3. FURTHER CONDITIONS SUFFICIENT FOR NORMAL STRUCTURE AND RELATED CONDITIONS.

(3·1) The Condition of Opial.

A Banach space [dual space] X satisfies the weak [weak*] Opial condition if whenever (x_n) converges weakly [weak*] to x_∞ and $x_0 \neq x_\infty$ we have

$$\liminf_n \|x_n - x_\infty\| < \liminf_n \|x_n - x_0\|$$

If equality is allowed in the above inequality we will say X satisfies the non-strict weak [weak*] Opial condition.

In the weak* case it is natural to allow (x_n) to be a net. Otherwise we would need to restrict attention to spaces with a weak*-sequentially compact ball [For example; the dual of a separable space, or more generally the dual of any smoothable space - Sullivan and Hagler, 1979].

Zdzisław Opial [1967] introduced the weak condition to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a non-expansive map on a closed convex subset to a fixed point.

A more extensive examination of the condition was made by Gossez and Lami-Dozo [1972]. In particular they prove the following.

3.1.1 THEOREM. If X is a Banach space satisfying the weak Opial condition then X has weak normal structure, in particular then X has the ω-F.P.P.

Proof. Suppose X fails to have ω-normal structure then by Remark (1.18) X contains a weakly null sequence (x_n) satisfying
$$\lim_n \operatorname{dist}(x_{n+1}, \overline{co}\{x_k\}_{k=1}^n) = d,$$
where $d = \operatorname{diam} \overline{co}\{x_k\}_{k=1}^\infty$.
Now given any $\varepsilon > 0$ there exists, by Mazur, $\lambda_1, \lambda_2, \cdots, \lambda_{n_0} \geq 0$ with $\sum_{i=1}^{n_0} \lambda_i = 1$ such that
$$\left\| \sum_{i=1}^{n_0} \lambda_i x_i \right\| < \varepsilon/2.$$

Also, there exists $N > n_0$ so that for $m \geq N$ we have $\operatorname{dist}(x_m, \overline{co}\{x_k\}_{k=1}^{n_0}) > d - \varepsilon/2$
But then,
$$\begin{aligned} d \geq \|x_m\| &\geq \left\|x_m - \sum_{i=1}^{n_0} \lambda_i x_i\right\| - \varepsilon/2 \\ &\geq \operatorname{dist}(x_m, \overline{co}\{x_k\}_{k=1}^{n_0}) - \varepsilon/2 \\ &> d - \varepsilon. \end{aligned}$$
Thus $x_m \xrightarrow{\omega} 0$ while for any $x_0 \in co\{x_k\}_1^\infty$ we have

$$\liminf_n \|x_n\| = d \geq \liminf_n \|x_n - x\|$$

contradicting the weak opial condition. ▨

The use of maguri's result in the above proof leaves open the following.

QUESTION: Does a dual space with the weak* opial condition have w*-normal structure?

None-the-less we do have the following result, proved indirectly by Karlovitz [1976].

B·1·2 PROPOSITION If X is a dual space satisfying the weak* opial condition, then X has the w*-F.P.P.

Proof [van Dulst, 1982]. Let C be any w*-compact convex subset of X and let $T: C \to C$ be a non-expansive map. Choose $(x_n) \subset C$ so that $\|x_n - Tx_n\| \to 0$ (approximate fixed point sequence for T). Passing to a subnet if necessary we may assume that $x_n \xrightarrow{w^*} x_\infty$.

Then $\liminf_n \|Tx_\infty - x_n\|$

$$= \liminf_n \|Tx_\infty - Tx_n\|$$

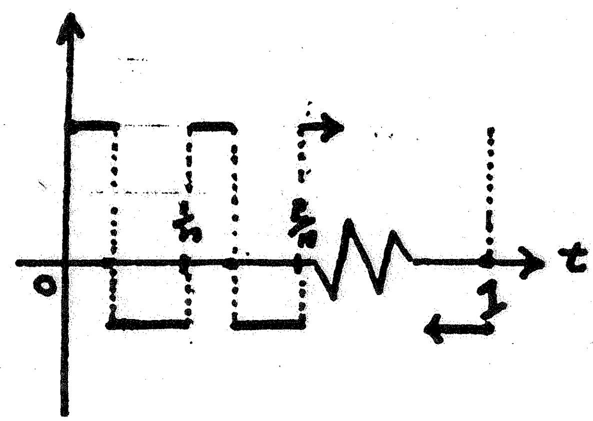
$$\leq \liminf \|x_\infty - x_n\|$$

contradicting the w^* Opial condition unless $Tx_\infty = x_\infty$. ∎

QUESTION: <u>What can be said of spaces satisfying the non-strict Opial conditions?</u>

We <u>remark</u> that $\mathcal{L}_1[0,1]$ fails the non-strict weak Opial condition. To see this, let f_n be the function obtained by extending periodically to all of $[0,1]$ the function defined on $[0, 1/n]$ by

$$f_n(t) = \begin{cases} 2 & 0 \le t \le \frac{1}{3n} \\ -1 & \frac{1}{3n} < t \le \frac{1}{n} \end{cases}$$



It is readily seen that $f_n \xrightarrow{w} 0$, $\|f_n\|_1 = 4/3$, for all n, while $\|f_n - f_0\|_1 = 1$ where $f_0(t) = -1$. Thus the weak non-strict Opial condition is violated.

In order to facilitate the investigation of specific spaces we consider <u>the relationship between the Opial conditions and other properties of the space.</u>

Karlovitz [1976] established intimate relationships between weak [and weak*] Opial condition and "approximate symmetry" in the Birkhoff-James notion of orthogonality, we however will persue a different line.

Recall, the <u>duality map</u> for a Banach space X is

$$\mathcal{D} : X \longrightarrow 2^{X^*} : x \longmapsto \{ f \in X^* : f(x) = \|f\|^2 = \|x\|^2 \}$$

By an (extended) <u>support mapping</u> for X we shall understand a mapping of the form

$$x \longmapsto \mu(\|x\|) f$$

where μ is a strictly increasing continuous gauge function with $\mu(0) = 0$ and f is a selection from $\mathcal{D}(x/\|x\|)$.

Opial [1967] observed that a uniformly convex space which admits a weak to weak* sequentially continuous support mapping satisfies the weak Opial condition. That uniform convexity need not imply the existence of such a support mapping had been observed by Browder [1966] and Hayes and Sims (in connection with operator numerical ranges). Indeed $L_4[0,1]$ does not have a weak to weak (equal to weak*) continuous support mapping. Opial extended this to $L_p[0,1]$ for all $p \neq 2$. In fact, Fixman and Rao have characterised $L_p(\mathcal{S}, \Sigma, \mu)$ spaces with a weak to weak continuous support mapping as those spaces for which every $A \in \Sigma$ with $0 < \mu(A) < \infty$ contains an atom.

The early results were substantially improved by Gossez and Lami-Dozo [1972]. They showed that the assumption of uniform convexity is unnecessary:
A Banach space [dual space] with a weak [weak*] to weak* sequentially continuous support mapping satisfies the weak [weak*] Opial condition. However this condition is not necessary: For $1 < p < q < \infty$ the space $(\ell_p \oplus \ell_q)_2$ satisfies the weak Opial condition, but [Bruck, 1969] no support mapping is weak to weak continuous.

(*) with w*-seq compact ball

The result of Gossez and Lami-Dozo is an immediate corollary of

(3.1.3) THEOREM [Sims, 1985]: The Banach space [dual space(*)] X satisfies the weak [weak*] Opial condition if and only if whenever (x_n) converges weakly [weak*] to a non-zero limit x_∞ there is a $\delta > 0$ so that eventually $\mathcal{D}(x_n)x_\infty \subset [\delta, \infty)$.

Proof. ($\Rightarrow$) Assume this were not the case,

(and scalling)
by passing to a subsequence we can find (x_n) converging weakly [weak*] to x_∞ with $1 = \|x_n\| \geq \|x_\infty\| > 0$ and $f_n \in \mathcal{D}(x_n)$ such that $\lim_n f_n(x_\infty) \leq 0$.

But
$$\begin{aligned} 1 &= \liminf_n \|x_n - 0\| \\ &> \liminf_n \|x_n - x_\infty\| \\ &\geq \liminf_n f_n(x_n - x_\infty) \\ &= \liminf_n (1 - f_n(x_\infty)) \\ &= 1 - \lim_n f_n(x_\infty) \end{aligned}$$
whence $\lim_n f_n(x_\infty) > 0$, a contradiction.

($\Leftarrow$) [a modification of the proof in Gossez and Lami-Dozo 1972]

Using the integral representation for the convex function $t \mapsto \frac{1}{2}\|x + ty\|^2$ [Roberts and Varberg, 1973, 12 Theorem A] we have

$$\tfrac{1}{2}\|x+y\|^2 = \tfrac{1}{2}\|x\|^2 + \int_0^1 g^+(x+ty;y)\,dt$$

where
$$g^+(u;y) := \operatorname*{limit}_{h \to 0+} \frac{\frac{1}{2}\|u+hy\|^2 - \frac{1}{2}\|u\|^2}{h}.$$

To establish the weak [weak*] Opial condition it suffices to show that if y_n converges weakly [weak*] to $y_\infty \neq 0$, then
$$\liminf_n \tfrac{1}{2}\|y_n\|^2 > \liminf_n \tfrac{1}{2}\|y_n - y_\infty\|^2.$$

Now,
$$\tfrac{1}{2}\|y_n\|^2 = \tfrac{1}{2}\|y_n - y_\infty\|^2 + \int_0^1 g^+(y_n - y_\infty + ty_\infty; y_\infty)\,dt$$

So

$$\liminf_n \tfrac{1}{2}\|y_n\|^2 \geq \liminf_n \tfrac{1}{2}\| y_n - y_\infty \|^2 + \liminf_n \int_0^1 g^+(y_n - y_\infty + t y_\infty ; y_\infty)\, d$$

By Fatou's lemma [see for example, Halmos 1950] it is therefore sufficient to prove that

$$\liminf_n g^+(y_n - y_\infty + t y_\infty ; y_\infty) > 0$$

for each $t \in (0,1)$.

But, by a well known characterization of the upper Gateaux derivative [see for example, Barbu and Precupanu, 1978, §2·1 example 2° and proposition 2·3] we have

$$g^+(y_n - y_\infty + t y_\infty ; y_\infty) = \max\{f(y_\infty) : f \in \mathcal{D}(y_n - y_\infty + t y_\infty$$

and so since $y_n - y_\infty + t y_\infty$ converges weakly [weak*] to $t y_\infty \neq 0$ we have for n sufficiently large and some $\delta > 0$ that

$$f(t y_\infty) > \delta \quad \text{for all } f \in \mathcal{D}(y_n - y_\infty + t y_\infty)$$

That is, for n sufficiently large (depending on t)

$$g^+(y_n - y_\infty + t y_\infty ; y_\infty) \geq \delta/t > 0. \quad \blacksquare$$

<u>(4·1·3 a) REMARK</u>: For the ($\Longrightarrow$) part of the above proof we see it is only necessary that X satisfy <u>If (x_n) converges weakly [weak*] to $x_\infty \neq 0$ we have $\liminf_n \max\{f(x_\infty) : f \in \mathcal{D}(x_n)\} > 0$</u>.

Thus for a space satisfying this we also have
$\liminf_n \min\{f(x_\infty) : f \in \mathcal{D}(x_n)\} > 0$.

(3·1·4) EXAMPLES.

1) For $1 \le p < \infty$ the space l_p satisfies the weak Opial condition. For $p=1$ this follows by the Schur property. For $p>1$ the duality map is single valued and given by
$x = (x(1), x(2), \dots) \longmapsto \|x\|_p^{2-p}(|x(1)|^{p-1}\operatorname{sgn} x(1), \dots)$
Thus, if $x_n \xrightarrow{\omega} x_\infty \neq 0$ we have
$\|x_n\|_p^{p-2} f_n \xrightarrow{\omega} \|x_\infty\|_p^{p-2} f_\infty$ where $f_i = \mathcal{D}(x_i)$.
Consequently, since $\liminf_n \|x_n\|_p \ge \|x_\infty\|_p$, we have

$$\liminf_n f_n(x_\infty) \ge \|x_\infty\|_p^2 > 0.$$

Further, if $p=1$ and $x_n \xrightarrow{\omega^*} x_\infty \neq 0$, choosing $f_n \in \mathcal{D}(x_n)$ so that

$$f_n(i) = |x_n(i)| \operatorname{sgn} x_n(i)$$

we see that:
given $\varepsilon > 0$ there exist n_0 so that $\sum_{i=n_0+1}^{\infty} |x_\infty(i)| < \varepsilon$
and so

$$f_n(x_\infty) \geqslant \sum_{i=1}^{n_0} f_n(i)\, x_\infty(i) - \|f_n\|\, \varepsilon .$$

Since $\{\|f_n\|\}_{n=1}^{\infty}$ is bounded and $x_n(i) \to x_\infty(i)$ it follows that

$$\liminf_n f_n(x_\infty) \geqslant \|x_\infty\|^2 .$$

Thus $\lim_n f_n(x_\infty) = \|x_\infty\|^2 = f_\infty(x_\infty) > 0$ and so

<u>ℓ_1 satisfies the weak* Opial condition.</u>
In particular <u>ℓ_1 has the w^*-F.P.P.</u>

2) <u>For $p \neq 2$ the space $\mathcal{L}_p[0,1]$ fails to satisfy the weak Opial Condition.</u> The case $p=1$ has already been considered. The same example works in $\mathcal{L}_p[0,1]$ for all $p \neq 2$. Indeed for the sequence f_n defined previously we have for any number $c \in [-2,1]$

$$\|f_n + c\|_p^p = \tfrac{1}{3}(2+c)^p + \tfrac{2}{3}(1-c)^p$$

is a minimum at c_0 satisfying

$$\frac{2+c_0}{1-c_0} = 2^{1/p-1}.$$

That is;

$$p=1 \Longrightarrow c_0 = 1$$
$$1<p<2 \Longrightarrow 0<c<1$$
$$p=2 \Longrightarrow c=0$$
$$p>2 \Longrightarrow -\tfrac{1}{2} \leq c < 0.$$

In particular then for $p \neq 2$

$$\|f_n + c_0\|_p < \|f_n\| \qquad (\text{as } c_0 \neq 0)$$

and so the space fails to satisfy the weak Opial condition.

<u>(3·1·5) REMARKS:</u> 1) <u>Relationship to U.C.E.D.</u>
The uniformly convex space $\mathcal{L}_4[0,1]$ of example 2) above amply demonstrates that:
U.C.E.D. is not sufficient for the weak Opial

condition to be satisfied.
On the other hand l_1 has the weak Opial condition but is not even strictly convex, so weak Opial $\not\Rightarrow$ U.C.E.D.

Thus U.C.E.D. and the weak [weak*] Opial conditions are effectively independent conditions sufficient to ensure weak [weak*] normal structure.

2) van Dulst [1982] has shown that every separable Banach space admits an equivalent norm with respect to which the weak Opial condition is satisfied. He also gives an equivalent renorming for any separable dual space with respect to which the space satisfies the weak* Opial condition. (This was the basis for remark (2·11·2) 3).

(3·2) Uniform Kadec – Klee Conditions

The material of this section is a development of ideas in van Dulst – Sims [1983] which are based on notions introduced by Huff [1980].

Recall a Banach space has the <u>property of Kadec – Klee</u> (Property H, perhaps more properly termed the

Radon-Riesz property) if whenever $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$ we have $\|x_n - x\| \to 0$.

This may be reformulated as stating: <u>every weakly compact subset of the unit sphere;</u>

$$\underline{S_X := \{x \in X : \|x\| = 1\}},$$

<u>is norm compact.</u>

Define the <u>measure of compactness</u> of a subset S by

$$\gamma(S) := \sup_{(x_n) \subseteq S} \inf_{m \neq n} \|x_m - x_n\|$$

(The supremum is taken over all infinite sequences of points in S).

<u>REMARKS</u>: 1) γ is equivalent to the "usual" measure of compactness;

$$K(S) := \inf\{\varepsilon > 0 : S \text{ has a finite } \varepsilon\text{-cover}\},$$

indeed $K(S) \le \gamma(S) \le 2K(S)$.

2) γ enjoys the following properties.

(a) $\gamma(S) = 0$ if and only if S is (norm) compact

(b) If $S_1 \subseteq S_2$ then $\gamma(S_1) \le \gamma(S_2)$

(c) $\gamma(S_1 \cup S_2) = \max\{\gamma(S_1), \gamma(S_2)\}$

(d) [Kuratowski] If $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$ is a nested sequence of closed non-empty sets with $\gamma(S_n) \to 0$, then

(i) $K = \bigcap_{n=1}^{\infty} S_n$ is non-empty and compact and

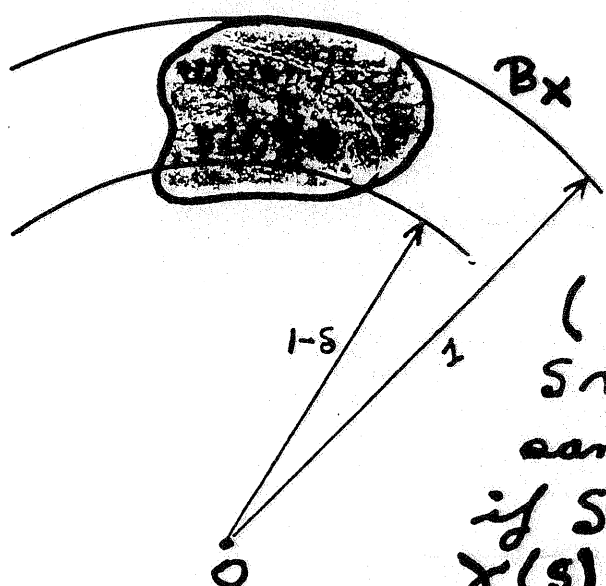
(ii) For any $\varepsilon > 0$ and n sufficiently large $S_n \subseteq K + \varepsilon B_X$.

3) By theorem (1.16) a diametral set D has $\gamma(D) = \operatorname{diam}(D)$.

4) In terms of γ the Kadec-Klee property becomes: <u>If S is a weakly compact subset of B_X with $\gamma(S) > 0$, then $\operatorname{dist}(S, 0) \leq 1$.</u>

$<$

Given $\varepsilon \in (0,1)$ we shall say that X is <u>ε-Uniformly Kadec-Klee</u> (<u>ε-UKK</u>) if there exists $\delta > 0$ so that whenever S is a weakly compact subset of B_X with $\gamma(S) > \varepsilon$ we have $\operatorname{dist}(S, 0) \leq 1 - \delta$ (or eqivalently $S \cap (1-\delta) B_X \neq \emptyset$).



<u>REMARK</u>: ε-UKK may be compared to the notion of ε-inquadrate, where if S is "metrically big" ($\operatorname{diam} S > \varepsilon$) we have $S \cap (1-\delta) B_X \neq \emptyset$. Here the same conclusion follows if S is "topologically large": $\gamma(S) > \varepsilon$.

<u>Note</u>: Our ε-UKK is the w-UKK of van Dulst-Sims [83]

<u>(3·2·1) Proposition</u>: For a Banach space X t.f.a.e.

i) X is ε-UKK

ii) whenever $C \subseteq B_X$ is a weak compact <u>convex</u> set with $\gamma(C) > \varepsilon$ we have $C \cap (1-\delta)B_X \neq \phi$.

iii) whenever $(x_n) \subset B_X$ has $sep(x_n) := \inf_{m \neq n} \|x_n - x_m\| > \varepsilon$ and $x_n \xrightarrow{\omega} x$ we have $\|x\| \leqslant 1-\delta$.

<u>Proof</u>. Clearly, i) $\Rightarrow$ ii) and iii) $\Rightarrow$ i).

<u>ii) $\Rightarrow$ iii)</u> Suppose there exists $(x_n) \subset B_X$ with $sep(x_n) > \varepsilon$, $x_n \xrightarrow{\omega} x$ and $\|x\| > 1-\delta$. Let f be a norm one linear functional which strictly separates x from $(1-\delta)B_X$ and let n_0 be such that for $n \geqslant n_0$ we have $f(x_n) \geq \frac{1}{2}(f(x)+1-\delta)$.

By Krein-Šmulian ← Then, $C = \overline{co}\{x_n\}_{n=n_0}^{\infty}$ is a weak compact convex set with $\gamma(C) > \varepsilon$ and $f(y) \geq \frac{1}{2}(f(x)+1-\delta) > 1-\delta$ for all $y \in C$. That is $C \cap (1-\delta)B_X = \phi$. ■

If X is ε-UKK for all $\varepsilon \in (0,1)$ we say X is <u>UKK</u>. Huff's original definition of this notion is the equivalent reformulation resulting from proposition (3·2·1, iii).

(3·2·2) THEOREM [van Dulst - Sims, 83]: If X is ε-UKK (see 1.1), then X has w-normal structure. In particular X has the w-F.P.P.

Proof. Suppose X contains a weak compact convex diametral set containing more than one point. Then, by Theorem (1·16) and the ensuing remark (1·18), there exists $(x_n) \subset X$ with $x_n \xrightarrow{w} 0$ and $\lim_n \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^{n}) = 1$. Since $0 \in \overline{\operatorname{co}}\{x_k\}_{k=1}^{\infty}$, by Mazur, it follows that $\|x_n\| \to 1$. Let δ be as in the definition of ε-UKK and choose n_0 so that for $n \geq n_0$ we have $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^{n}) > \varepsilon$ and $\|x_n\| > 1-\delta$. Let $y_n = x_{n_0+n} - x_{n_0}$, then $\|y_n\| \leq 1$, $\operatorname{sep}(y_n) > \varepsilon$ and $y_n \xrightarrow{w} -x_{n_0}$. But $\|x_{n_0}\| > 1-\delta$, contradicting ε-UKK by part (iii) of the previous proposition. ▨

The above theorem may be strengthened as follows.

(3·2·3) THEOREM [van Dulst - Sims, 83]: Let X be UKK and let C be a weak compact convex set. Then, the chebyshev centre of C, $\mathcal{C}(C)$, is norm compact (convex and non-empty).

<u>Proof.</u> Suppose $\mathcal{C}(C)$ is not compact, then it contains a sequence (x_n) with $\operatorname{sep}(x_n) \geq \varepsilon_0$, for some $\varepsilon_0 > 0$. By passing to a subsequence we may assume $x_n \xrightarrow{\omega} x$. Let δ be as in the definition of $\varepsilon_0/\operatorname{rad} C$ – UKK and fix $y \in C$, and let $y_n = (x_n - y)/\operatorname{rad}(C)$. Then $\|y_n\| \leq 1$, $\operatorname{sep}(y_n) \geq \varepsilon_0/\operatorname{rad} C$ and $y_n \xrightarrow{\omega} (x-y)/\operatorname{rad}(C)$, so by UKK $\|x-y\| \leq (1-\delta) \operatorname{rad} C$. Since y is arbitrary this gives $\operatorname{rad} C \leq (1-\delta) \operatorname{rad} C$, a clear contradiction. ∎

<u>3.2.4 EXAMPLES.</u>

1) Vacuously every finite dimensional space and every Schur space has UKK. In particular l_1 has UKK. Indeed any l_1 sum of finite dimensional spaces has UKK [Huff, 80]. This shows that in general UKK does not imply U.C.E.D. The next example establishes the essential independence of these two properties even in the presence of reflexivity.

2) By theorem (2.11) the space

$$X := (l_2 \oplus l_3 \oplus \cdots \oplus l_n \oplus \cdots)_2$$

is a reflexive space which can be given an equivalent U.C.E.D. norm. However, as we now show, it admits no equivalent UKK norm [Huff, 1980].
For any Banach space X and $\varepsilon > 0$ define for $S \subseteq X$ by

$$\beta_\varepsilon(S) := \{x : \text{there exists } (x_n) \subseteq S \text{ with } \operatorname{sep}(x_n) > \varepsilon \text{ and } x_n \xrightarrow{\omega} x\}$$

<u>Claim</u>: If X has an equivalent UKK norm, then for each $\varepsilon > 0$ there exists n_0 such that $\beta_\varepsilon^{n_0}(B_X) = \phi$.

Since the conclusion is isomorphically invariant we might as well assume that the UKK norm is the given one. Now let δ be that associated with ε in the definition of UKK, then by (3.2.1) (iii)

$\beta_\varepsilon(B_X) \subseteq (1-\delta) B_X$, iterating

$\beta_\varepsilon^n(B_X) \subseteq (1-\delta)^n B_X$. Choosing n_0 so that $(1-\delta)^{n_0 - 1} < \varepsilon/2$ we see from the definition of $\beta_\varepsilon(S)$ that $\beta_\varepsilon^{n_0}(B_X)$ must be empty ($\beta_\varepsilon^{n_0 - 1}(B_X)$ has diameter less than ε and so cannot contain any sequence with a separation constant of ε or more).

To see that $X = (l_2 \oplus \cdots \oplus l_n \oplus \cdots)_2$ cannot be equivalently renormed to be UKK it suffices, in the light of the above claim, to show

$$\beta_{\frac{1}{2}}^{2^p}(B_{\ell_p}) \neq \phi.$$

Let $e_{n_1}, \dots, e_{n_{2^p-1}}$ be any 2^p-1 basis vectors in ℓ_p, then for $m > \max\{n_1, \dots, n_{2^p-1}\}$

$$y_m = \tfrac{1}{2}(e_{n_1} + \dots + e_{n_{2^p-1}} + e_m)$$

is such that

$$\|y_m\|_p = 1 \;, \quad \|y_m - y_n\|_p = \tfrac{1}{2}\|e_m - e_n\|_p > \tfrac{1}{2}$$

and $y_m \xrightarrow{w} \frac{1}{2}(e_{n_1} + \dots + e_{n_{2^p-1}})$.

Thus one half the sum of any 2^p-1 basis vectors in in $\beta_{\frac{1}{2}}(B_{\ell_p})$.

An identical calculation yields that one-half the sum of any 2^p-2 basis vectors is in $\beta^2_{\frac{1}{2}}(B_{\ell_p})$.

Continuing in this way we eventually arrive at $\frac{1}{2}e_n \in B_{\frac{1}{2}}^{2^p-1}(B_{\ell_p})$ for any n

and so $0 = w\text{-}\lim_n \frac{1}{2}e_n \in B_{\frac{1}{2}}^{2^p}(B_{\ell_p})$.

3) The space $L_4[0,1]$ shows that u.K.K. need not imply the weak Opial condition. The previous example shows that the converse implication may also fail: $X := (\ell_2 \oplus \cdots \oplus \ell_n \oplus \cdots)_2$ has the weak-Opial Condition.

Let $x_n = (x_n^{(k)})$, $x_n^{(k)} \in l_k$, converge weakly to $x_0 \neq 0$. Let $f_n^{(k)}$ be the unique (by smoothness) element of $D(x_n^{(k)})$, then $f_n := (f_n^{(k)})$ is the unique element of $D(x_n)$. Choose k_0 such that $x_0^{(k_0)} \neq 0$, then $x_n^{(k_0)} \xrightarrow{\omega} x_0^{(k_0)}$ and so by theorem (3·1·3) we can find n_0 and $\delta > 0$ so that for $n > n_0$ we have $f_n^{(k_0)}(x_0^{(k_0)}) > \delta$.

Now we can find a finite subset N of $\mathbb{N}$ so that $k_0 \in N$,

$$\left(\sup_n \|f_n\|\right) \cdot \left(\sum_{k \notin N} \|x_0^{(k)}\|^2\right)^{\frac{1}{2}} < \delta/2$$

and, again by theorem (3·1·3), there is an $n_1 \geq n_0$ so that $n \geq n_1$ implies

$$f_n^{(k)}(x_0^{(k)}) \geq 0, \quad k \in N.$$

It now follows that for $n > n_1$

$$\begin{aligned} f_n(x_0) &= \sum f_n^{(k)}(x_0^{(k)}) \\ &\geqslant \sum_{k \in N} f_n^{(k)}(x_0^{(k)}) - \delta/2 \\ &\geqslant f_n^{(k_0)}(x_0^{(k_0)}) - \delta/2 \\ &\geq \delta/2. \end{aligned}$$

That X has the weak Opial condition now follows from (3·1·3).

if we like

we now turn to

<u>The Weak*-case</u>, when X is a dual space.

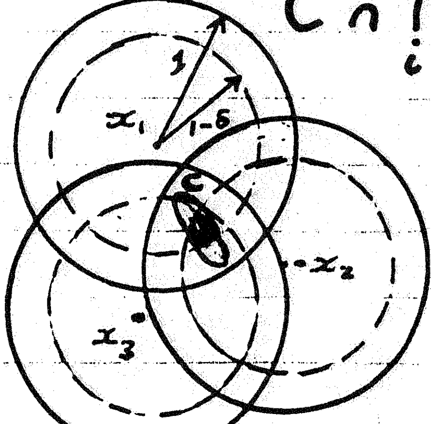
In view of the previous discussion it seems natural to say for $\varepsilon>0$ that X is ε-UKK* if there exists $\delta>0$ ($\delta\in(0,1)$) so that whenever C is a w^*-compact convex subset of B_X with $\gamma(C)>\varepsilon$ we have $C\cap(1-\delta)B_X\neq\emptyset$.

X is UKK* if it is ε-UKK* for all $\varepsilon>0$.

The appeal to Mazur's theorem in (3·2·2) precludes a similar argument for the w^*-case, none-the-less the conclusions remain valid. To see this we make use of the following.

<u>(3·2·5) LEMMA.</u> Let X be an ε-UKK* dual space and let δ be that associated with ε by the definition of ε-UKK*. Then, if C is a w^*-compact convex subset of X with $\gamma(C)>\varepsilon$ and if $x_1,x_2,\cdots,x_n$ are points of X with $C\subseteq B_1[x_i]$ for $i=1,2,\cdots,n$ we have

$$C\cap\bigcap_{i=1}^{n}B_{1-\delta}[x_i]\neq\emptyset.$$



Proof: By the definition of ε-UKK* the result is true for $n=1$. Suppose the result were to fail, then there is a largest n (≥ 1) for which it is true. Denote this largest n by n_0, then there exists a w^*-compact convex set $C \subseteq X$ with $\gamma(C) > \varepsilon$ and points $x_0, x_1, \ldots, x_n$ with $C \subseteq B_1[x_i]$ for $i = 0, 1, \ldots, n$ but for which

$$C \cap \bigcap_{i=0}^{n} B_{1-\delta}[x_i] = \phi .$$

Let $C_0 := C \cap \bigcap_{i=1}^{n_0} B_{1-\delta}[x_i]$, then by the definition of n_0, $C_0 \neq \phi$. Further $C_0 \cap B_{1-\delta}[x_0] = \phi$ so there exists a w^*-continuous linear functional f and k with

$$\inf f(C_0) > k > \sup f(B_{1-\delta}[x_0]).$$

Let $C_1 := \{x \in C : f(x) \geq k\}$ and let

$$C_2 := \{x \in C : f(x) \leq k\} .$$

Then, $C_1 \subseteq C \subseteq B_1[x_0]$

while $C_1 \cap B_{1-\delta}[x_0] = \phi$.

Since C_1 is w^*-compact and convex it follows from ε-UKK* that $\gamma(C_1) \leq \varepsilon$. and so since $C = C_1 \cup C_2$ we must have (by p78, remark 2(c)) that $\gamma(C_2) > \varepsilon$.

But then, C_2 is a w^*-compact convex set with $\gamma(C_2) > \varepsilon$ such that

$$C_2 \cap \bigcap_{i=1}^{n_0} B_{1-\delta}[x_i] \subseteq C_2 \cap C_0 = \phi$$

contradicting our choice of n_0 as the largest value for which the implication held. ∎

<u>(3·2·6) THEOREM</u> [van Dulst - Sims, 83]:
If X is a dual space with the ε-UKK* property for some $\varepsilon \in (0,1)$, then X has w^*-normal structure and hence in particular X has the w^*-F.P.P.

<u>Proof</u>. Suppose not, then we can find a diametral w^*-compact convex subset of X K, with diam $K = 1$. Then $\gamma(K) = 1 > \varepsilon$ and for each $x \in K$, $K \subset B_1[x]$.
Let $E_x = K \cap B_{1-\delta}[x]$, then E_x is a w^*-compact subset of K which is non-empty by the ε-UKK* property. Further the above lemma ensures that the family E_x has the finite intersection property, and so by the w^*-compactness of K there exists $x_0 \in \bigcap_{x \in K} E_x$,

but then for any $x \in K$ we have
$$x_0 \in E_x \subset B_{1-\delta}[x].$$
So $\|x - x_0\| \leq 1 - \delta$, contradicting the diametrality of K. ∎

Indeed the stronger analogue of theorem (3·2·3) is true.

(3·2·7) THEOREM [van Dulst - Sims, 83]:

Let X be a UKK* dual space and C be a weak*compact convex subset of X. Then, $\mathcal{E}(C)$ is norm-compact.

(stronger than FPP*)

Proof. Suppose this were not this case, then we can find a weak* compact convex subset of X with rad $C = 1$ and $\gamma(\mathcal{E}(C)) > \varepsilon_0$ for some $\varepsilon_0 > 0$. From the definition of rad C it follows that

$$\mathcal{E}(C) \subseteq B_1[x] \quad \text{for each } x \in C.$$

Let δ correspond with ε_0 in the definition of UKK* then

$$E_x := \mathcal{E}(C) \cap B_{1-\delta}[x]$$

is a non-empty weak* compact convex subset of C for each $x \in C$. The argument now proceeds along the same lines as those of the last part of the proof for theorem (3·2·6). ∎

We now consider necessary and sufficient conditions for a dual space to be ε-UKK*. Some conditions will be sufficient others necessary.

Our first result shows that for the w^*-compact convex sets in the definition it is sufficient to consider "w^*-slices of B_X".

<u>(3.2.8) PROPOSITION</u>: A dual space X has the ε-UKK* property if and only if there exists $k \in (0,1)$ such that for every norm one weak*-continuous linear functional f the <u>(weak*-) slice</u> of the unit ball

$$\underline{S[f,k]} := \{ x \in B_X : f(x) \geq k \}$$

has $\gamma(S[f,k]) \leq \varepsilon$.

<u>Proof.</u> ($\Rightarrow$) Obvious, since for any $k > 1-\delta$, where δ is that given in the definition of ε-UKK*, $S[f,k]$ is a weak*-closed convex subset of B_X disjoint from $B_{1-\delta}[0]$.

($\Leftarrow$) Let C be a weak*-compact convex subset of B_X with $\gamma(C) > \varepsilon$ we show that $C \cap (1-\delta) B_X \neq \phi$ where $\delta = 1-k$. Suppose not, then there exists a norm one weak*-continuous linear functional separating C from $(1-\delta) B_X$. That is, $\inf f(C) > \sup f((1-\delta)B_X)$

$$= 1-\delta = k.$$

Thus $C \subseteq S[f,k]$ and so

$$\gamma(S[f,k]) \geq \gamma(C) > \varepsilon$$

contradicting our hypothesis. ∎

Our next result depends on the characterization of upper-semi-continuity of the duality map in terms of slices given in Giles, Gregory & Sims [1978].

Recall the duality map
$$\mathcal{D} : x \longmapsto \mathcal{D}(x) := \{ f \in X^* : f(x) = \|f\|^2 = \|x\|^2 \}$$
is <u>norm to norm upper semi-continuous</u> if given $\varepsilon > 0$ and $x \in S_X$ there exists $\delta > 0$ such that for all $y \in S_X$ with $\|x - y\| < \delta$ we have $\mathcal{D}(y) \subseteq \mathcal{D}(x) + B_\varepsilon[0]$. $\mathcal{D}$ is <u>uniformly norm to norm upper semi-continuous</u> if there exists a common δ for all $x \in S_X$.

<u>(3.2.9) LEMMA:</u> $\mathcal{D}$ is norm to norm upper semi-continuous if and only if for each $\varepsilon > 0$ and $x \in S_X$ there exists a $k \in (0,1)$ so that the slice $S[x, k] \subseteq \mathcal{D}(x) + B_\varepsilon[0]$ (The 'continuity' is uniform if and only if k may be choosen independent of $x \in S_X$.)

<u>Proof.</u> ($\Leftarrow$) Suppose $\mathcal{D}(x) + B_\varepsilon[0]$ contains the slice $S[x, k]$ determined by x. Then, setting $\delta = 1 - k$, for $y \in B_\delta[x] \cap S_X$ we have
$$|f(x) - 1| = |f(x) - f(y)| < \delta$$
for all $f \in \mathcal{D}(y)$. That is, $\mathcal{D}(y) \subseteq S[x, k]$ and the result follows.

($\Rightarrow$) Suppose $\mathcal{D}$ is norm to norm upper semi-continuous then there exists $\delta' > 0$ so that $\mathcal{D}(y) \subseteq \mathcal{D}(x) + B_{\varepsilon/2}[0]$ whenever $y \in B_{\delta'}(x) \cap S_X$.

Let $\delta = \min\{\delta', \varepsilon/2\}$ and let $k = 1 - \delta^2/4$. Then, for $f \in S[x, k]$ we have $|f(x) - 1| \leq \delta^2/4$, so by the Bishop-Phelps-

Bollobás theorem, there exists $y \in S(X)$ and $g \in \mathcal{D}(y)$ such that $\|x-y\| < \delta$ and $\|f-g\| < \delta$. But then, $\mathcal{D}(y) \subseteq \mathcal{D}(x) + B_{\varepsilon/2}[0]$ and so

$$f \in g + B_{\frac{\varepsilon}{2}}[0] \subseteq \mathcal{D}(y) + B_{\frac{\varepsilon}{2}}[0]$$

$$\subseteq \mathcal{D}(x) + B_{\varepsilon}[0].$$

That is $S[x,k] \subseteq \mathcal{D}(x) + B_{\varepsilon}[0]$. ∎

(3·2·10) COROLLARY: For the conditions listed below we have i) ⇒ ii) ⇒ iii).

i) (a) $\mathcal{D}$ is norm to norm uniformly upper semi-continuous

and

(b) For each $x \in S_X$, $\mathcal{D}(x)$ is norm compact.

ii) X^* has UKK*.

iii) (a) $\mathcal{D}$ is norm to norm upper semi-continuous

and

(b) For each $x \in S_X$, $\mathcal{D}(x)$ is norm compact.

Proof. i) ⇒ ii) Given any $\varepsilon > 0$, from (3·2·9) there exists $k \in (0,1)$ such that for all $x \in S_X$ $S[x,k] \subseteq \mathcal{D}(x) + B_{\frac{\varepsilon}{4}}[0]$, from this and i)(b) it follows easily that $\gamma(S[x,k]) < \varepsilon$ and hence ii) follows by (3·2·8).

ii) $\Rightarrow$ iii) From ii) via (3·2·8) we have $\gamma(S[x, 1-\frac{1}{n}]) \longrightarrow 0$ as $n \longrightarrow \infty$, hence by (d) of Remark 1)

$$\mathcal{D}(x) = \bigcap_{n=1}^{\infty} S[x, 1-\tfrac{1}{n}] \text{ is normcompact}$$

(giving iii)(b)) and further for any $\varepsilon > 0$ we have for n sufficiently large that

$$S[x, 1-\tfrac{1}{n}] \subseteq \mathcal{D}(x) + B_\varepsilon[0],$$

from which iii)(a) follows by (3·2·9). $\blacksquare$

(3·2·11) EXAMPLES.

Note Swami Maineh
H_1 is not UKK*-able
(no equiv UKK* norm) for UKK norm

The most obvious example is

1) ℓ_1 has UKK*: we prove this using the characterization given in (3·2·8). First observe: In ℓ_1 let $\underset{\sim}{f} = (f_i)_{i=1}^{\infty} \in S[\underset{\sim}{x}, k]$ where $\underset{\sim}{x} = (x_i) \in c_0$ with $\|\underset{\sim}{x}\|_\infty = 1$. That is;

$$\sum_{i=1}^{\infty} |f_i| \leq 1 \text{ and } \sum_{i=1}^{\infty} x_i f_i \geq k \quad \text{where } x_i \to 0$$

and $\operatorname{Max} |x_i| = 1$.

Given any $\varepsilon > 0$, let M be such that $|x_i| < \varepsilon$ for $i > M$, then

$$\begin{aligned}
\sum_{i=M+1}^{\infty} |f_i| &\leq 1 - \sum_{i=1}^{M} |f_i| \\
&\leq 1 - \sum_{i=1}^{M} x_i f_i \\
&\leq 1 - \Big(k - \sum_{i=M+1}^{\infty} x_i f_i\Big) \\
&\leq (1-k) + \varepsilon .
\end{aligned}$$

Qu.
Is $L_{p,1}$ UKK* in its usual norm?

Now, let $\underset{\sim}{f}_n = (f_i^n)$ be a sequence in $S[\underset{\sim}{x}, k]$ converging weak* to $\underset{\sim}{f} \in S[\underset{\sim}{x}, k]$

Then $|f_i^n - f_i^m| \to 0$ as $m, n \to \infty$ and

$$\|\underset{\sim}{f}_n - \underset{\sim}{f}_m\| = \sum_{i=1}^{M} |f_i^n - f_i^m| + \sum_{i=M+1}^{\infty} |f_i^n - f_i^m|$$

$$\leq \sum_{i=1}^{M} |f_i^n - f_i^m| + 2(1-k+\varepsilon).$$

Thus $\inf\limits_{m \neq n} \|\underset{\sim}{f}_n - \underset{\sim}{f}_m\| \leq \limsup\limits_{m,n} \|\underset{\sim}{f}_n - \underset{\sim}{f}_m\|$

$$\leq 2(1-k+\varepsilon).$$

Since ε was arbitrary it follows that

$\gamma(S[\underset{\sim}{x}, k]) \leq 2(1-k)$. Index of C

Given $\eta \in (0,2)$ Set $k = 1 - \eta/2$ $k \in (0,1)$ & so

l_1 is η-UKK* $\forall \eta \in (0,2)$ $\Rightarrow l_1$ is UKK*.

2) An easy calculation establishes that if $(X, \|\cdot\|_1)$ has the ε-uKK [ε-uKK*] property (with corresponding δ), and $\|\cdot\|_2$ is an equivalent norm on X with $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$, then $(X, \|\cdot\|_2)$ has the ε'-uKK [ε'-uKK*] property where $\varepsilon' = \frac{m}{M}\varepsilon$ (corresponding $\delta' = 1 - \frac{M}{m}(1-\delta)$), provided ε and $1-\delta$ are both less than $\frac{m}{M}$.

In particular it follows that the space $X_\alpha := (l_2, \|\cdot\|_\alpha)$ where $\|x_\alpha\| := \alpha\|x\|_2 \vee \|x\|_\infty$ is ε-uKK (some $\varepsilon > 0$, which varies with α) for $1 \geq \alpha > 2/\sqrt{5}$.

REMARKS: 1) The space l_1 shows that the implication ii) $\Rightarrow$ i) of ~~corollary~~ (3·2·10) is not generally valid. The space $X = (l_2 \oplus l_3 \oplus \cdots + l_n \oplus \cdots)_2$ provides a counter example to iii) $\rightarrow$ ii) of the same corollary.

2) The conclusions of theorems (3·2·6) and (3·2·7) for l_1 were first proved by Lim [1980]. Indeed for this space ~~he~~ obtains a stronger conclusion than that of (3·2·7), namely that the asymptotic centres with respect to a non-empty subset of l_1 are norm compact.

For C a non-empty subset of a Banach space X and $(A_\alpha : \alpha \in \Delta)$ a decreasing net of bounded non-empty subsets of X, let

$$r(x) := \inf_\alpha \operatorname{rad}(A_\alpha, x) = \lim_\alpha \operatorname{rad}(A_\alpha, x),$$

$$r := \inf_{x \in C} r(x)$$

and

$$\mathcal{A} := \{ x \in C : r(x) = r \}.$$

$\mathcal{A}$ is the <u>asymptotic centre</u> of $(A_\alpha : \alpha \in \Delta)$ with respect to C.

If $A_\alpha \equiv C$ we obtain the chebyshev centre of C, $\mathcal{C}(C)$.

We say X has w (w^*) - asymptotic

<u>normal structure</u> if for every non-empty $\omega(\omega^*)$ compact convex subset K of X containing more than one point the asymptotic centre of any decreasing net of non-empty subsets of K with respect to K is a proper subset of K. Since K is diametral if and only if $C(K) = K$ we see that $\omega(\omega^*)$-asymptotic normal structure implies $\omega(\omega^*)$-normal structure.

In 1974 Lim proved the equivalence of ω-asymptotic normal structure and ω-normal structure, however no such equivalence seems known in the ω^*-case. None-the-less Lim's 1980 results verifies that l_1 has ω^*-asymptotic normal structure. This suggests;

* QUESTION: Does UKK* imply ω^*-asymptotic normal structure?

3) We note that from corollary (3-2-10) and the results of section 3 in Giles, Gregory and Sims [1978] we have that X^* has the Radon-Nikodym property whenever X^* has UKK*. Further X is reflexive whenever X^{**} has UKK*.

We also remark that a result of Lima [1981] establishes a connection between UKK* and approximation theory (more precisely, the theory of M-ideals).

4) Based on the analogy with $\ell_1(S)$ Lau and Mah [1986] ask the following.

<u>QUESTION:</u> Does the trace-class of operators on a Hilbert space, $\mathcal{T}(H)$, with the trace norm (which can be identified with $K(H)^*$, where $K(H)$ is the ideal of compact operators) have UKK*, w*-normal structure or the w*-F.P.P.?

It is known that $\mathcal{T}(H)$ has the Kadets-Klee property [Arazy, 1981]. Lau and Mah show that $\mathcal{T}(H)$ has w*-quasi-normal structure as introduced by Soardi [1972]: A dual space has <u>w*-quasi-normal</u> structure if for every w*-compact convex subset with more than two points, there exists $x \in C$ so that

$$\|x - y\| < \operatorname{diam} C \quad \text{for all } y \in C.$$

has ANS if and only if $1 \geq \alpha > \frac{1}{2}$, while it has normal structure if and only if $1 \geq \alpha > \frac{1}{\sqrt{2}}$. Thereby establishing that ANS is genuinely weaker than normal structure. They also establish the F.P.P. for $X_{\frac{1}{2}}$. Since we will obtain the F.P.P. for all $\alpha \in [1,0)$ in chapter , we will not pursue the details here.

Bynum [1980 and ?] shows that the reflexive space

$\ell_{p,\infty} = (\ell_p, |||\cdot|||)$, where $|||x||| = \|x^+\|_p \vee \|x^-$

lacks ANS for all p, though it has the F.P.P. for $1 < p < \infty$.

(A)

<u>(e_n) is diametral in ℓ_2 with the Smith-Turett renorming:</u>

Recall; Days renorming of c_0 is given by

$$\|x\|_D := \|Dx\|_2$$

where

$$Dx(n) = \begin{cases} x(n_k)/2^k & \text{if } n = n_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

here $(n_k)_{k=1}^{\infty}$ is an enumeration of the support of x so that $|x(n_k)| \geq |x(n_{k+1})|$.

Since, $\frac{1}{2}\|x\|_{c_0} \leq \|x\|_D \leq \frac{1}{\sqrt{3}}\|x\|_{c_0}$, we have

$$\frac{1}{4}\|x\|_2 \leq \frac{1}{2}\|Tx\|_{c_0} \leq \|x\| \leq \frac{1}{\sqrt{3}}\|Tx\|_{c_0} \leq \frac{1}{\sqrt{3}}\|x\|$$

Now for $n < m$ we have

$$\|e_n - e_m\| = \|\underbrace{\tfrac{1}{\sqrt{2}}, 0, \cdots, \overbrace{1, 1, \cdots, 1}^{n}, 0, \cdots, 0, \overbrace{-1, -1, \cdots, -1}^{m}, 0 \cdots}_{x}\|$$

$$\|Dx\|_2 = \left\|\left(\frac{1}{2^{n+m+1}\sqrt{2}}, 0, \cdots; \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, 0, \cdots, 0, -\frac{1}{2^{n+1}}, \cdots, -\frac{1}{2^{n+m}}, \right)\right\|_2$$

$$= \left(\sum_{k=1}^{n+m} \frac{1}{4^k} + \frac{1}{2 \cdot 4^{n+m+1}}\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{\infty} \frac{1}{4^k}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}.$$

So $\operatorname{diam} \{e_n\}_{n=1}^{\infty} \leq \frac{1}{\sqrt{3}}$.

On the other hand for $x_1, x_2, \dots x_n \geq 0$ $\sum_1^n x_k$ we have

$$\|\sum_1^n x_k e_k - e_{n+1}\|$$

$$= \|(\tfrac{1}{2}\|\sum x_k e_k - e_{n+1}\|_2, x_1, x_2, x_2, \cdots, \overbrace{x_n, \cdots, x_n}^{n}, \overbrace{-1, -1, \cdots}^{n+1}$$

$< \frac{1}{\sqrt{2}}$

K'

$$\geq \sum_{k=1}^{n+1} \frac{1}{4^k}$$

there are at least $n+1$ entries of abs. value 1 (and all others have smaller abs. values).

So $\operatorname{rad} co(e_n) \geq \inf_n \sum_{k=1}^{n+1} \frac{1}{4^k}$

$$= \frac{1}{\sqrt{3}}$$

a sample:

(K)

Let $P_n : \ell^2 \to \ell^2$ be defined by

$$P_n(\alpha_1, \alpha_2, \ldots) = (\underbrace{0, \ldots, 0}_{n-1}, \alpha_n, \alpha_{n+1}, \ldots)$$

Define a norm $\|\cdot\|$ on ℓ^2 by

$$\|x\| = \sqrt{\sum_{n=1}^{\infty} 2^{-n}\left[\|(P_n x)^+\|_2^2 \vee \|(P_n x)^-\|_2^2\right]}$$

Then $\|\cdot\|$ is not strictly convex. For, if

$$x = (\alpha_1, \alpha_2, \ldots) \text{ where } \begin{cases} \alpha_{2n-1} = 2^{-n} \\ \alpha_{2n} \quad = 0 \end{cases},$$

and

$$y = (\beta_1, \beta_2, \ldots) \text{ where } \begin{cases} \beta_{2n-1} = 2^{-n} \\ \beta_{2n} \quad = -10^{-n} \end{cases},$$

then $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$.

(3·3) The asymptotic Normal Structure of Baillon and Schöneberg.

In 1981 Baillon + Schöneberg introduced a weakening of normal structure which they also called asymptotic normal structure (cf. Remark 2 at the end of (3·2)):
The Banach space X [dual space] has <u>w (w^*) - ANS</u> if whenever C is a non-trivial weak (weak*) compact convex subset of X and $(x_n) \subset C$ is a sequence satisfying $\|x_{n+1} - x_n\| \to 0$, then there exists an $x \in C$ such that $\liminf_n \|x_n - x\| < \operatorname{diam} C$.

Clearly w (w^*) - normal structure implies w (w^*) - ANS.

Since an approximate fixed point sequence (x_n) in a w (w^*) - compact convex minimal invariant set (for a non-expansive mapping) satisfies $\|x_{n+1} - x_n\| \to 0$ (Proposition 1·3) but $\lim_n \|x - x_n\| = \operatorname{diam} C$ (theorem 1·8), we have:-

<u>(3·3·1) PROPOSITION:</u> w (w^*) - ANS implies the w (w^*) - F.P.P.

Baillon & Schöneberg [1981] show that the reflexive space
$X_\alpha := (l_2, \|\cdot\|_\alpha)$, where $\|x\|_\alpha = \alpha\|x\|_2 \vee \|x\|_\infty$