convex function $f(y) = \|y - x\|^2/2$ attains a minimum at \bar{x} over set C , this directly follows from the Pshenichnii–Rockafellar condition (Theorem 4.3.6):

$$0 \in \partial f(\bar{x}) + N(C; \bar{x}).$$

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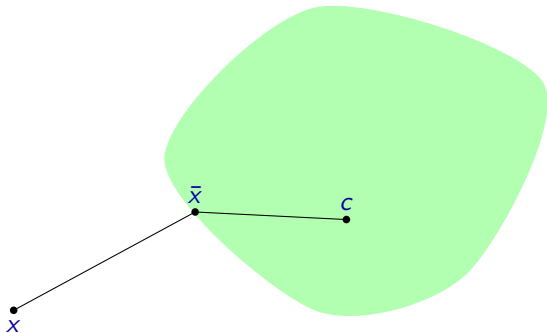
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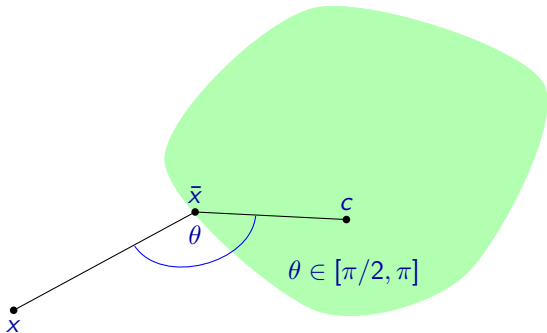


Geometrically, the normal cone characterization is:



$$x - \bar{x} \in N(C; \bar{x}) \iff \langle x - \bar{x}, c - \bar{x} \rangle \leq 0 \text{ for all } c \in C.$$

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Definition 4.5.3 (Projection)

Let X be a Hilbert space and let C be a closed convex subset of X . For any $x \in X$ the unique nearest point $y \in C$ is called the projection of x on C and we define the projection mapping P_C by $P_C x = y$.

We summarize some useful properties of the projection mapping in the next proposition whose elementary proof is left as an exercise.

Proposition 4.5.4 (Properties of Projection)

Let X be a Hilbert space and let C be a closed convex subset of X . Then the projection mapping P_C has the following properties.

- (i) for any $x \in C$, $P_C x = x$;
- (ii) $P_C^2 = P_C$;
- (iii) for any $x, y \in X$, $\|P_C y - P_C x\| \leq \|y - x\|$.

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Theorem 4.5.5 (Potential Function of Projection)

Let X be a Hilbert space and let C be a closed convex subset of X . Define

$$f(x) = \sup \left\{ \langle x, y \rangle - \frac{\|y\|^2}{2} \mid y \in C \right\}.$$

Then f is convex, $P_C(x) = f'(x)$, and therefore P_C is a monotone operator.

Proof. It is easy to check that f is convex and

$$f(x) = \frac{1}{2}(\|x\|^2 - \|x - P_C(x)\|^2).$$

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We need only show $P_C(x) = f'(x)$.

Fix $x \in X$. For any $y \in X$ we have

$$\|(x + y) - P_C(x + y)\| \leq \|(x + y) - P_C(x)\|,$$

so

$$\begin{aligned} \|(x + y) - P_C(x + y)\|^2 &\leq \|x + y\|^2 - 2\langle x + y, P_C(x) \rangle + \|P_C(x)\|^2 \\ &= \|x + y\|^2 + \|x - P_C(x)\|^2 - \|x\|^2 \\ &\quad - 2\langle y, P_C(x) \rangle, \end{aligned}$$

hence $f(x + y) - f(x) - \langle P_C(x), y \rangle \geq 0$. On the other hand, since $\|x - P_C(x)\| \leq \|x - P_C(x + y)\|$ we get

$$\begin{aligned} f(x + y) - f(x) - \langle P_C(x), y \rangle &\leq \langle y, P_C(x + y) - P_C(x) \rangle \\ &\leq \|y\| \times \|P_C(x + y) - P_C(x)\| \\ &\leq \|y\|^2, \end{aligned}$$

which implies $P_C(x) = f'(x)$. ●

Projection Algorithms as Minimization Problems

We start with the simple case of the intersection of two convex sets. Let X be a Hilbert space and let C and D be two closed convex subsets of X . Suppose that $C \cap D \neq \emptyset$. Define a function

$$f(c, d) := \frac{1}{2} \|c - d\|^2 + \iota_C(c) + \iota_D(d).$$

We see that f attains a minimum at (\bar{c}, \bar{d}) if and only if $\bar{c} = \bar{d} \in C \cap D$. Thus, the problem of finding a point in $C \cap D$ becomes one of minimizing function f .

We consider a natural descending process for f by alternately minimizing f with respect to its two variables. More concretely, start with any $x_0 \in D$. Let x_1 be the solution of minimizing

$$x \rightarrow f(x, x_0).$$

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Projection Algorithms as Minimization Problems

It follows from Theorem 4.5.2. that

$$x_0 - x_1 \in N(C; x_1).$$

That is to say $x_1 = P_C x_0$. We then let x_2 be the solution of minimizing

$$x \rightarrow f(x_1, x).$$

Similarly, $x_2 = P_D x_1$. In general, we define

$$x_{i+1} = \begin{cases} P_C x_i & i \text{ is even,} \\ P_D x_i & i \text{ is odd.} \end{cases} \quad (1)$$

This algorithm is a generalization of the classical von Neumann projection algorithm for finding points in the intersection of two subspaces.

We will show that in general x_i weakly converge to a point in $C \cap D$ and when $\text{int}(C \cap D) \neq \emptyset$ we have norm convergence.

Attracting Mappings and Fejér Sequences

We discuss two useful tools for proving the convergence.

Definition 4.5.6 (Nonexpansive Mapping)

Let X be a Hilbert space, let C be a closed convex nonempty subset of X and let $T: C \rightarrow X$. We say that T is nonexpansive provided that $\|Tx - Ty\| \leq \|x - y\|$.

Definition 4.5.7 (Attracting Mapping)

Let X be a Hilbert space, let C be a closed convex nonempty subset of X and let $T: C \rightarrow C$ be a nonexpansive mapping. Suppose that D is a closed nonempty subset of C . We say that T is attracting with respect to D if for every $x \in C \setminus D$ and $y \in D$,

$$\|Tx - y\| \leq \|x - y\|.$$

We say that T is k -attracting with respect to D if for every $x \in C \setminus D$ and $y \in D$,

$$k\|x - Tx\|^2 \leq \|x - y\|^2 - \|Tx - y\|^2.$$

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Attracting Mappings and Fejér Sequences

Lemma 4.5.8 (Attractive Property of Projection)

Let X be a Hilbert space and let C be a convex closed subset of X . Then $P_C: X \rightarrow X$ is 1-attracting with respect to C .

Proof. Let $y \in C$. We have

$$\begin{aligned}\|x - y\|^2 - \|P_C x - y\|^2 &= \langle x - P_C x, x + P_C x - 2y \rangle \\ &= \langle x - P_C x, x - P_C x + 2(P_C x - y) \rangle \\ &= \|x - P_C x\|^2 + 2\langle x - P_C x, P_C x - y \rangle \\ &\geq \|x - P_C x\|^2.\end{aligned}$$

Note that if T is attracting (k -attracting) with respect to a set D , then it is attracting (k -attracting) with respect to any subset of D .

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Attracting Mappings and Fejér Sequences

Definition 4.5.9 (Fejér Monotone Sequence)

Let X be a Hilbert space, let C be closed convex set and let (x_i) be a sequence in X . We say that (x_i) is Fejér monotone with respect to C if

$$\|x_{i+1} - c\| \leq \|x_i - c\|, \quad \text{for all } c \in C \text{ and } i = 1, 2, \dots$$

Next we summarize properties of Fejér monotone sequences.

Theorem 4.5.10 (Properties of Fejér Monotone Sequences)

Let X be a Hilbert space, let C be a closed convex set and let (x_i) be a Fejér monotone sequence with respect to C . Then

- (i) (x_i) is bounded and $d(C; x_{i+1}) \leq d(C; x_i)$.
- (ii) (x_i) has at most one weak cluster point in C .
- (iii) If the interior of C is nonempty then (x_i) converges in norm.
- (iv) $(P_C x_i)$ converges in norm.

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- (iii) If the interior of C is nonempty then (x_i) converges in norm.
- (iv) $(P_C x_i)$ converges in norm.

Proof. (i) is obvious.

Observe that, for any $c \in C$ the sequence $(\|x_i - c\|^2)$ converges and so does

$$(\|x_i\|^2 - 2\langle x_i, c \rangle). \quad (2)$$

Now suppose $c_1, c_2 \in C$ are two weak cluster points of (x_i) . Letting c in (2) be c_1 and c_2 , respectively, and taking limits of the difference, yields $\langle c_1, c_1 - c_2 \rangle = \langle c_2, c_1 - c_2 \rangle$ so that $c_1 = c_2$, which proves (ii).

To prove (iii) suppose that $B_r(c) \subset C$. For any $x_{i+1} \neq x_i$, simplifying

$$\left\| x_{i+1} - \left(c - h \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \right) \right\|^2 \leq \left\| x_i - \left(c - h \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \right) \right\|^2$$

we have

$$2h\|x_{i+1} - x_i\| \leq \|x_i - c\|^2 - \|x_{i+1} - c\|^2.$$

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we have

$$2h\|x_{i+1} - x_i\| \leq \|x_i - c\|^2 - \|x_{i+1} - c\|^2.$$

For any $j > i$, adding the above inequality from i to $j - 1$ yields

$$2h\|x_j - x_i\| \leq \|x_i - c\|^2 - \|x_j - c\|^2.$$

Since $(\|x_i - c\|^2)$ is a convergent sequence we conclude that (x_i) is a Cauchy sequence.

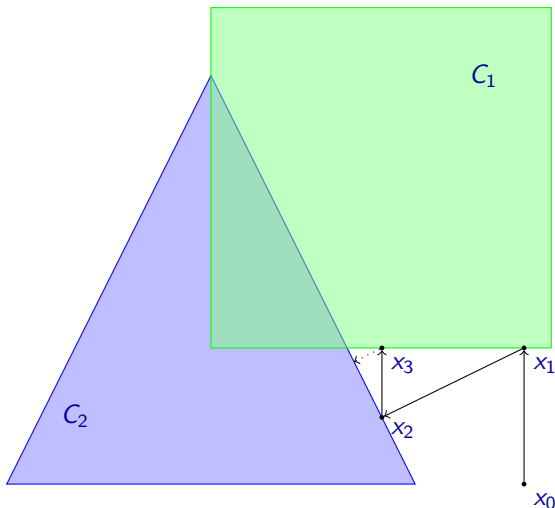
Finally, for natural numbers i, j with $j > i$, apply the parallelogram law $\|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a + b\|^2$ to $a := P_C x_j - x_j$ and $b := P_C x_i - x_j$ we obtain

$$\begin{aligned}\|P_C x_j - P_C x_i\|^2 &= 2\|P_C x_j - x_j\|^2 + 2\|P_C x_i - x_j\|^2 \\ &\quad - 4\left\|\frac{P_C x_j + P_C x_i}{2} - x_j\right\|^2 \\ &\leq 2\|P_C x_j - x_j\|^2 + 2\|P_C x_i - x_j\|^2 \\ &\quad - 4\|P_C x_j - x_j\|^2 \\ &\leq 2\|P_C x_i - x_j\|^2 - 2\|P_C x_j - x_j\|^2 \\ &\leq 2\|P_C x_i - x_i\|^2 - 2\|P_C x_j - x_j\|^2.\end{aligned}$$

We identify $(P_C x_i)$ as a Cauchy sequence, because $(\|x_i - P_C x_i\|)$ converges by (i).

Attracting Mappings and Fejér Sequences

The following example shows the first few terms of a sequence $\{x_n\}$ which is Fejér monotone with respect to $C = C_1 \cap C_2$.



Convergence of Projection Algorithms

Let X be a Hilbert space. We say a sequence (x_i) in X is *asymptotically regular* if

$$\lim_{i \rightarrow \infty} \|x_i - x_{i+1}\| = 0.$$

Lemma 4.5.11 (Asymptotical Regularity of Projection Algorithm)

Let X be a Hilbert space and let C and D be closed convex subsets of X . Suppose $C \cap D \neq \emptyset$. Then the sequence (x_i) defined by the projection algorithm

$$x_{i+1} = \begin{cases} P_C x_i & i \text{ is even,} \\ P_D x_i & i \text{ is odd.} \end{cases}$$

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is asymptotically regular.

Proof. By Lemma 4.5.8 both P_C and P_D are 1-attracting with respect to $C \cap D$. Let $y \in C \cap D$. Since x_{i+1} is either $P_C x_i$ or $P_D x_i$ it follows that

$$\|x_{i+1} - x_i\|^2 \leq \|x_i - y\|^2 - \|x_{i+1} - y\|^2.$$

Since $(\|x_i - y\|^2)$ is a monotone decreasing sequence, therefore the right-hand side of the inequality converges to 0 and the result follows. ●

Convergence of Projection Algorithms

Now, we are ready to prove the convergence of the projection algorithm.

Theorem 4.5.12 (Convergence for Two Sets)

Let X be a Hilbert space and let C and D be closed convex subsets of X . Suppose $C \cap D \neq \emptyset$ ($\text{int}(C \cap D) \neq \emptyset$). Then the projection algorithm

$$x_{i+1} = \begin{cases} P_C x_i & i \text{ is even,} \\ P_D x_i & i \text{ is odd.} \end{cases}$$

converges weakly (in norm) to a point in $C \cap D$.

Proof. Let $y \in C \cap D$. Then, for any $x \in X$, we have

$$\begin{aligned}\|P_C x - y\| &= \|P_C x - P_C y\| \leq \|x - y\|, & \text{and} \\ \|P_D x - y\| &= \|P_D x - P_D y\| \leq \|x - y\|.\end{aligned}$$

Since x_{i+1} is either $P_C x_i$ or $P_D x_i$ we have that

$$\|x_{i+1} - y\| \leq \|x_i - y\|.$$

That is to say (x_i) is a Fejér monotone sequence with respect to $C \cap D$. By item (i) of Theorem 4.5.10 the sequence (x_i) is bounded, and therefore has a weakly convergent subsequence. We show that all weak cluster points of (x_i) belong to $C \cap D$. In fact, let (x_{i_k}) be a subsequence of (x_i) converging to x weakly.

Taking a subsequence again if necessary we may assume that (x_{i_k}) is a subset of either C or D . For the sake of argument let us assume that it is a subset of C and, thus, the weak limit x is also in C . On the other hand by the asymptotical regularity of (x_j) in Lemma 4.5.11 $(P_D x_{i_k}) = (x_{i_k+1})$ also weakly converges to x . Since $(P_D x_{i_k})$ is a subset of D we conclude that $x \in D$, and therefore $x \in C \cap D$. By item (ii) of Theorem 4.5.10 (x_j) has at most one weak cluster point in $C \cap D$, and we conclude that (x_j) weakly converges to a point in $C \cap D$. When $\text{int}(C \cap D) \neq \emptyset$ it follows from item (iii) of Theorem 4.5.10 that (x_j) converges in norm. ●

Convergence of Projection Algorithms

Whether the alternating projection algorithm converged in norm without the assumption that

$$\text{int}(C \cap D) \neq \emptyset,$$

or more generally of metric regularity, was a long-standing open problem.

Recently Hundal constructed an example showing that the answer is negative [5].

The proof of Hundal's example is self-contained and elementary. However, it is quite long and delicate, therefore we will be satisfied in stating the example.

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Convergence of Projection Algorithms

Example 4.5.13 (Hundal)

Let $X = \ell_2$ and let $\{e_i \mid i = 1, 2, \dots\}$ be the standard basis of X . Define $v: [0, +\infty) \rightarrow X$ by

$$v(r) := \exp(-100r^3)e_1 + \cos((r - [r])\pi/2)e_{[r]+2} + \sin((r - [r])\pi/2)e_{[r]+3},$$

where $[r]$ signifies the integer part of r and further define

$$C = \{e_1\}^\perp \text{ and } D = \text{conv}\{v(r) \mid r \geq 0\}.$$

Then the hyperplane C and cone D satisfies $C \cap D = \{0\}$. However, Hundal's sequence of alternating projections x_i given by

$$x_{i+1} = P_D P_C x_i$$

starting from $x_0 = v(1)$ (necessarily) converges weakly to 0 , but not in norm.

Convergence of Projection Algorithms

A related useful example is the moment problem.

Example 4.5.14 (Moment Problem)

Let X be a Hilbert lattice¹ with lattice cone $D = X^+$. Consider a linear continuous mapping A from X onto \mathbb{R}^N . The moment problem seeks the solution of $A(x) = y \in \mathbb{R}^N, x \in D$.

Define $C = A^{-1}(y)$. Then the moment problem is feasible iff

$$C \cap D \neq \emptyset.$$

A natural question is whether the projection algorithm converges in norm.

This problem is answered affirmatively in [1] for $N = 1$ yet remains open in general when $N > 1$.

¹All Hilbert lattices are realized as $L_2(\Omega, \mu)$ in the natural ordering for some measure space.

Projection Algorithms for Multiple Sets

We now turn to the general problem of finding some points in

$$\bigcap_{n=1}^N C_n,$$

where C_n , $n = 1, \dots, N$ are closed convex sets in a Hilbert space X .

Let a_n , $n = 1, \dots, N$ be positive numbers. Denote

$$X^N := \{x = (x_1, x_2, \dots, x_N) \mid x_n \in X, n = 1, \dots, N\}$$

the product space of N copies of X with inner product

$$\langle x, y \rangle = \sum_{n=1}^N a_n \langle x_n, y_n \rangle.$$

Then X^N is a Hilbert space.

Projection Algorithms for Multiple Sets

Define

$$C := C_1 \times C_2 \times \cdots \times C_N, \quad \text{and}$$

$$D := \{(x_1, \dots, x_N) \in X^N : x_1 = x_2 = \cdots = x_N\}.$$

Then C and D are closed convex sets in X^N and

$$x \in \bigcap_{n=1}^N C_n \iff (x, x, \dots, x) \in C \cap D.$$

Applying the projection algorithm (1) to the convex sets C and D defined above we have the following generalized projection algorithm for finding some points in

$$\bigcap_{n=1}^N C_n,$$

as we shall now explain.

Projection Algorithms for Multiple Sets

Define

$$C := C_1 \times C_2 \times \cdots \times C_N, \quad \text{and}$$
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as we shall now explain.

Projection Algorithms for Multiple Sets

Denote $P_n = P_{C_n}$. The algorithm can be expressed by

$$x_{i+1} = \left(\sum_{n=1}^N \lambda_n P_n \right) x_i, \quad (3)$$

where $\lambda_n = a_n / \sum_{m=1}^N a_m$.

In other words, each new approximation is the convex combination of the projections of the previous step to all the sets $C_n, n = 1, \dots, N$. It follows from the convergence theorem in the previous subsection that the algorithm (3) converges weakly to some point in $\bigcap_{n=1}^N C_n$ when this intersection is nonempty.

Projection Algorithms for Multiple Sets

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Projection Algorithms for Multiple Sets

Theorem 4.5.15 (Weak Convergence for N Sets)

Let X be a Hilbert space and let $C_n, n = 1, \dots, N$ be closed convex subsets of X . Suppose that $\bigcap_{n=1}^N C_n \neq \emptyset$ and $\lambda_n \geq 0$ satisfies $\sum_{n=1}^N \lambda_n = 1$. Then the projection algorithm

$$x_{i+1} = \left(\sum_{n=1}^N \lambda_n P_n \right) x_i,$$

converges weakly to a point in $\bigcap_{n=1}^N C_n$.

Proof. This follows directly from Theorem 4.5.12. ●

Projection Algorithms for Multiple Sets

When the interior of $\bigcap_{n=1}^N C_n$ is nonempty we also have that the algorithm (3) converges in norm. However, since D does not have interior this conclusion cannot be derived from Theorem 4.5.12. Rather it has to be proved by directly showing that the approximation sequence is Fejér monotone w.r.t. $\bigcap_{n=1}^N C_n$.

Theorem 4.5.16 (Strong Convergence for N Sets)

Let X be a Hilbert space and let $C_n, n = 1, \dots, N$ be closed convex subsets of X . Suppose that $\text{int} \bigcap_{n=1}^N C_n \neq \emptyset$ and $\lambda_n \geq 0$ satisfies $\sum_{n=1}^N \lambda_n = 1$. Then the projection algorithm

$$x_{i+1} = \left(\sum_{n=1}^N \lambda_n P_n \right) x_i,$$

converges to a point in $\bigcap_{n=1}^N C_n$ in norm.

Projection Algorithms for Multiple Sets

When the interior of $\bigcap_{n=1}^N C_n$ is nonempty we also have that the algorithm (3) converges in norm. However, since D does not have interior this conclusion cannot be derived from Theorem 4.5.12. Rather it has to be proved by directly showing that the approximation sequence is Fejér monotone w.r.t. $\bigcap_{n=1}^N C_n$.

Theorem 4.5.16 (Strong Convergence for N Sets)

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$$x_{i+1} = \left(\sum_{n=1}^N \lambda_n P_n \right) x_i,$$

converges to a point in $\bigcap_{n=1}^N C_n$ in norm.

Proof. Let $y \in \bigcap_{n=1}^N C_n$. Then

$$\begin{aligned}\|x_{i+1} - y\| &= \left\| \left(\sum_{n=1}^N \lambda_n P_n \right) x_i - y \right\| = \left\| \sum_{n=1}^N \lambda_n (P_n x_i - P_n y) \right\| \\ &\leq \sum_{n=1}^N \lambda_n \|P_n x_i - P_n y\| \leq \sum_{n=1}^N \lambda_n \|x_i - y\| = \|x_i - y\|.\end{aligned}$$

That is to say (x_i) is a Fejér monotone sequence with respect to $\bigcap_{n=1}^N C_n$. The norm convergence of (x_i) then follows directly from Theorems 4.5.10 and 4.5.15. ●

Commentary and Open Questions

- We have proven convergence of the **projection algorithm**. It can be traced to von Neumann, Weiner and before, and has been studied extensively.
- We emphasize the relationship between the projection algorithm and variational methods in Hilbert spaces:
 - While projection operators can be defined outside of the setting of Hilbert space, they are **not necessarily non-expansive**.
 - In fact, **non-expansivity of the projection operator characterizes Hilbert space** in two more dimensions.
- The **Hundal example** clarifies many other related problems regarding convergence. Simplifications of the example have since been published.
 - What happens if we only allow “nice” cones?
- **Bregman distance** provides an alternative perspective into many generalizations of the projection algorithm.

- 1 Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be nonexpansive and let $\alpha \in [-1, 1]$. Show that $(I + \alpha T)$ is a *maximally monotone* continuous operator.
- 2 (Common projections) Prove formula for the projection onto each of the following sets:

- 1 Half-space: $H := \{x \in \mathcal{H} : \langle a, x \rangle = b\}$, $0 \neq a \in \mathcal{H}$, $b \in \mathbb{R}$.

- 2 Line: $L := x + \mathbb{R}y$ where $x, y \in \mathcal{H}$.

- 3 Ball: $B := \{x \in \mathcal{H} : \|x\| \leq r\}$ where $r > 0$.






- 4 Ellipse in \mathbb{R}^2 : $E := \{(x, y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 = 1\}$.

Hint: $P_E(u, v) = \left(\frac{a^2 u}{a^2 - t}, \frac{b^2 v}{b^2 - t} \right)$ where t solves

$$\frac{a^2 u^2}{(a^2 - t)^2} + \frac{b^2 v^2}{(b^2 - t)^2} = 1.$$

- 3 (Non-existence of best approximations) Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of an infinite dimensional Hilbert space. Define the set $A := \{e_1/n + e_n : n \in \mathbb{N}\}$. Show that A is norm closed and bounded but $d_A(0) = 1$ is not attained. Is A weakly closed?

References

-  H.H. Bauschke & J.M. Borwein. **On the convergence of von Neumann's alternating projection algorithm for two sets.** *Set-Valued Analysis* 1(2):185–212 (1993).
-  J.M. Borwein & Q.J. Zhu. **Techniques of Variational Analysis.** Springer (2013).
-  Y. Censor & A. Cegielski. **Projection methods: an annotated bibliography of books and reviews.** *Optimization*, accepted for publication. Preprint PDF.
-  R. Escalante & M. Raydan, **Alternating projection methods.** SIAM Fundamentals of Algorithms (2011)
-  H.S. Hundal. **An alternating projection that does not converge in norm.** *Nonlinear Analysis: Theory, Methods & Applications* 57(1):35–61 (2004).

Many resources (and definitions) available at:

<http://www.carma.newcastle.edu.au/jon/ToVA/>