2. NORMAL STRUCTURE

(2.1) Definitions

We say that a Banach space X has:

normal structure if it contains no closed bounded convex diametral sets with more than one point;

w (w^*)-normal structure if it contains no w (w^*)-compact convex diametral sets with more than one point.

The normal structure constant of X is

$$N(X) := \sup_C \frac{\mathrm{rad}(C)}{\mathrm{diam}(C)},$$

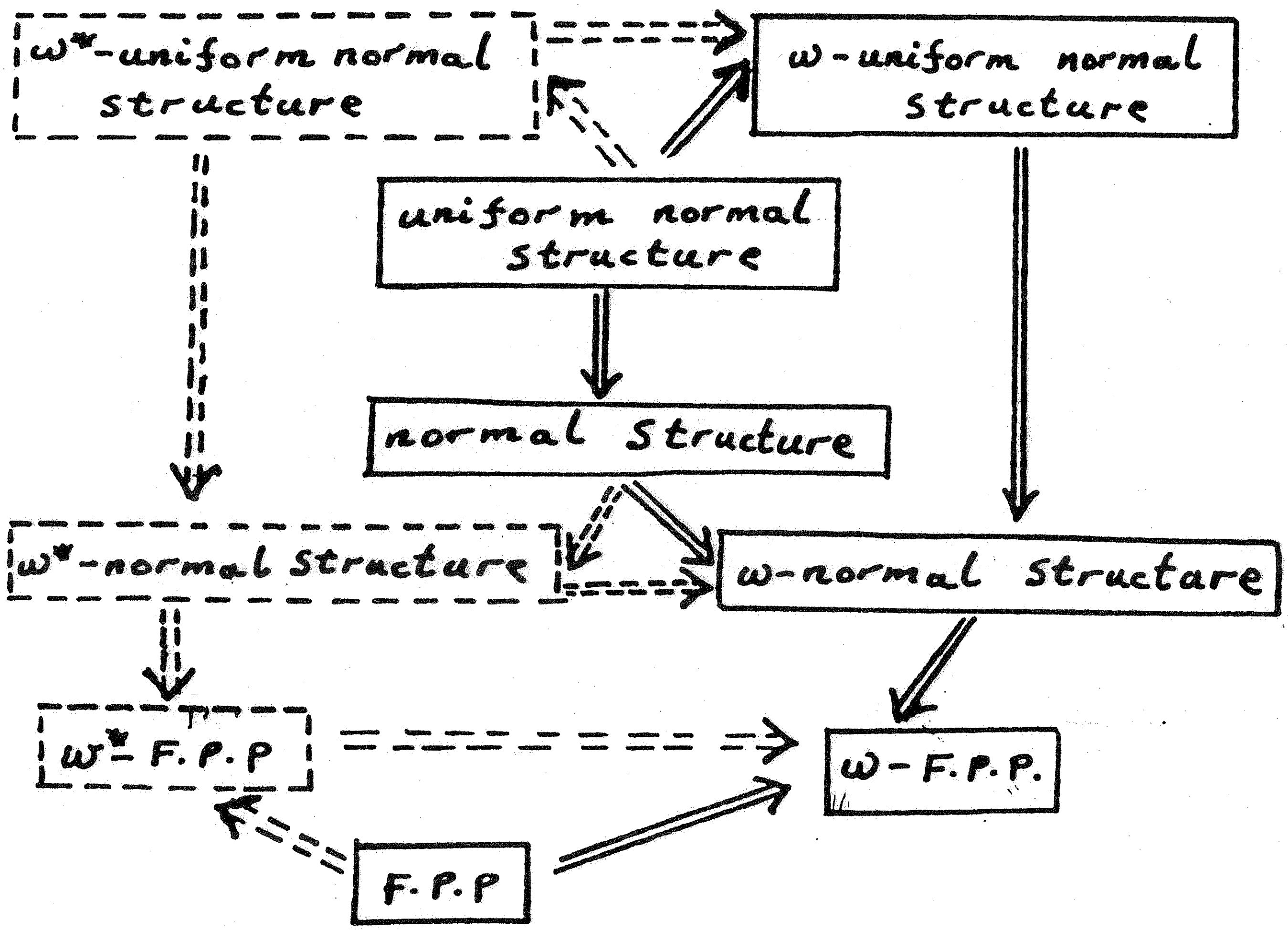
where the supremum is taken over all closed bounded convex sets C with more than one point. If the admissible sets C are further required to be w (w^*)-compact we obtain the w (w^*)-normal structure constant, $N_w(X)$ $(N_{w^*}(X))$.

When the space X is clear from the context we will drop it from the notations above and write N for $N(X)$, and so on.

X has <u>uniform normal structure</u>, <u>$\omega(\omega^*)$-uniform normal structure</u> if we have respectively $N(X) < 1$, $N_{\omega(\omega^*)}(X) < 1$

With the aid of theorem (1·5) we have the following <u>implications</u>

<u>(2·2)</u>



Broken lines indicate concepts and implications which only apply when X is a dual space, Many of these concepts coalesce when X is reflexive.

The <u>modulus of convexity</u> for the Banach space X is

$$\delta(\varepsilon) := \inf\left\{ 1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1 \text{ \& } \|x-y\| \ge \varepsilon \right\}$$

δ is an increasing function on $[0,2]$ with $\delta(0) = 0$.

We say X is <u>ε_0-inquadrate</u> if $\delta(\varepsilon_0) > 0$ and <u>uniformly convex</u> if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. Equivalently, X is uniformly convex if and only if whenever (x_n) and (y_n) are such that $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $\left\|\frac{x_n + y_n}{2}\right\| \to 1$

we have $\|x_n - y_n\| \to 0$.

The spaces l_p and $L_p(\mu)$ are uniformly convex for $1 < p < \infty$, with

$$\delta(\varepsilon) = \begin{cases} 1 - \left(1 - \left(\varepsilon/2\right)^p\right)^{\frac{1}{p}} & \text{for } 2 \le p < \infty \\ \xi \; [= (p-1)\varepsilon^2/8 + o(\varepsilon^2)], \text{ where} & \\ \quad (1 - \xi + \frac{\varepsilon}{2})^p + (1 - \xi - \frac{\varepsilon}{2})^p = 2, & \text{for } 1 \le p \le 2. \end{cases}$$

<u>(2.3) PROPOSITION</u> (Edelstein, 1963):
Spaces which are ε_0-inquadrate for some $\varepsilon_0 < 1$, in particular uniformly convex spaces, have uniform normal structure with a normal structure constant $N \le 1 - \delta(1)$.

Proof. Let C be a closed convex subset of X with $\operatorname{diam}(C) = 1$. For any $t \in [0,1)$ choose $x_1, x_2 \in C$ with $\|x_1 - x_2\| \geq t$ and let $x_0 := \frac{1}{2}(x_1 + x_2)$. Then for any $x \in C$ we have

$\|x - x_1\| \leq 1$, $\|x - x_2\| \leq 1$ and

$\|(x - x_1) - (x - x_2)\| = \|x_1 - x_2\| \geq t$.

It follows from the definition of δ that

$$\|x - x_0\| = \left\| \frac{(x - x_1) + (x - x_2)}{2} \right\| \leq 1 - \delta(t).$$

Thus $\operatorname{rad}(C) \leq \inf\limits_{0 \leq t < 1} [1 - \delta(t)] = 1 - \delta(1)$. ■

<u>(2.3.1) Remark</u> In general proposition (2.3) does not give a sharp estimate for N. In the case of Hilbert space it yields $N \leq \sqrt{3}/2 \doteq 0.866$. In fact $N = 1/\sqrt{2} \doteq 0.707$.

$N \leq \sqrt{3}/2 \doteq 0.866$. In fact we have:

$$\underline{\underline{N(\ell_2) = 1/\sqrt{2} \doteq 0.707.}}$$

Let C be a closed bounded convex subset in a Hilbert space. Choose $x_0 \in C$ so that $\operatorname{rad}(C) = \sup\limits_{x \in C} \|x_0 - x\|$. (That is $x_0 \in \mathcal{C}(C)$, the Chebyshev centre of C which is non-empty by the weak compactness.) Let $x, y \in C$ then for any $\lambda \in [0,1)$ we have

$$\|\lambda x_0 + (1-\lambda)x - y\|^2 = \|\lambda(x_0-y)+(1-\lambda)(x-y)\|^2$$
$$= \lambda^2\|x_0-y\|^2 + (1-\lambda)^2\|x-y\|^2 - 2\lambda(1-\lambda)(x_0-y, x-y)$$
$$= \lambda\|x_0-y\|^2 + (1-\lambda)\|x-y\|^2 - \lambda(1-\lambda)\|x_0-x\|^2$$
(by the Polarization Identity).

Now, since $\lambda x_0 + (1-\lambda)x \in C$ taking the supremum over $x, y \in C$ we obtain
$$(\mathrm{rad}\, C)^2 \leqslant \lambda(\mathrm{rad}\, C)^2 + (1-\lambda)(\mathrm{diam}\, C)^2 - \lambda(1-\lambda)(\mathrm{rad}\, C)^2$$
or
$$\left(\frac{\mathrm{rad}\, C}{\mathrm{diam}\, C}\right)^2 \leqslant \frac{1}{1+\lambda} \quad \text{for all } \lambda \in [0,1)$$
whence
$$N(\ell_2) \leqslant \frac{1}{\sqrt{2}}.$$
To establish equality, consider $C := \overline{co}\{e_k\}_{k=1}^{\infty}$ for which $\mathrm{diam}\, C = \sqrt{2}$ while $\mathrm{rad}\, C = 1$ (with $0 \in C$ as centre). ∎

Lim [1983] developed the above argument for ℓ_p with $2<p<\infty$ to obtain upper bounds for $N(\ell_p)$, and hence as we shall see $N(\mathcal{L}_p)$.

For $2<p<\infty$, $N(\ell_p) \leqslant \left(1 + \dfrac{1+\xi_0^{p-1}}{(1+\xi_0)^{p-1}}\right)^{-1/p}$

where ξ_0 is the unique positive solution of $(p-2)\xi^{p-1} + (p-1)\xi^{p-2} = 1$.

The set $C := \overline{co}\{e_n\}_{n=1}^{\infty}$ gives a lower bound of $N(\ell_p) \geqslant 2^{-1/p}$.

Question: Is a better estimate for N in terms of δ possible?

In particular determine sharper (precise) estimates for $N(\ell_p)$, $1 < p < \infty$, $p \neq 2$. Here the work of Bynum [1980], Maluta [1984] and in particular Amir [1982/83] and Amir and Franchetti [1982/83] are relevant.

Note: Our $N(X)$ is the reciprocal of Bynum's and is half the self Jung constant of X.

The following Corollary seems to have been noticed by many authors; Browder [196], Edelstein [ ], Göhde [ ], Kirk [1965]

<u>(2·4) Corollary</u>

Spaces which are ε_0-inquadrate for some $\varepsilon_0 < 1$, in particular uniformly convex spaces, have the F.P.P.

Proof. This is immediate from proposition (2·3), (2·2) and the fact that such spaces are reflexive (Mil'man – Pettis theorem). ∎

The dual (equivalently predual, by the reflexivity) of a Banach space X is uniformly convex if and only if the space is <u>uniformly smooth</u> (the norm is uniformly Fréchet differentiable on the unit sphere S_X).
Baillon [1978-79] established the F.P.P. for reflexive spaces whose <u>modulus of smoothness</u>

$$\rho(\tau) := \sup\left\{ \tfrac{1}{2}\left(\|x+\tau y\| + \|x-\tau y\|\right) - 1 : x, y \in S_X \right\}$$

satisfies

$$\lim_{\tau \downarrow 0} \rho(\tau)/\tau < \tfrac{1}{2}.$$

In particular then, uniformly smooth Banach spaces have the F.P.P.
Barry Turett [1982] observed that

X satisfies

$$\lim_{\tau \downarrow 0} \rho(\tau)/\tau < \tfrac{1}{2}$$

if and only if X^* is ε_0-inquadrate for $\varepsilon_0 < 1$ — see also [Giles, Gregory and Sims, 1978]. Hence the assumption of reflexivity is redundant. Turett that spaces with an ε_0-inquadrate dual ($\varepsilon_0 < 1$) have normal structure.

<u>(2.5) Lemma [Turett, 1982]</u>: If X fails ω-normal structure, then given $\varepsilon > 0$ there exists $f, g \in S_{X^*}$ and $x \in S_X$ such that $\|f-g\|, f(x), g(x) > 1-\varepsilon$. That is, the dual ball contains arbitrarily "thin" ω^*-slices with diameter near one or more.

Proof. By Corollary 1.11 and Proposition 0.6 if X fails ω-normal structure we can find $(x_n) \subset X$ with $x_n \xrightarrow{\omega} 0$ such that $\operatorname{diam} \overline{co}\{x_n\}_{n=1}^{\infty} = 1$ and $\operatorname{dist}(x_{n+1}, \overline{co}\{x_k\}_{k=1}^{n}) \to 1$.

<u>Observation</u>: without loss of generality we may take $x_1 = 0$; choose $x_2, x_3, \ldots$ as above. Since $x_n \xrightarrow{\omega} 0$, by Mazur, $\operatorname{dist}(0, \overline{co}\{x_k\}_{k=2}^{n}) \to 0$ as $n \to \infty$ and so for n sufficiently large we have $\operatorname{dist}(x_{n+1}, \overline{co}(\{0\} \cup \{x_k\}_{k=2}^{n})) \sim \operatorname{dist}(x_{n+1}, \overline{co}\{x_k\}_{k=2}^{n})$.

Note, this argument also shows that $\|x_n\| \to 1$.

Since for each n, $B_{1-\frac{1}{n}}(x_{n+1}) \cap \overline{co}\{x_k\}_{k=1}^{n} = \emptyset$ we may apply the Eidelheit separation theorem to obtain a norm one functional f_{n+1} such that

$$1 \geq f_{n+1}(x_{n+1} - x_k) > 1 - \tfrac{1}{n}, \text{ for all } k \leq n.$$

In particular $k=1$ gives $f_{n+1}(x_{n+1}) > 1 - \frac{1}{n}$.

Now choose $j_0 \in \mathbb{N}$ with $1 < j_0 < 2/\varepsilon$.

Since $x_n \xrightarrow{w} 0$ there exists $n_0 > j_0$ so that $|f_{j_0}(x_{n_0})| < \varepsilon/2$.

But then,

$$-f_{j_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) \geq -f_{j_0}(x_{n_0} - x_{j_0}), \ \|x_{n_0} - x_{j_0}\| \leq 1$$

$$\geq 1 - \tfrac{1}{n_0} - \tfrac{\varepsilon}{2}$$

$$> 1 - \varepsilon,$$

while

$$f_{n_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) \geq f_{n_0}(x_{n_0} - x_{j_0})$$

$$> 1 - \tfrac{1}{n_0} > 1 - \varepsilon.$$

and

$$\|f_{n_0} - (-f_{j_0})\| \geq (f_{n_0} + f_{j_0})(x_{n_0})$$

$$> 1 - \tfrac{1}{n_0} - \varepsilon/2$$

$$> 1 - \varepsilon.$$

The result now follows by taking $f = f_{n_0}$, $g = -f_{j_0}$ and $x = \dfrac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}$. ∎

(2·5·1) Corollary: If X^* is ε_0-inquadrate for some $\varepsilon_0 < 1$, then X has normal structure.

Proof. Since X is reflexive it suffices to note that if X^* is ε_0-inquadrate then choosing ε in lemma 2·5 so that $\varepsilon < \min\{\delta(\varepsilon_0), 1-\varepsilon_0\}$, if $f, g \in S_{X^*}$ and $x \in S_X$ are such that $f(x), g(x) > 1-\varepsilon$ then $\|\frac{f+g}{2}\| > 1-\varepsilon$ and so $\|f-g\| < \varepsilon_0 < 1-\varepsilon$. ∎

(2·6) THEOREM: If X^* is ε_0-inquadrate for some $\varepsilon_0 < 1$, then X has uniform normal structure. In particular uniformly smooth spaces have uniform normal structure.

Proof. Suppose X fails to have uniform normal structure, then we can find a sequence (C_n) of diameter one subsets of X with $\mathrm{rad}(C_n) \to 1$.

For any ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $C = (C_n)_{\mathcal{U}} := \{(x_n)_{\mathcal{U}} : x_n \in C_n\}$, then C is a convex subset of the ultrapower $(X)_{\mathcal{U}}$ with $\mathrm{diam}\, C = 1$ and $\mathrm{rad}(C) = 1$. Thus $(X)_{\mathcal{U}}$ fails to have normal structure. However, since X is superreflexive we have $(X)^*_{\mathcal{U}} = (X^*)_{\mathcal{U}}$ is ε_0-inquadrate [Sims, 82; §10 proposition 6] which contradicts corollary 2·5·1 ∎

Proposition 2.3 and the last theorem suggest the following.

QUESTIONS: Is uniform normal structure a self-dual property?

Is uniform normal structure a super-property?

The next proposition shows that the answer to this second question would be "yes" if spaces with uniform normal structure were superreflexive, however this appears not to be known. What is know (see below) is that uniform normal structure implies reflexivity.

(2.7) PROPOSITION: The weak-normal structure constant is "finitely determined". That is, given any $\varepsilon > 0$ there exists a finite subset F

with $$\frac{\mathrm{rad}(\mathrm{co}(F))}{\mathrm{diam}(\mathrm{co}(F))} \geq (1-\varepsilon) N_w$$

Proof. Let C be any weakly-compact convex set with $\mathrm{diam}(C) = 1$ and let $r < \mathrm{rad}(C)$. Then
$$\bigcap_{x \in C} B_r[x] \cap C = \phi$$
(if x_0 were in this intersection then x_0 would be a point of C with $\|x - x_0\| \leq r$ for all $x \in C$; that is $\mathrm{rad}(C) \leq r$.)

Since each of the sets $B_r[x] \cap C$ is a weakly compact subset of C there exists a finite subset F of C with

$$\bigcap_{x \in F} B_r[x] \cap C = \phi .$$

But then

$$\bigcap_{x \in F} B_r[x] \cap co(F) \subseteq \bigcap_{x \in F} B_r[x] \cap C = \phi$$

and it follows that $r \leq rad(co(F))$.
Thus

$$\frac{rad(C)}{diam(C)} \leq \sup\left\{ \frac{rad(co(F))}{diam(co(F))} : F \subseteq C \text{ is finite} \right\}$$

establishing the proposition. $\blacksquare$

The following is useful - to estimate N_ω (N in the case of a reflexive space).

<u>2.7.1 Corollary</u> [Amir, 82-83]: Let $(X_\alpha)_{\alpha \in \Lambda}$ be a net of <u>subspaces</u> directed by inclusion with $X = \overline{\bigcup_{\alpha \in \Lambda} X_\alpha}$, then

$$N_\omega(X) = \sup_{\alpha \in \Lambda} N_\omega(X_\alpha) = \lim_{\alpha} N_\omega(X_\alpha).$$

<u>Proof.</u> By (2-7) it is enough to consider sets of the form $C = co\{x_1, x_2, \cdots, x_n\}$. Now, we can find $y_1, y_2, \cdots, y_n \in X_\alpha$, for some α, with $\|x_i - y_i\| < \varepsilon$ $(i = 1, 2, \cdots, n)$, $\varepsilon > 0$.
Let $K = co\{y_1, \cdots, y_n\} \subseteq X_\alpha$, then

$$N_\omega(X_\alpha) \geq \frac{\text{rad}\, K}{\text{diam}\, K} \geq \frac{\text{rad}\, C - \varepsilon}{\text{diam}\, C + \varepsilon}$$

and the result follows. ∎

2.7.2 Corollary: For $1 < p < \infty$ we have
$$N(\mathcal{L}_p(\mu)) = N(\ell_p) = \lim_n N(\ell_p^n).$$

Proof. $\mathcal{P} := \mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n$ be a measurable partition of the measure space $(\mathcal{S}, \Sigma, \mu)$ with $\mu(\mathcal{S}_i) > 0$ $(i = 1, 2, \cdots, n)$ and let $X_\mathcal{P} = \langle \chi_{\mathcal{S}_i} \rangle_{i=1}^n$ — the subspace spanned by the characteristic functions $\chi_{\mathcal{S}_i}$.

Then, $X_\mathcal{P}$ is isometric to ℓ_p^n and $(X_\mathcal{P})$ is clearly a net directed by inclusion whose union is dense in $\mathcal{L}_p(\mu)$. ∎

For further results along these lines see Amir [82-83].

That spaces with uniform normal structure are reflexive was proved by BAE [83]. Independently it would seem, Malta [84] observed that the result is an immediate consequence of an earlier result of Mil'man and Mil'man [1965] which represents a mild strengthening of a contemporaneous result of R.C. James [64, see for example Beauzamy, 1982]. Since the result of Milman and Milman is of independent interest we choose to develop it here, although the proof is somewhat more intricate and long than is necessary for our purpose.

(2·8) THEOREM [Mil'man & Mil'man, 65]: If X is a non-reflexive Banach space then for every $\varepsilon > 0$ there exists a sequence of unit vectors (x_n) such that for all $m \in \mathbb{N}$ if

$$y \in co\{x_1, \dots, x_m\} \text{ and } y' \in co\{x_k\}_{k=m+1}^{\infty}$$

then we have

$$1-\varepsilon \leq \|y - y'\| \leq 1+\varepsilon.$$

Taking $C = \overline{co}\{x_k\}_{k=1}^{\infty}$ we have $\text{diam}\, C \leq 1+\varepsilon$ while $\text{rad}\, C \geq 1-\varepsilon$ and so we obtain

2·8·1 Corollary [Maluta, 84]: If X has uniform normal structure then X is reflexive.

Proof (of theorem 2·8). Since X is non-reflexive it contains a non-reflexive separable subspace E. We will carry out the construction inside E. Since B_E is not weakly compact we can find a nested family $\mathcal{F}$ of non-empty closed bounded convex subsets

$$K_1 \supset K_2 \supset K_3 \supset \dots$$

with

$$\bigcap^{\infty} K = \emptyset$$

T°

Further

$d(\mathcal{F}) := \lim_n \operatorname{diam} K_n$ exists and is strictly greater than zero (otherwise Cantor's intersection theorem would apply to give a non-empty intersection). Similarly the sequence $(\operatorname{dist}(x, K_n))_{n=1}^{\infty}$ is increasing and bounded above for each $x \in E$, so

$r(x, \mathcal{F}) := \lim_n \operatorname{dist}(x, K_n)$ exists, and can be regarded as the "distance from x to the vacuum of the nested family $\mathcal{F}$," in particular $r(x, \mathcal{F}) > 0$.

The proof now proceeds through a series of steps.

Step 1) Given $\mathcal{F} = (K_n)_{n=1}^{\infty}$ as above we can find a nested family $\mathcal{F}' = (K_n')_{n=1}^{\infty}$ of closed convex sets which is subordinate to $\mathcal{F}$ in the sense that $K_n' \subseteq K_n$ for all n, and such that for each $x \in E$

$$r(x, \mathcal{F}) = r(x, \mathcal{F}') := \lim_n \operatorname{dist}(x, K_n')$$

$$= \lim_n \|x - x_n\|$$

for any sequence (x_n) with $x_n \in K_n'$. ($\mathcal{F}'$ is closely-flattened.).

Proof of Step 1. Let (y_n) be a dense sequence in the separable space E.

Let $\mathcal{F}_1 = (K_n^1)_{n=1}^{\infty}$ where

$$K_n^1 = K_n \cap B_{(1+\frac{1}{n})r(y_1, \mathcal{F})}[y_1].$$

$\mathcal{F}_1$ is a nested family of non-empty closed convex sets subordinate to $\mathcal{F}$. Further for $x_n \in K_n^1$,

$$\begin{aligned} \operatorname{dist}(y_1, K_n) &\leq \operatorname{dist}(y_1, K_n^1) \\ &\leq \|y_1 - x_n\| \\ &\leq (1+\tfrac{1}{n})\, r(y_1, \mathcal{F}), \end{aligned}$$

so

$$\lim_n \|y_1 - x_n\| = r(y_1, \mathcal{F}) = r(y_1, \mathcal{F}_1).$$

Continuing in this way we construct a sequence $\mathcal{F}_m = (K_n^m)$ of nested families of decreasing subordination so that

$$\lim_n \|y_m - x_n\| = r(y_m, \mathcal{F}_m) = r(y_m, \mathcal{F}).$$

for any $x_n \in K_n^m$.

Let $\mathcal{F}' = (K_m^m)_{m=1}^{\infty}$, then $K_m^m \subseteq K_m^{m-1} \subseteq \cdots \subseteq K_m$ so $\mathcal{F}'$ is subordinate to $\mathcal{F}$.

Further, for any n and $x_m \in K_m^m$ $m \geq n$,

$$\begin{aligned} \operatorname{dist}(y_n, K_m) &\leq \operatorname{dist}(y_n, K_m^m) \\ &\leq \|y_n - x_m\| \\ &\leq (1+\tfrac{1}{m})\, r(y_n, \mathcal{F}) \end{aligned}$$

as $K_m^m \subseteq K_m^{m-1} \subseteq \cdots \subseteq K_m^n \subseteq B_{(1+\frac{1}{m})r(y_n, \mathcal{F})}[y_n]$.

So letting $m \to \infty$ we have

$$r(y_n, \mathcal{F}) = r(y_n, \mathcal{F}') = \lim_m \|y_n - x_m\|$$

That $\mathcal{F}'$ is the desired family now follows since $|\operatorname{dist}(x, C) - \operatorname{dist}(y, C)| \leq \|x-y\|$ for any set C and $\{y_k\}$ dense in E.

<u>Note.</u> The sequence of numbers

$$\left(\inf_{x\in K_n} r(x, \mathcal{F})\right)_{n=1}^{\infty}$$

is increasing for any nested family $\mathcal{F} = (K_n)$ and so converges.

Let

$$r(\mathcal{F}) := \lim_n \inf_{x\in K_n} r(x, \mathcal{F})$$

Clearly, $r(\mathcal{F}) \leq r(\mathcal{F}')$ if $\mathcal{F}'$ is subordinate to $\mathcal{F}$ and $r(\mathcal{F}) \leq d(\mathcal{F})$.

<u>Step 2)</u> For $\mathcal{F}'$ as in step 1) we can find a nested family $\mathcal{F}''$ subordinate to $\mathcal{F}'$ for which

$$0 < d(\mathcal{F}'') = r(\mathcal{F}'') = r(\mathcal{F}') = r(\mathcal{F})$$

<u>Proof of step 2)</u>. Choose $x_n \in K_n'$ $(:= K_n^n)$ so that

$$\inf_{x\in K_n'} r(x, \mathcal{F}') > r(x_n, \mathcal{F}') - \tfrac{1}{n}$$

then

$$r(\mathcal{F}') = \lim_{n\to\infty} r(x_n, \mathcal{F}')$$

Further, by the flattening

$$r(x_n, \mathcal{F}') = \lim_{m\to\infty} \|x_n - x_{n+m}\|$$

Now let $m_1 = 1$ and choose m_2 so that for $m > m_2$ we have $|r(x_{m_1}, \mathcal{F}') - \|x_{m_1} - x_m\|| < 1$.
Then choose m_3 so that for $m > m_3$
$|r(x_{m_2}, \mathcal{F}') - \|x_{m_2} - x_m\|| < \frac{1}{2}$
etc.
In this way we obtain integers $m_1 < m_2 < \cdots$
so that
$$|r(x_{m_k}, \mathcal{F}') - \|x_{m_k}, x_m\|| < \tfrac{1}{k} \text{ for } m > m_{k+1}$$

In particular then,
$$|r(x_{m_k}, \mathcal{F}') - \|x_{m_k} - x_{m_r}\|| < \tfrac{1}{k} \text{ for } r > k$$
and so
$$r(\mathcal{F}') = \lim_k r(x_{m_k}, \mathcal{F}')$$
$$= \lim_{\substack{k\to\infty \\ r>k}} \|x_{m_k} - x_{m_r}\| = \lim_{\substack{r,k\to\infty \\ r\neq k}} \|x_{m_k} - x_{m_r}\|$$

Now let $K_n'' = \overline{co}\{x_{m_k}\}_{k=n+1}^{\infty}$

and let $\mathcal{F}'' = (K_n'')_{n=1}^{\infty}$.
Then
$$d(\mathcal{F}'') = \lim_n \operatorname{diam}(K_n'')$$
$$= \lim_n \sup_{\substack{r,k>n \\ r\neq k}} \|x_{m_k} - x_{m_r}\|$$
$$= r(\mathcal{F}') \leq r(\mathcal{F}'') \leq d(\mathcal{F}'')$$
establishing the result.

<u>Step 3)</u> For the nested family $\mathcal{F}''=(K''_n)$ of Step 2, if $x_n \in K''_n$, given $\varepsilon_1>0$ we can find a subsequence $y_k := x_{n_k}$ so that for any sequence $(n_m) \subseteq \mathbb{N}$ with $n_m \geq m$ and for any $u_m \in co\{y_k\}_{k=m}^{n_m}$ and $u'_m \in co\{y_k\}_{k=n_m+1}^{\infty}$ we have:

i) For all m, $d(\mathcal{F}'') - \varepsilon_1 \leq \|u_m - u'_m\|$

and

ii) $\limsup_m \|u_m - u'_m\| \leq \lim_m \operatorname{diam}(\{y_k\}_{k=m}^{\infty}) \leq d(\mathcal{F}'')$.

<u>Proof of Step 3)</u>. Since $d(\mathcal{F}'') = r(\mathcal{F}'') := \lim_n \inf_{x\in K''_n} r(x,\mathcal{F}'')$

there exists n_1 so that

$$d(\mathcal{F}'') - \varepsilon_1/2 \leq \inf_{x\in K''_{n_1}} r(x,\mathcal{F}'').$$

Further, $r(x,\mathcal{F}'') := \lim_n \operatorname{dist}(x, K''_n)$ so there exists n_2 such that

$$r(x_{n_1},\mathcal{F}'') \leq \operatorname{dist}(x_{n_1}, K''_{n_2}) + \varepsilon_1/2.$$

Thus for any element $u' \in K_{n_2}$, in particular any element of $\overline{co}\{x_k\}_{k=n_2}^{\infty}$, we have

$$d(\mathcal{F}'') - \varepsilon_1 \leq \|x_{n_1} - u'\|.$$

Now let $u \in co\{x_{n_1}, x_{n_2}\}$ then, as above, there exists $n_3(u)$ so that

$$d(\mathcal{F}'') - \varepsilon_1/2 \leq \operatorname{dist}(u, K''_{n_3(u)})$$

<u>Claim</u> There exists n_3 so that for all

$u \in co\{x_{n_1}, x_{n_2}\}$ we have
$$d(\mathcal{F}'') - \varepsilon_1 \leq dist(u, K''_{n_3}).$$

Assume not, then there exists a sequence (m_k) with $m_k \to \infty$ and points $u_k \in co\{x_{n_1}, x_{n_2}\}$ so that $dist(u_k, K''_{m_k}) < d(\mathcal{F}'') - \varepsilon_1$. Since $co\{x_{n_1}, x_{n_2}\}$ is compact there exists a subsequence $u_{k_j} \to u$, but then for $m_{k_j} > m_3(u)$ we have
$$dist(u_{k_j}, K''_{m_{k_j}}) \geq dist(u_{k_j}, K''_{m_3(u)}) \longrightarrow dist(u, K''_{m_3(u)}) \geq d(\mathcal{F}'') - \frac{\varepsilon}{2}$$
which is impossible.

Continuing in this way we obtain a sequence $n_1 < n_2 < \cdots$ such that for any $u \in co\{x_{n_k}\}_{k=1}^{m}$
$$d(\mathcal{F}'') - \varepsilon_1 \leq dist(u, K''_{n_{m+1}})$$

In particular, $d(\mathcal{F}'') - \varepsilon_1 \leq \|u - u'\|$ for any $u' \in co\{x_{n_k}\}_{k=m+1}^{\infty}$, establishing the lower inequality.

Now, since $u_m, u'_m \in \{y_k\}_{k=m}^{\infty} \subseteq K'''_m := co\{y_k\}_{k=m}^{\infty}$ we have
$$\limsup_m \|u_m - u'_m\| \leq \lim_m diam(\{y_k\}_{k=m}^{\infty}) = \lim_m diam(K'''_m) = d(\mathcal{F}''') \leq d(\mathcal{F}'')$$
as $\mathcal{F}''' := (K'''_n)$ is subordinate to $\mathcal{F}''$.

Step 4) (the last). The sequence (x_n) of the theorem is constructed from the sequence (y_k) of Step 3 as follows.

From step 2) for any $\varepsilon_2 > 0$ we can find n_0 so that for any m,

$$d - \varepsilon_1 \leq \| u - u' \| \leq d + \varepsilon_2 ,$$

where $u \in co\{y_k\}_{k=n_0}^{n_0+m}$

and

$$u' \in co\{y_k\}_{k=n_0+m+1}^{\infty}, \text{ and } d := d(\mathcal{F}'') > 0$$

Let $z_k := (y_{n_0+k} - y_{n_0})/d$ then

$$1 - \varepsilon_1/d \leq \| \omega - \omega' \| \leq 1 + \varepsilon_2/d$$

for $\omega \in co\{z_k\}_{k=1}^{m}$, $\omega' \in \overline{co}\{z_k\}_{k=m+1}^{\infty}$.

Further, from above with $m=1$

$$1 - \varepsilon_1/d \leq \| z_k \| \leq 1 + \varepsilon_2/d .$$

Since ε_1 & ε_2 are arbitrary it is clear that for any $\varepsilon > 0$ we can choose them so that $x_k := z_k / \| z_k \|$ satisfies the claim of the theorem. ∎

REMARKS: 1) The converse of theorem (2-8) is also true (and easier to show): If there exists a sequence $(x_n) \subset S_X$ satisfying the condition in theorem (2-8) for some $c \in [0,1)$, then X is non-reflexive

2) We can deduce corollary (2·8·1) from step 2) of the above proof, this is essentially Bae's Proof: From step 2) we have

$$0 < d := \lim_n \operatorname{diam}(K_n'') = \lim_n \inf_{x \in K_n''} \lim_m \operatorname{dist}(x, K_m).$$

Thus, given any $\varepsilon > 0$ we have for n sufficiently large;

$$| d - \operatorname{diam}(K_n'') | < \varepsilon$$

while

$$\begin{aligned} \operatorname{rad}(K_n'') &:= \inf_{x \in K_n''} \sup_{y \in K_n''} \|x - y\| \\ &\geq \inf_{x \in K_n''} \operatorname{dist}(x, K_m''), \quad \text{all } m > n \\ &\geq d - \varepsilon . \end{aligned}$$

Hence $\dfrac{\operatorname{rad}(K_n'')}{\operatorname{diam}(K_n'')} \geq \dfrac{d - \varepsilon}{d + \varepsilon}$ can be chosen arbitrarily close to 1.

3) The converse to corollary (2·8·1) is spectacularly false. Indeed reflexivity is far from sufficient even for normal structure.

(2.9) <u>There exist reflexive spaces lacking normal structure.</u>

<u>(2.9.1) EXAMPLE: For $\alpha \in (0,1)$ let X_α denote l_2 with norm $\|x\|_\alpha := (\alpha\|x\|_2) \vee \|x\|_\infty$</u>

$\|\cdot\|_\alpha$ is an equivalent norm for l_2, indeed $\alpha\|x\|_2 \leq \|x\|_\alpha := \max\{\alpha\|x\|_2, \|x\|_\infty\} \leq \|x\|_2$. Hence, X_α is reflexive. None-the-less, <u>for $\alpha \leq 1/\sqrt{2}$ X_α fails to have normal structure:</u>

$C := \overline{co}\{e_n\}_{n=1}^\infty$ has diameter $\sqrt{2}$ in the 2-norm and diameter 1 in the ∞-norm. So, for $\alpha \leq \frac{1}{\sqrt{2}}$, C has diameter 1 in X_α. Now, for any $x \in C$ we have $\|x - e_n\|_\alpha \geq \|x - e_n\|_\infty \longrightarrow 1$ (as (x_n) is a null sequence). That is; C is diametral and so X_α fails to have normal structure.

The space $X_{\frac{1}{2}}$ was introduced into the field by R.C. James, for the purpose of this example, and has played an important role. Karlovitz [1976 (b)] demonstrated the F.P.P. for $X_{\frac{1}{2}}$. Indeed as we will subsequently see X_α has the F.P.P. for all $\alpha > 0$. This will provide us with the first of many examples which demonstrate that normal structure, while sufficient, is not necessary for the w-F.P.P. (F.P.P. in reflexive spaces).

A localized version of uniform convexity was introduced by Lovaglia [1955]. Say the Banach space X is <u>locally uniformly convex</u> if given x with $\|x\|=1$ and a sequence (y_n) with $\|y_n\|=1$ such that $\left\|\frac{x+y_n}{2}\right\| \to 1$ we have that

$\|x-y_n\| \to 0$.

We show that local uniform convexity, even with the additional assumption of reflexivity, is not sufficient to ensure normal structure.

<u>2.9.2 EXAMPLE.</u> On $X=\ell_2$ define

$$\|x\| := \left(\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \|Q_m x^+\|_2^2 \vee \|Q_m x^-\|_2^2 \right)^{\frac{1}{2}}$$

where $Q_m(x) := (0, \cdots, 0, x_{m+1}, x_{m+2}, \cdots)$ for $x=(x_1, x_2, \cdots)$.

It is easily seen that

$$\tfrac{1}{2}\|x\|_2 \leq \|x\| \leq \|x\|_2 ,$$

so <u>$\|\cdot\|$ is an equivalent norm on ℓ_2</u>.

<u>Claim 1)</u> <u>$(\ell_2, \|\cdot\|)$ does not have normal structure</u>:

Let $C := \overline{co}\{e_k\}_{k=1}^{\infty}$.

Since $\|e_n - e_m\| = \sqrt{\sum_{k=0}^{m-1} \frac{1}{2^{k+1}}}$

$$= \sqrt{1 - 1/2^m} \quad \text{for } m > n$$

we see that $\operatorname{diam}(C) = 1$.

On the other hand, for any $x \in \operatorname{co}\{e_k\}_k$ we have

$$\|x - e_m\| = \sqrt{\sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \left(x_{k+1}^2 + \cdots + x_n^2\right) \vee 1 + \sum_{k=n}^{m-1} \frac{1}{2^{k+1}}}$$

$$= \sqrt{\sum_{k=0}^{m-1} \frac{1}{2^{k+1}}}$$

$$= \sqrt{1 - 1/2^m} \qquad \text{for } m > n$$

and so $\operatorname{rad}(C) = 1$.

<u>Claim 2)</u> <u>$(l_2, \|\cdot\|)$ is locally uniformly rotund:</u>

This example may derive from ideas of E. Lami-Dozo. An alternative example has been given by Smith and Turett [ ]: Define $T: l_2 \to c_0$ by $Tx := (\frac{1}{2}\|x\|_2, x_1, x_2, x_2, x_3, x_3, x_3, \ldots)$, $x = (x_1, x_2, \ldots)$. Then $\|x\| := \|Tx\|_D$, where $\|\cdot\|_D$ is Day's locally uniformly convex norm on c_0 (see Rainwater [1969]), satisfies

$$\tfrac{1}{4}\|x\|_2 \leq \|x\| \leq \|x\|_2 / \sqrt{3}.$$

So $\|\cdot\|$ is an equivalent locally uniformly convex norm for l_2. Smith and Turett show that in $(l_2, \|\cdot\|)$ the sequence (e_n) is diametral.

A convexity condition weaker than uniform convexity which does imply normal structure was introduced by Garkavi [1962]: X is <u>uniformly convex in every direction</u> (U.C.E.D) if whenever (x_n) and (y_n) are sequences in B_X for which $\|x_n + y_n\| \to 2$ and for some $z \neq 0$ $x_n - y_n = \lambda_n z$ we then have $\lambda_n \to 0$. Geometrically this means that for each "direction" $z \neq 0$, the collection of all chords of the unit ball which are parallel to z and whose lengths are bounded away from zero have midpoints which lie uniformly deep inside the ball.

<u>(2·10) THEOREM</u> [Garkavi, 1962]: X is U.C.E.D. if and only if the Chebyshev centre of each closed bounded convex subset of X consists of at most one point. In particular, such spaces have normal structure.

<u>Proof.</u> ($\Rightarrow$) Let C be a closed bounded convex subset of X and suppose $x_1, x_2 \in \mathcal{C}(C)$, clearly $x_0 = \frac{1}{2}(x_1 + x_2) \in \mathcal{C}(C)$ also.

Choose $(y_n) \in C$ so that

$$\|y_n - x_0\| \longrightarrow r = \mathrm{rad}(C).$$

Let $u_n = \frac{1}{r}(x_1 - y_n)$

and $v_n = \frac{1}{r}(x_2 - y_n)$

then we have

$$\|u_n\|, \|v_n\| \leq 1, \quad \|u_n + v_n\| = \frac{2}{r}\|x_0 - y_n\| \longrightarrow 2$$

and so, since X is U.C.E.D. and $u_n - v_n = \frac{1}{r}(x_1 - x_2)$ for all n we conclude that $x_1 = x_2$.

($\Leftarrow$) Suppose X is not U.C.E.D., then there exists $z \neq 0$ and $(x_n), (y_n) \subset B_X$ with $\|x_n + y_n\| \longrightarrow 2$ and $x_n - y_n = \lambda_n z$ where $\lambda_n \geq \lambda > 0$.

Let $C = \overline{co}\{\lambda z/2, \pm\frac{1}{2}(x_n + y_n) : n = 1, 2, \ldots\}$.

Clearly $\mathrm{rad}(C) = 1$ with $0 \in \mathcal{C}(C)$, further

$$\mathrm{rad}(C, \lambda z/2) = \sup_n \left\| \lambda z/2 \pm \frac{x_n + y_n}{2} \right\|$$

$$= \sup_n \left\| \left(\frac{\lambda}{2\lambda_n} \pm \frac{1}{2}\right) x_n - \left(\frac{\lambda}{2\lambda_n} \mp \frac{1}{2}\right) y_n \right\|$$

$$\leq \left(\frac{1}{2} \pm \frac{\lambda}{2\lambda_n}\right) + \left(\frac{1}{2} \mp \frac{\lambda}{2\lambda_n}\right) = 1$$

so $\lambda z/2 \in \mathcal{C}(C)$.

A detailed study of U.C.E.D. was made by Day, James and Swaminathan [1971]. They obtain several equivalent formulations of the property, investigate the stability of U.C.E.D. under products and obtain a number of results concerning spaces which admit an equivalent U.C.E.D. norm.

We will be interested in a special case, previously obtained by Zizler [ ].

<u>(2.11) THEOREM:</u> Every separable Banach space admits an equivalent U.C.E.D. norm.

<u>Proof.</u> Let $(x_n) \subset S_X$ be a dense sequence in the unit sphere of the separable Banach space X. Choose $f_n \in S_{X^*}$ such that $f_n(x_n) = 1$. Then (f_n) is a strictly norming subset of X^* for X, so the mapping

$$T : X \longrightarrow \ell_2 : x \longmapsto \left(f_n(x)/2^n \right)_{n=1}^{\infty}$$

is continuous, <u>1-1</u> and linear.

Hence

$$|||x||| := \left(\|x\|^2 + \|Tx\|_2^2 \right)^{\frac{1}{2}}$$

is an equivalent norm on X. We show it is U.C.E.D. Thus, let $(x_n), (y_n)$ be such that $x_n - y_n = \lambda_n z$ with $z \neq 0$ and for which $|||y_n||| \leq 1$, $|||x_n||| = |||y_n + \lambda_n z||| \leq 1$ while $|||x_n + y_n||| = |||2y_n + \lambda_n z||| \longrightarrow 2$.

Then we must have

$$2|||y_n|||^2 + 2|||y_n+\lambda_n z|||^2 - |||2y_n+\lambda_n z|||^2 \longrightarrow 0$$

or

$$\begin{aligned}&(2\|y_n\|^2+2\|y_n+\lambda_n z\|^2-\|2y_n+\lambda_n z\|^2)\\ &\qquad + (2\|Ty_n\|_2^2+2\|T(y_n+\lambda_n z)\|_2^2-\|T(2y_n+\lambda_n z)\|_2^2)\\ &= (2\|y_n\|^2+2\|y_n+\lambda_n z\|^2-\|2y_n+\lambda_n z\|^2)+\|T(\lambda_n z)\|_2^2\\ &\qquad\qquad\longrightarrow 0,\end{aligned}$$

and so, since the first term is positive (convexity of the function $\frac{1}{2}\|\cdot\|^2$), we have $\lambda_n\|Tz\|_2 \to 0$. Since T is 1-1 it follows that $\lambda_n \to 0$ and $(X, |||\cdot|||)$ is U.C.E.D. ∎

<u>2·11·1 Corollary</u>: There exist non-reflexive spaces which are U.C.E.D. In particular <u>there exist non-reflexive spaces with normal structure</u>.

<u>(2·11·2) REMARKS</u>: 1) A more specific example of the last corollary is obtained by taking $X = \ell_1$ and noting that the argument in the proof of theorem (2·11) with $T = I$ shows that $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}$ is an equivalent U.C.E.D. norm on ℓ_1. Hence $(\ell_1, \|\cdot\|)$ has normal structure, however it fails to have the F.P.P.: Let $C := \{x \in \ell_1 : \|x\|_1 = 1, x_k \geq 0 \text{ all } k\}$, then the right shift $T : x \mapsto (0, x_1, x_2, \ldots)$ is a fixed point free isometry on C in $(\ell_1, \|\cdot\|)$. It does of course have the ω-F.P.P. (see 2·2). This example appears due to E. Lami-Dozo.

2) In case X is the dual of a separable space the proof of theorem (2·11) is

2·11·1 Corollary: There exist non-reflexive spaces which are U.C.E.D. In particular there exist non-reflexive spaces with normal structure.

(2·11·2) REMARKS: 1) A more specific example of the last corollary is obtained by taking $X = \ell_1$ and noting that the argument in the proof of theorem (2·11) with $T = I$ shows that $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}$ is an equivalent U.C.E. norm on ℓ_1. Hence $(\ell_1, \|\cdot\|)$ has normal structure, however it fails to have the F.P.P. Let $C := \{x \in \ell_1 : \|x\|_1 = 1, x_k \geq 0 \text{ all } k\}$, then the right shift $T : x \mapsto (0, x_1, x_2, \dots)$ is a fixed point free isometry on C in $(\ell_1, \|\cdot\|)$. It does of course have the ω-F.P.P. (see 2·2). This example appears due to E. Lami-Dozo.

2) In case X is the dual of a separable space the proof of theorem (2·11) is readily adapted to obtain an equivalent dual U.C.E.D. norm.

3) An alternative equivalent renorming of separable spaces (separable dual spaces) which ensures ω-normal structure (ω^*-normal structure) is given by van Dulst [1982].

Van Dulst [1982] also shows that every Banach space can be equivalently renormed to fail normal structure.

<u>(2·12) THEOREM</u>: Every Banach space X admits an equivalent norm $|||\cdot|||$ so that $(X, |||\cdot|||)$ fails to have normal structure.

<u>Proof.</u> Let (b_n) be a normalized basic sequence in X (see Beauzamy [ ] ch. II §1) with coefficient functionals c_n extended to X. That is; $c_n \in X^*$, $c_n(b_m) = \delta_{mn}$ and $\|c_n\| \leq K$ for some $K > 0$ and all n.

Let $x_n := b_0 + 2b_n$

and $f_n := c_0 - c_n$ for $n = 1, 2, \ldots$.

Then $f_n(x_m) = \begin{cases} 1 & \text{if } n \neq m \\ -1 & \text{if } n = m. \end{cases}$

Define

$$|||x||| = \left(\sup_n |f_n(x)|\right) \vee \left(\tfrac{1}{3}\|x\|\right),$$

then $\frac{1}{3}\|x\| \leq |||x||| \leq K\|x\|$, so $|||\cdot|||$ is an equivalent norm on X. Further, since $\|x_n\| \leq 3$, we have $|||x_n||| = 1$, and so $1 \geq |||f_n||| \geq |f_n(x_n)| = 1$.

Now let $C = \overline{co}\{x_n\}_{n=1}^{\infty}$, clearly $\mathrm{diam}(C) \leq 2$ in $(X, |||\cdot|||)$, while for any $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ we have

$$\left|\left|\left| \sum_{i=1}^{n} \lambda_i x_i - x_{n+1} \right|\right|\right| \geq f_{n+1}\left(\sum_{i=1}^{n} \lambda_i x_i - x_{n+1}\right) = 2.$$

Thus C is diametral with (x_n) a diameterizing sequence. ∎

We conclude this chapter with a few observations concerning <u>normal structure as a "Banach space property"</u>. As we shall see our knowledge in this direction is unsatisfyingly incomplete.

Isomorphic stability (or lack there-of) has already been considered (2·10, 2·11, 2·12). Clearly (uniform) normal structure is inherited by subspaces, however the following seems unknown.

<u>QUESTION</u>: Is (uniform) normal structure inherited by Quotients.

Regarding stability under substitution, the only result I can verify is an early one of Belluce, Kirk and Steiner [1968], however the comprehensive survey of Kirk [1983] makes mention of several additional recent results by T. <u>Landes</u> (Pacific J 110 1984)

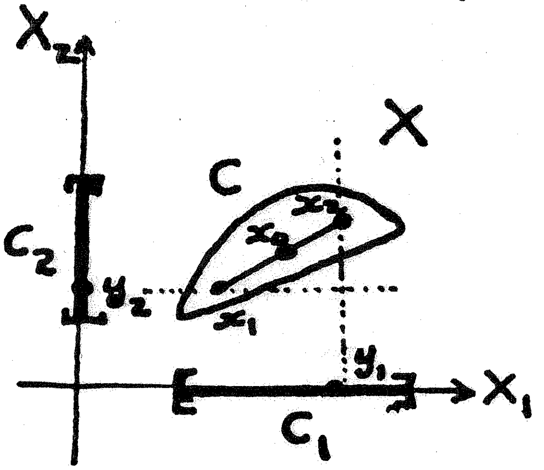
<u>(2·13) THEOREM</u> [Belluce, Kirk & Steiner, 68]: The l_∞ direct sum of a finite family of spaces each of which has normal structure also has normal structure.

<u>Proof.</u> Clearly it suffices to prove the result for two space X_1, X_2 each of which has

normal structure. Let $\mathfrak{X} = (X_1 \oplus X_2)_\infty$ and let $P_i : \mathfrak{X} \to X_i : (x(1), x(2)) \mapsto x(i)$ be the natural coordinate projection. Now, if C is a closed bounded convex subset of $\mathfrak{X}$ and $C_i = P_i(C)$ we have for $i = 1, 2$

$$\operatorname{diam}(C_i) \le \operatorname{diam}(C) \le \|(\operatorname{diam}(C_1), \operatorname{diam}(C_2))\|_\infty$$

so $\operatorname{diam}(C) = \max\{\operatorname{diam}(C_1), \operatorname{diam}(C_2)\}$. On the other hand; for $\varepsilon > 0$ choose $y_i \in C_i$ such that $\|y_i - z\| \le \operatorname{rad}(C_i) + \varepsilon$ for all $z \in C_i$. Select $x_i \in P_i^{-1}(y_i)$ and set $x_0 = \frac{1}{2}(x_1 + x_2)$. Then for any $x \in C$ we have



$$\|x - x_0\| = \max_{i=1,2}\{\|x(i) - x_0(i)\|_i\}$$

Now $\|x(1) - x_0(1)\|_1$

$$= \|x(1) - \tfrac{1}{2}(x_1(1) + x_2(1))\|_1$$
$$= \|x(1) - \tfrac{1}{2}(y_1 + x_2(1)\|_1$$
$$\le \tfrac{1}{2}\left[\|x(1) - y_1\|_1 + \|x_1 - x_2(1)\|_1\right]$$
$$\le \tfrac{1}{2}\left[\operatorname{rad}(C_1) + \varepsilon + \operatorname{diam}(C_1)\right]$$

Similarly $\|x(2) - x_0(2)\|_2$

$$\le \tfrac{1}{2}\left[\operatorname{diam}(C_2) + \operatorname{rad}(C_2) + \varepsilon\right].$$

Since ε is arbitrary it follows that

$$\operatorname{rad}(C) \le \operatorname{rad}(C, x_0)$$
$$\le \tfrac{1}{2}\left[\operatorname{diam}(C) + \operatorname{rad}(C_1) \vee \operatorname{rad}(C_2)\right]$$
$$< \operatorname{diam}(C). \quad \blacksquare$$

REMARK: From the above proof we see

that

$$N(X) \leq \tfrac{1}{2}\left[1 + N(X_1) \vee N(X_2)\right].$$

So a finite ℓ_∞ sum of spaces, each of which has uniform normal structure, also has <u>uniform normal structure</u>.

Finally we inquire about <u>3-space properties</u> for (uniform) normal structure. As a very partial result in this direction we have the following.

(2·

<u>(2·14) PROPOSITION</u> [Giles, Sims & Swaminathan, 85]

Let M be a complemented subspace of X such that M has finite codimension in X (or more generally M has a complement that is a Schur space). Then X has (weak) normal structure if M has uniform normal structure.

<u>Proof.</u> Since M has uniform normal structure it is reflexive (Corollary 2·8·1) and so, since M is of finite codimension, X too is reflexive. Now suppose X does not have (weak) normal structure, then, by Remark 1·18(1), X contains a weakly null sequence (x_n) satisfying:

$$\mathrm{dist}\left(x_{n+1}, \mathrm{co}\{x_k\}_{k=1}^{n}\right) \to 1.$$

Let P be the linear projection from X onto M with $I-P$ a projection onto the (Schur) complement of X. Since $x_n \xrightarrow{\omega} 0$ we therefore have $\|x_n - Px_n\| \to 0$.

Choose $\varepsilon > 0$ so that $\varepsilon < {1-N}/{4(1+N)}$ where $N = N(M)$, then there exists n_0 so that

$$\|x_n - Px_n\| \leq \varepsilon \text{ for all } n \geq n_0.$$

Let $C := \overline{co}\{x_n\}_{n=n_0}^{\infty}$, then since $\|x - Px\| \leq \varepsilon$ for all $x \in C$ we have

$$\operatorname{diam} P(C) \leq \operatorname{diam} C + 2\varepsilon$$
$$= 1 + 2\varepsilon.$$

From the uniform normal structure of M we can find a point $x_0 \in C$ such that

$$\|Px_0 - Px\| \leq N(1+2\varepsilon) \quad \text{for all } x \in C.$$

But then, for each $x \in C$ we have

$$\|x - x_0\| \leq \|x - Px\| + \|Px - Px_0\| + \|Px_0 - x_0\|$$
$$\leq 2\varepsilon + N(1+2\varepsilon)$$
$$< \tfrac{1}{2}(1+N) \quad \text{, by the choice of } \varepsilon.$$
$$< 1$$

contradicting the diametrality of C. ∎

(2.15) REMARK: We close this chapter with the observation that uniform normal structure is stable under small perturbations of the space. This follows from the readily seen inequality: For two spaces X, Y we have:

$$\frac{1}{d(X,Y)} N(X) \leq N(Y) \leq d(X,Y)\, N(X)$$

where $d(X,Y)$ is the Banach-Mazur distance between X and Y,

$$d(X,Y) := \text{infimum } \|U^{-1}\| \|U\|$$

where the infimum is taken over all linear isomorphisms U of X onto Y.

In particular then, if $|||\cdot|||$ is an equivalent norm of X satisfying

$$m\|x\| \le |||x||| \le M\|x\| \quad \text{for all } x \in X$$

we have

$$\frac{m}{M} N(X, \|\cdot\|) \le N(X, |||\cdot|||) \le \frac{M}{m} N(X, \|\cdot\|).$$

$\Rightarrow$ $N(\ell_2, (\alpha\|x\|_2) \vee \|x\|_\infty)$ has u. n. str. for $\alpha \geq \frac{1}{\sqrt{2}}$ (precisely same range as it has normal str.