THE EXISTENCE QUESTION FOR FIXED POINTS OF NONEXPANSIVE MAPS

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0. Statement of the problem; Examples.

The best known fixed point results in infinite dimensional spaces are undoubtedly the Banach contraction mapping principle and the Schauder-Tycknoff fixed point theorem.

(0·1) **THEOREM** (Banach contraction mapping principle): Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a *strict contraction*; that is, for some $k \in [0, 1)$ $d(Tx, Ty) \leq k\, d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point x_0 in X. Further, for any point $x_1 \in X$ and $n \in \mathbb{N}$ we have

$$d(T^n x_1, x_0) \leq \frac{k^n}{1-k} d(x_1, Tx_1).$$

(0·2) **THEOREM** (Schauder-Tychnoff fixed point theorem): Let C be a compact convex subset of a locally convex topological vector space. Then every continuous mapping $T : C \rightarrow C$ has a fixed point.

In theorem (0·1) a stringent form of continuity is imposed on the mapping T, while the assumption on the domain X is minimal for the existence of a fixed point On the contrary, in theorem (0·2) a minimal condition is imposed on the mapping while the nature of the domain C is heavily constrained.

The questions with which we will be concerned are in a sense intermediate to these two results. More specifically we will be interested in identifying Banach spaces X with one or other of the following properties.

A mapping $T: C \subseteq X \rightarrow C$ is non-expansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$. C

FPP, *the fixed point property*: X has the FPP if every non-expansive self-mapping of a non empty closed convex subset of X has a fixed point.

ω-FPP, *the weak fixed point property*: X has the ω-FPP if every non-expansive self-mapping of a non-empty weakly compact convex subset of X has a fixed point.

ω*-FPP, *the weak* fixed point property*: a dual space X^* has the ω*-FPP if every non-expansive self-mapping of a non-empty weak*-compact convex subset of X^* has a fixed point.

One reason why deciding which spaces have the ω (ω^*)-FPP is both intreaguing and difficult is that the continuity condition on the mapping is in a different (stronger) topology than the compactness of the domain.

For a reflexive space all three properties coincide. In general FPP $\Rightarrow$ ω-FPP and for a dual space ω^*-FPP $\Rightarrow$ ω-FPP. A natural advantage of the ω^*-FPP is the ready supply of ω^*-compact subsets guaranteed by the Banach-Alaoglu theorem.

Some Examples

(0·3) <u>c_0 fails the FPP</u>

Let $C = B^+_{c_0} := \{(x_n) \in c_0 : 0 \leq x_n \leq 1, \text{all } n\}$

and define T by
$$T(x_n) := (1 - x_1, x_1, x_2, \dots).$$

Then for any $\underset{\sim}{x}, \underset{\sim}{y} \in c_0$ we have $\|T\underset{\sim}{x} - T\underset{\sim}{y}\| = \|\underset{\sim}{x} - \underset{\sim}{y}\|$, so T is non-expansive and T maps C into C. On the other hand the only possible fixed point for T is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) \notin c_0$.

Q: If X has FPP
If all non-exp. T $B(X) \to B(X)$ have FPs.
is X reflexive.

(0·4) [Lim, 1980] ℓ_1 with the equivalent dual norm $\|\underset{\sim}{f}\|' := \|\underset{\sim}{f}^+\|_1 \vee \|\underset{\sim}{f}^-\|_1$ fails the w^*-FPP

We first show that $\|\cdot\|$ is indeed an equivalen dual norm for ℓ_1. To this end, for $\underset{\sim}{x} \in c_0$ define

$$\|\underset{\sim}{x}\| := \|\underset{\sim}{x}^+\|_\infty + \|\underset{\sim}{x}^-\|_\infty .$$

Then $\|\cdot\|$ is an equivalent norm on c_0 satisfyin $\|\underset{\sim}{x}\|_\infty \leq \|\underset{\sim}{x}\| \leq 2\|\underset{\sim}{x}\|_\infty$ and so it suffices to show that for $\underset{\sim}{f} \in \ell_1$ we have

$$\|\underset{\sim}{f}\|' = \sup\{\underset{\sim}{f}(\underset{\sim}{x}) : \underset{\sim}{x} \in c_0, \|\underset{\sim}{x}\| \leq 1\}.$$

For $\underset{\sim}{x} \in c_0$ with $\|\underset{\sim}{x}\| \leq 1$ let

$$y_i = \begin{cases} x_i & \text{if } f_i x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

then $\|\underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| \leq 1$ and

$$\begin{aligned}
\underset{\sim}{f}(\underset{\sim}{x}) &:= \sum_{i=1}^{\infty} f_i x_i \\
&\leq \sum_{i=1}^{\infty} f_i y_i \\
&\leq \|\underset{\sim}{y}^+\|_\infty \|f^+\|_1 + \|\underset{\sim}{y}^-\|_\infty \|\underset{\sim}{f}^-\|_1 \\
&= \underbrace{\left(\frac{\|\underset{\sim}{y}^+\|_\infty}{\|\underset{\sim}{y}\|} \|\underset{\sim}{f}^+\|_1 + \frac{\|\underset{\sim}{y}^-\|_\infty}{\|\underset{\sim}{y}\|} \|\underset{\sim}{f}^-\| \right)}_{\text{a convex combination of } \|\underset{\sim}{f}^+\|_1 \text{ and } \|\underset{\sim}{f}^-\|_1 .} \|\underset{\sim}{y}\|
\end{aligned}$$

$$\leq \left(\|\underset{\sim}{f}^{+}\|_1 \vee \|\underset{\sim}{f}^{-}\|_1\right) \|\underset{\sim}{y}\|$$

$$\leq \|\underset{\sim}{f}\|' .$$

To see the reverse inequality note that $\|\underset{\sim}{f}^{+}\|_1'$ (or $\|\underset{\sim}{f}^{-}\|_1'$) can be approximated arbitrarily well by $\underset{\sim}{f}(\underset{\sim}{x})$ where the x_i are a suitable choice of 0 or 1 (0 or -1) and so $\|\underset{\sim}{x}\| \leq 1$.

Now let $C = \{ \underset{\sim}{f} \in \ell_1 : f_i \geq 0 , \|\underset{\sim}{f}\| \leq 1 \}$ and define T by

$$T\underset{\sim}{f} := \left(1 - \sum_{i=1}^{\infty} f_i , f_1 , f_2 , \cdots \right).$$

Then C is a weak*-compact convex subset of ℓ and it is readily verified that T is a fixed point free affine mapping of C into C. We conclude by showing that T is an isometry (hence certainly non-expansive mapping) on C.

Given $\underset{\sim}{f}, \underset{\sim}{g} \in C$ let $P := \{ i : f_i - g_i \geq 0 \}$ and $N := \{ i : f_i - g_i < 0 \}$.
In the case that $\sum_{i \in P} (f_i - g_i) \geq \sum_{i \in N} (g_i - f_i)$ we have $\|\underset{\sim}{f} - \underset{\sim}{g}\|' = \sum_{i \in P} (f_i - g_i)$ and

$$\|T\underset{\sim}{f} - T\underset{\sim}{g}\|' = \left\| \left(\sum_{i=1}^{\infty} (g_i - f_i) , f_1 - g_1 , f_2 - g_2 , \cdots \right) \right\|$$

$$= \| \underbrace{\sum_{i\in N}(g_i - f_i) - \sum_{i\in P} f_i - g_i}_{\text{negative}},\ f_1 - g_1,\ f_2 - g_2, \ldots)\|'$$

$$= \operatorname{Max}\left\{ \sum_{i\in P}(f_i - g_i),\ \sum_{i\in P}(f_i - g_i) \right\}$$

$$= \|\underset{\sim}{f} - \underset{\sim}{g}\| .$$

The equality follows similarly in the case when $\|\underset{\sim}{f} - \underset{\sim}{g}\|' = \sum_{i\in N}(g_i - f_i)$.

(0·5) $\mathcal{L}_1(\mu)$ fails the ω-FPP.

Although the question had been raised more than twenty years earlier it was not until 1981 that Dale Alspach gave an example, drawn from ergodic theory, showing that not all Banach spaces enjoy the ω-FPP.

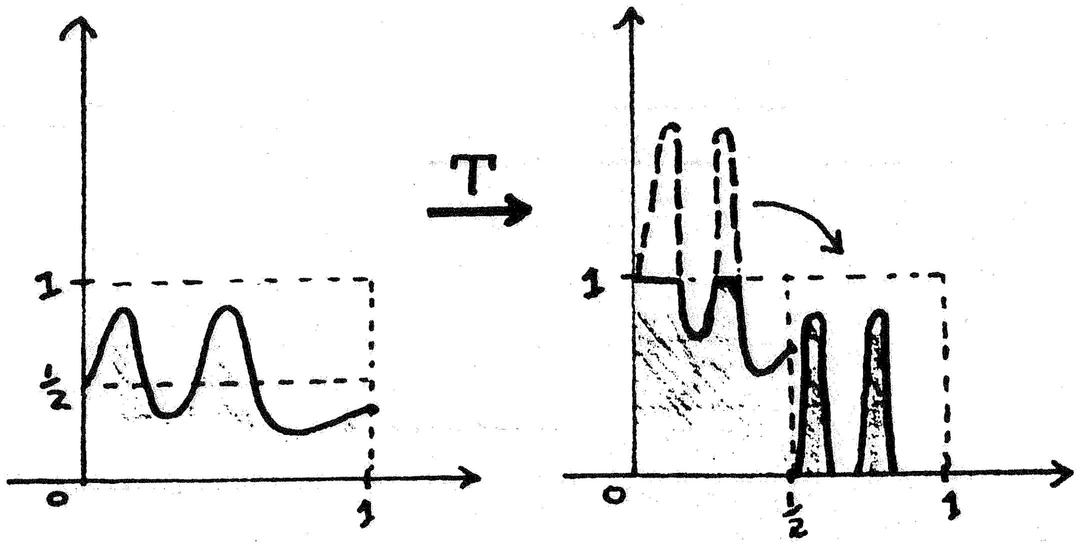
(0·5·1) Alspach's example [Alspach, 1981]

Here we take C to be the set

$$C := \left\{ f \in \mathcal{L}_1[0,1] : 0 \leq f \leq 1 ,\ \int_0^1 f = \tfrac{1}{2} \right\}$$

As the section of an order interval by a hyperplane in an order continuous Banach lattice, C is weakly compact.

The mapping T is essentially the baker transform of ergodic theory illustrated below.



Formally, for $f \in C$

$$Tf(t) := \begin{cases} (2f(2t)) \wedge 1 & \text{for } 0 \le t \le \frac{1}{2} \\ (2f(2t-1)-1) \vee 0 & \text{for } \frac{1}{2} < t \le 1 . \end{cases}$$

It is clear from the above description that <u>T is an isometry on C</u>.

We now show that <u>T is fixed point free</u>.

Intuitively the idea is simple. First observe that the successive iterates of any point in C under T assume values closer to 0 or 1. Hence any fixed point for T must be a function which assumes only the values 0 or 1. By the "ergodic" nature of T it then follows that such a function must be either constantly 0 or constantly 1, and neither of these functions lie in C.

The details follow.

For any $f \in C$ we have $Tf(t) = 1$ if and only if either
$0 \leq t \leq \frac{1}{2}$ and $\frac{1}{2} \leq f(2t) \leq 1$
or
$\frac{1}{2} < t \leq 1$ and $f(2t-1) = 1$.

Now, suppose f is a fixed point for T then

$$
\begin{aligned}
A &:= \{t : f(t) = 1\} \\
&= \{t : Tf(t) = 1\} \\
&= \{t : 0 \leq t \leq \tfrac{1}{2} \,\&\, \tfrac{1}{2} \leq f(2t) \leq 1\} \cup \{t : \tfrac{1}{2} < t \leq 1 \,\&\, f(2t-1) = 1\} \\
&= \{t/2 : \tfrac{1}{2} \leq f(t) \leq 1\} \cup \{\tfrac{1}{2} + t/2 : f(t) = 1\} \\
&= \tfrac{1}{2}\{t : \tfrac{1}{2} \leq f(t) < 1\} \cup \tfrac{1}{2}A \cup (\tfrac{1}{2} + \tfrac{1}{2}A)
\end{aligned}
$$

Since the three sets in the above union are mutually disjoint and each of the last two sets has measure one half that of A it follows that $B_1 := \{t : \frac{1}{2} \leq f(t) < 1\}$ is a null set.
But, then
$$
\begin{aligned}
B_1 &= \{t : \tfrac{1}{2} \leq Tf(t) < 1\} \\
&\supset \{t/2 : \tfrac{1}{4} \leq f(t) < \tfrac{1}{2}\}
\end{aligned}
$$
and so $B_2 := \{t : \frac{1}{4} \leq f(t) < \frac{1}{2}\}$ is also a null set.
Continuing in this way we have
$B_n := \{t : 1/2^n \leq f(t) < 1/2^{n-1}\}$ is a null set
for $n = 1, 2, \cdots$

hence
$\{t : 0 < f(t) < 1\} = \bigcup_{n=1}^{\infty} B_n$ is null
and
$$f \equiv \chi_A \text{ (where meas}(A) = \tfrac{1}{2}\text{)}.$$

From the definition of T we have
$$T(\chi_A) = (\chi_{\frac{1}{2}A} + \chi_{(\frac{1}{2}+\frac{1}{2}A)})$$

so, up to sets of measure zero,

(#)
$$A = \tfrac{1}{2}A \cup (\tfrac{1}{2}+\tfrac{1}{2}A).$$
Continuing to iterate under T yields
$$A = \tfrac{1}{4}A \cup (\tfrac{1}{4}+\tfrac{1}{4}A) \cup (\tfrac{1}{2}+\tfrac{1}{4}A) \cup (\tfrac{3}{4}+\tfrac{1}{4}A)$$
$$A = \tfrac{1}{8}A \cup (\tfrac{1}{8}+\tfrac{1}{8}A) \cup (\tfrac{1}{4}+\tfrac{1}{8}A) \cup \cdots$$
et hoc genus omne.

error here

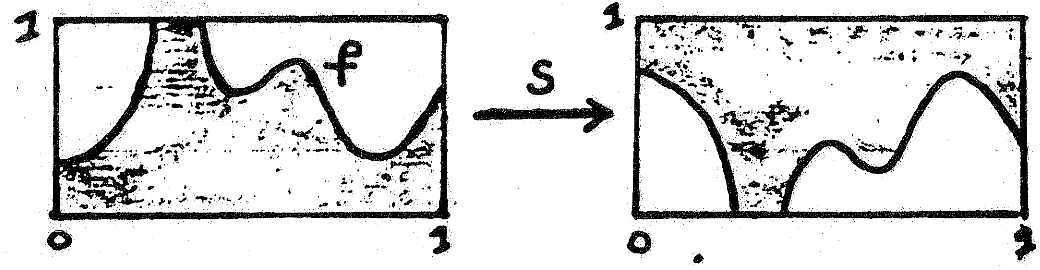
Thus the intersection of A with any non-empty open interval of [0,1] has non-zero measure, an impossibility for a set which is not of full measure.

(#) leads to a contradiction, & so T has no fixed point

(0·5·2) Sine's modification of the Alspach Example.

Robert Sine [1981] gave the following modification to the example of (0·5·1) which allows us to take as the domain C of our fixed point free non-expansive mapping the whole order interval $0 \leq f \leq 1$.

For $f \in C := \{g : 0 \leq g \leq 1\}$ let $Sf := \chi_{[0,1]} - f$

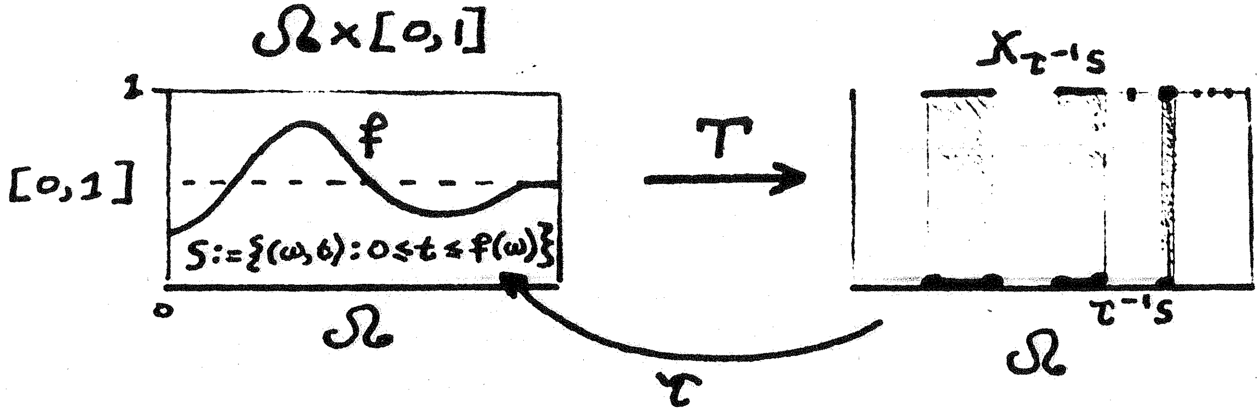


then S defines a mapping of C onto C with $\|Sf - Sg\| = \|f - g\|$ for all $f, g \in C$.

An argument similar to that for Alspach example shows that the composition ST, where T is the baker transform of 0.5. is an isometry on the order interval $0 \leq f \leq$ with χ_A where $A = [0,1]$ or ϕ the on possible fixed points. However the acti of ST is to map each of these functions onto the other, hence ST is fixed point free on the order interval $0 \leq f \leq 1$.

(0.5.3) Schechtman's Construction.

$$T f := \chi_{\tau^{-1}\{(\omega,t)\,:\, 0\le t\le f(\omega)\}}$$



Clearly T is an isometry on C and $f\in C$ is a fixed point for T if and only if $f=\chi_A$ where $A\in\Sigma$ is such that $\mu(A)=\frac{1}{2}$ and $\hat{\tau}(A):=\tau^{-1}(A\times[0,1])=A$ a.e

Thus if τ is further chosen so that $\hat{\tau}$ is "ergodic"; that is $\hat{\tau}(A)=A$ a.e. if and only if $A=\Omega$ or $A=\emptyset$, then T is an example of a fixed point free non-expansive mapping on C.

Perhaps the simplest example of an (Ω,Σ,μ) and τ suitable for the above construction is the following.

Let $\Omega=[0,1]^{N_0}$ with product Lebesgue measure and define τ by

$$\tau^{-1}((\omega_1,\omega_2,\cdots),t):=(t,\omega_1,\omega_2,\cdots).$$

Clearly τ is measure preserving, further if $A\ne\emptyset$ and $\hat{\tau}(A)=A$, then for any $(\omega_1,\omega_2,\cdots)\in A$ we see that

$(t, \omega_1, \omega_2, \cdots) \in A$ for any $t \in [0,1]$. Iterating under $\hat{\tau}$ gives $(t_1, t_2, \cdots, t_n, \omega_1, \omega_2, \cdots) \in A$ for any $n \in \mathbb{N}$ and $t_1, t_2, \cdots, t_n \in [0,1]$, and so we have $A = \mathcal{S}$.

An alternative example with $\mathcal{S} = [0,1]$ is obtained by taking

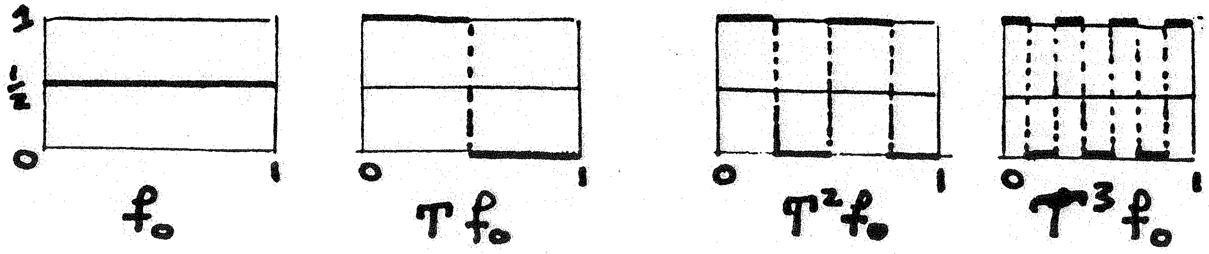
$$\tau^{-1}\left(\sum_{n=1}^{\infty} \varepsilon_n/2^n , \sum_{n=1}^{\infty} \delta_n/2^n \right) := \frac{\delta_1}{2} + \frac{\varepsilon_1}{2^2} + \frac{\delta_2}{2^3} + \frac{\varepsilon_2}{2^4} + \cdots$$

where $\varepsilon_n, \delta_n \in \{0,1\}$ for $n = 1, 2, \cdots$. A good way to view this example is via the correspondence

$$[0,1] \longleftrightarrow \{0,1\}^{\aleph_0} : \sum_{n=1}^{\infty} \varepsilon_n/2^n \longleftrightarrow (\varepsilon_1, \varepsilon_2, \cdots).$$

The measure of a set specified by prescribing precisely m of the ε_n's being $1/2^m$. It is then clear that the product of two such sets has measure $1/2^{m_1+m_2}$ where m_1+m_2 is also the number of digits prescribed for points in the τ^{-1} image of the product. It follows that τ is measure preserving. The ergodicity is established by iterating under $\hat{\tau}$ and an argument similar to that used for the conclusion of Alspach's example.

(0·5·4) REMARKS. Schechtman's construction is both simpler and more versatile than that of Alspach. None-the-less the Alspach example has some advantages. Besides permitting Sine's modification to obtain an example on an order interval, the relatively simple action of the baker transform permits detailed calculations. For example, it is possible to determine the *orbit* f_0, Tf_0, T^2 $T^3 f_0, \cdots$ of certain functions f_0 under T. If $f_0 = \frac{1}{2}\chi_{[0,1]}$ we obtain the iterates depicted below.



f_0 $\quad Tf_0$ $\quad T^2 f_0$ $\quad T^3 f_0$

Here we see that the sequence $T^n f_0 = \frac{1}{2}(r_n + 1)$, where r_n is the n'th *Rademacher function*, is an orbit under T.

PROBLEMS: These examples indicate an intimate connection between fixed point free isometries and ergodic transformations of the underlying measure space. In the true tradition of ergodic theory, *is the set of fixed point free isometries on the order interval* $[0 \leq f \leq 1]$

residual in an appropriate sense, at least among isometries which map into the set of 0,1-valued functions?

Clearly any space containing an isometric copy of $\mathcal{L}_1(\mu)$ also fails the ω-F.P.P. *What do the examples look like when translated into l_∞, $C[0,1]$?*

Examples 0·4 and 0·5 also suggest the following question. *If a space X fails the (ω, ω^*) F.P.P. does it necessarily fail with an isometry?*

All the examples presented in this section have been negative in nature, besides helping delineate the problem this seems only fair since the remainder of these notes are devoted to positive results. As a start in this direction we close the section with two simple observations.

(0·6) *PROPOSITION*: If X fails any of the fixed point properties, then given any point $x_0 \in X$ there exists a (ω^*, ω-compact) closed bounded convex set C_0

with $x_0 \in C_0$ and $\text{diam}(C_0) := \sup_{x,y \in C_0} \|x-y\| = 1$,

and there exists a fixed point free non-expansiv mapping $T_0 : C_0 \to C_0$.

Proof. If X fails the (ω, ω^*) F.P.P. then there exists a $(\omega, \omega^*$-compact$)$ closed bounded convex subset C and a fixed point free non-expansive mapping $T : C \to C$. C must contain more than one point (otherwis the solitary element would be fixed by T), hence $d = \text{diam}(C) > 0$. Choose $x_1 \in C$ and let $C_0 := \frac{1}{d}(C - x_1) + x_0$ and define

$$T_0(x) := \frac{1}{d}\left(T\left(d(x-x_0)+x_1\right) - x_1\right) + x_0$$

A simple calculation now verifies the claim. ■

Proposition (0·6) will be used throughout the sequel to simplify calculations. The next result should also be of use, though as far as I know it has played no significant role in the theory.

Clearly a space has the ω-F.P.P. (or F.P.P.) if and only if all of its closed subspaces do.

<u>(0.7) PROPOSITION</u>: The ω-F.P.P. and the F.P.P. are separably determined.

Proof. If X fails the (ω-) F.P.P. then there exists a (ω-compact) closed bounded convex subset C and a fixed point free non-expansive mapping $T: C \to C$. Choose any point $c \in C$. Let $K_1 = \{c\}$ and inductively define K_n by

$$K_{n+1} = \overline{co}\{T(K_n) \cup K_n\}.$$

Let $\hat{K} = \overline{\bigcup_{n=1}^{\infty} K_n}$, then $\hat{K}$ is a separable closed convex (and hence weakly compact if C is) subset of C.

<u>claim</u>: $\hat{K}$ is invariant under T. Let $x \in \hat{K}$. Given $\varepsilon > 0$ there exists $y \in K_n$ for some n with $\|x - y\| < \varepsilon$, but then $Ty \in K_{n+1} \subseteq \hat{K}$ and $\|Tx - Ty\| \leq \varepsilon$, so $Tx \in \hat{K}$ establishing the claim.

The result now follows by considering the pair $\hat{K}$, $T|_{\hat{K}}$ in the separable closed subspace spanned by $\hat{K}$. ■

An analogous result for the w^* F.P.P. in dual spaces <u>would</u> be useful but appears not to be known.

1. Minimal Invariant Sets

Given a weak (weak*)-compact convex set C and a non-expansive map $T: C \to C$ in a Banach space X, let $K \equiv K(C,T)$ denote the class of non-empty weakly (weak*)-compact convex subsets of C which are invariant under T (K is invariant under T if $T(K) \subseteq K$). If K is partially ordered by inclusion the weak (weak*)-compactness ensures that the intersection of any decreasing chain of sets is a lower bound for the chain in K. Thus we may apply Zorn's lemma to establish the existence of a minimal element of K. We shall refer to such a minimal element as a minimal invariant subset for T (minimality within the class K being understood).

OBSERVATIONS.

1.1) If C is weak-compact convex and D is a minimal invariant subset for T then $D = \overline{co}\, T(D)$. (In the weak* case we must take the ω^*-closed convex hull.)

1.2) X has the ω (ω^*) - F.P.P. if and only if all minimal invariant subsets for non-expansive mappings are "singleton" sets; that is, have only one element (or equivalently have zero diameter).

The $\omega(\omega^*)$ F.P.P. has been established by identifying specific properties of minimal invariant sets and then imposing conditions on the space which preclude the existence of sets, other than singleton sets, with the properties.

Another useful tool has been the existence of "<u>approximate fixed point sequences</u>": given a non-expansive map $T : C \to C$ on a closed bounded convex set C choose $x_0 \in C$ and for $\lambda \in [0,1)$ define $T_\lambda : C \to C$ by

$$T_\lambda(x) := \lambda T x + (1-\lambda) x_0 .$$

T_λ is a strict contraction and so by the Banach contraction mapping principle has a unique fixed point x_λ in C.

$$\| x_\lambda - T x_\lambda \| = (1-\lambda) \| x_0 - T x_\lambda \|$$
$$\leq (1-\lambda) \operatorname{diam}(C).$$

Thus $\| x_\lambda - T x_\lambda \| \to 0$ as $\lambda \to 1$. In particular letting $\lambda = (1 - \frac{1}{n})$ we obtain a sequence of <u>approximate fixed points</u> for T; (x_n) with $\| x_n - T x_n \| \to 0$.

(1.3) <u>PROPOSITION</u> The set $\{ x_\lambda : \lambda \in [0,1) \}$ is a connected arc in C.

__Proof.__ We show the mapping $[0,1) \to C : \lambda \mapsto x_\lambda$ is continuous. Given $\lambda, \beta \in [0,1)$

$$\|x_\lambda - x_\beta\| = \|\lambda T x_\lambda - \beta T x_\beta + (\lambda - \beta) x_0\|$$

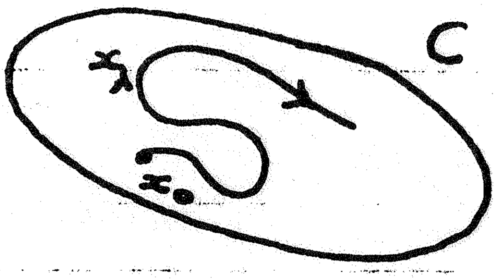
$$\leq \|\lambda T x_\lambda - \lambda T x_\beta\| + |\lambda - \beta| \|T x_\beta + x_0\|$$

$$\leq \lambda \|x_\lambda - x_\beta\| + 2B|\lambda - \beta||,$$

where B is a bound on the norms of elements in C

Thus,

$$\|x_\lambda - x_\beta\| \leq \frac{2B}{1-\lambda} |\lambda - \beta| \qquad \blacksquare$$



__Structure of minimal invariant sets__

Let T be a non-expansive mapping of a w (w^*)-compact convex subset let D denote a w (w^*)-compact minimal invariant set for T.

__(1.4) LEMMA:__ If $\psi : D \to \mathbb{R}$ is a weak (weak*) lower semi-continuous convex mapping with $\psi(Tx) \leq \psi(x)$ for all $x \in D$, then ψ is constant on D.

<u>Proof.</u> Since D is $w(w^*)$-compact and ψ is $w(w^*)$-lower semi-continuous ψ achieves its minimum on D. Let $x_0 \in D$ be such that $\psi(x_0) = \inf \psi(D)$ and let $E = \{x \in D : \psi(x) = \psi(x$ then E is a non-empty $w(w^*)$-closed convex set which is invariant under T.
Thus, by minimality $E = D$. ■

We will be particularly interested in three instances of such a ψ.

a) $\psi(x) := \sup\{\|x-y\| : y \in D\}$

b) $\psi(x) := \limsup_n \|x - x_n\|$,

where i) (x_n) is a sequence of approximate fixed points for T in D,

and ii) (x_n) is an orbit of T in D; that is $x_n = T^n x_0$ for some point $x_0 \in D$.

The function defined in a) is the <u>radius of the set D about x</u>;

$$\mathrm{rad}(D,x) := \sup\{\|x-y\| : y \in D\}.$$

A point of D about which the radius is the diameter of D is termed a <u>diametral point</u> of D.

The Chebyschev radius of D is

$$\mathrm{rad}(D) = \inf_{x \in D} \mathrm{rad}(D,x)$$

and the (possibly empty) <u>Chebyshev centre</u> of D is

$$\mathcal{C}(D) = \{x \in D : \mathrm{rad}(D,x) = \mathrm{rad}(D)\}.$$

$\mathcal{C}(D)$ is convex if D is and if D is weak (or weak*) compact then $\mathcal{C}(D)$ is non-empty.

We will say a set D is <u>diametral</u> if $\mathrm{rad}(D) = \mathrm{diam}(D)$; that is, if every point of D is a diametral point. Clearly this happens if and only if $D = \mathcal{C}(D)$.

That diametral sets consisting of more than one point can exist may at first seem strange. However it is readily seen that the set

$$B^+_{c_0} = \{(x_n) \in c_0 : 0 \le x_n \le 1, \text{all } n\}$$

is a diametral subset of c_0.

<u>**(1·5) THEOREM**</u> (Brodskii - Milman, 1948 / Garkavi, 1961 / Kirk, 1965) If D is a weak (weak*)-compact minimal invariant set for a non-expansive mapping then D is diametral.

Proof. It suffices to verify that

$$\psi(x) := \sup\{\|x-y\| : y \in D\}$$

satisfies the hypotheses for lemma (1·4), as then ψ is a constant on D with value equal to $\sup_{x \in D} \psi(x) = \sup_{x \in D} \sup_{y \in D} \|x-y\| = \mathrm{diam}(D)$.

To complete the proof note that since the norm is weak (weak*) lower-semi-continuous, the supremum in the definition of ψ is approached at extreme points of D and by $(1{\cdot}1\,;\ \overline{co}(T(D)) = D)$ the extreme points of D are contained in the closure (w^*-closure) of $T(D)$ we have that

$$\psi(x) = \sup \{ \|x-y\| : y \in T(D) \}.$$

It follows that $\psi(Tx) \leq \psi(x)$. ∎

SOME EXAMPLES

(1·6) *The domain $C := \{ f \in \mathcal{L}_1[0,1] : 0 \leq f \leq 1, \int f = \}$ in the Alspach example (0·5·1) is not a minimal invariant set for the baker's transform.* This follows since $\mathrm{diam}(C) = 1$

$$(1 \geq \mathrm{diam}(C) \geq \| \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]} \|_1 = 1),$$

while for any $f \in C$ we have $-\frac{1}{2} \leq f - \frac{1}{2} \leq \frac{1}{2}$ so

$$\| f - \tfrac{1}{2} \chi_{[0,1]} \|_1 = \int_0^1 | f - \tfrac{1}{2} | \leq \tfrac{1}{2}.$$

and C is not diametral.

Indeed the author knows of no example of a non-trivial minimal invariant set for a non-expansive map on a weak compact convex set.

(1·7) The domain C in Lim's example (0·4) is a w^*-compact minimal invariant subset.

To see this note that for any $\underset{\sim}{f} = (f_m) \in C$ we have as successive iterates

$$T\underset{\sim}{f} = (1 - \sum_1^\infty f_m, f_1, f_2, \cdots)$$

$$T^2\underset{\sim}{f} = (0, 1 - \sum_1^\infty f_m, f_1, f_2, \cdots)$$

$$T^3\underset{\wedge}{f} = (0, 0, 1 - \sum_1^\infty f_m, f_1, f_2, \cdots)$$

etc.

So $T^n \underset{\sim}{f} \xrightarrow{w^*} 0$.

Thus 0 belongs to any T invariant w^*-compact convex subset K of C.
Hence $T^n(0) = e_n$, the n'th basis vector is in K.
It follows that $C = \overline{co}\,\{e_n\} \subseteq K \subseteq C$, so $K = C$.

1·8 THEOREM (Goebel, 1975/Karlovitz, 1976): If (x_n) is a sequence of approximate fixed points for the non-expansive mapping T in the minimal invariant set D, then

$$\lim_n \|x - x_n\| = diam(D), \text{ for all } x \in D.$$

We will call any sequence of points with this property a <u>diameterizing sequence</u> for D.

<u>Proof.</u> Let (y_n) be any sequence of approximate fixed points for T in D and define

$$\psi(x) := \limsup_n \|x - y_n\|.$$

By lemma (1.4) ψ is constant on D with value k say. Let (y_{n_j}) be a subsequence (net) with $y_{n_j} \xrightarrow{\omega(\omega^*)} y_0$, then

$$k \geq \limsup_j \|x - y_{n_j}\| \geq \liminf_j \|x - y_{n_j}\|$$
$$\geq \|x - y_0\|.$$

Thus $k \geq \sup_{x \in D} \|x - y_0\| = \operatorname{diam}(D)$ by theorem (1.5).

Now taking as (y_n) any subsequence (x_{n_k}) of (x_n) we have

$$\limsup_k \|x - x_{n_k}\| = \operatorname{diam}(D) \text{ for all}$$

x in D and so $\lim_n \|x - x_n\| = \operatorname{diam}(D)$. ∎

<u>(1.9) Corollary</u>: Given any $\varepsilon > 0$ and $x \in D$ there exists $\Lambda \in [0, 1)$ such that the segment

$$\{x_\lambda : \lambda \in (\Lambda, 1)\}$$

of the arc described in proposition 1.3 sequence. Clearly the lies outside the ball of radius $(1-\varepsilon)\,\mathrm{diam}(D)$ centred at x.

<u>(1·10) Corollary:</u> If K is any compact subset of D then

$$\lim_n \mathrm{dist}(x_n, K) = \mathrm{diam}(D).$$

Proof. Assume not then there exists a subsequence (x_{n_k}) of (x_n) and points y_k of K with

$$\|x_{n_k} - y_k\| < (1-\varepsilon_0)\,\mathrm{diam}(D)$$

for some $\varepsilon_0 > 0$.

Passing to further subsequences if necessary we may, by the compactness of K, assume that $y_k \longrightarrow y_0$.

But then,

$$\limsup_k \|x_{n_k} - y_0\| \leqslant (1-\varepsilon_0)\,\mathrm{diam}(D)$$

contradicting theorem (1·8). ▨

<u>1·11 Corollary</u> The minimal invariant set D contains a sequence of approximate fixed points for T satisfying

$$\lim_n \mathrm{dist}\left(x_{n+1}, \overline{\mathrm{co}}\{x_j\}_{j=1}^{n}\right) = \mathrm{diam}(D).$$

We will call any such sequence a <u>diametral sequence</u>. Clearly the closed convex hull of a diametral sequence is a diametral set with the sequence as a diameterizing sequence.

Proof. Starting with any sequence of approximate fixed points proceed inductively to extract the subsequence (x_n) using $K = \overline{co}\{x_j\}_{j=1}^{n}$ in corollary (1.10). ▨

(1.12) REMARK: The argument in the first part of the proof to theorem (1.8) applied to the function $\psi(x) := \limsup_n \|x - T^n x_0\|$

establishes a similar result for orbits, namely;

$$\limsup_n \|x - T^n x_0\| = \operatorname{diam}(D)$$

for any x and $x_0 \in D$.

(1.13) Corollary: A non-trivial minimal invariant set D for the non-expansive mapping T contains no periodic point of T.

Proof. Suppose $x_0 \in D$ is a periodic point of T; that is, for some $N \in \mathbb{N}$ $T^N x_0 = x_0$. Let $x = \frac{1}{N}\sum_{n=1}^{N} T^n x_0$, then by (1.12) we have

$$\limsup_m \|x - T^m x_0\| = \operatorname{diam}(D).$$

however this is difficult to reconcile with the fact that $T^m x_0 \in \{T^n x_0\}_{n=1}^{N} \subset D$, unless diam$(D) = 0$. ■

Before proceeding to new developments we pause to note an intriguing result of Edelstein and O'Brien [1978]. It shows that without loss of generality we can assume that the sequence in (1·12) is not only an orbit for T but also an *approximate fixed point sequence*.

(1·14) THEOREM: Let C be a closed bounded convex subset of X and let $T : C \to C$ be a non-expansive mapping. Define $U : C \to C$ by

$$U(x) := \frac{1}{2}x + \frac{1}{2}Tx,$$

then $\| U^{n+1}x - U^n x \| \to 0$ uniformly for $x \in C$.

Remarks i) It follows that $(U^n x)$ is an approximate fixed point sequence for any $x \in C$.

ii) Since T and U have precisely the same (possibly empty) set of fixed points, we may without loss of generality replace T by U when considering the F.P.P.

iii) The conclusion remains valid if U is replaced by any of the non-expansive maps $\lambda I + (1-\lambda)T$, $0 < \lambda < 1$.

Proof_ Without loss of generality we take diam$(C)=1$. Suppose the conclusion were not true, then for some $\varepsilon_0 > 0$ and all $N_0 \in \mathbb{N}$ there exists an $N \geq N_0$ and $x \in C$ with

$$\| U^{N+1}x - U^N x \| \geq \varepsilon_0 \quad \cdots\cdots (1)$$

Choose $M \in \mathbb{N}$ with $M > 2/\varepsilon_0$ and let N, x be such that (1) holds with $N > 2^{M+1} M$.

Let $x_n = U^n x$ and $y_n = T x_n$ for $n=0,1,\cdots,N$ then we have, by the non-expansiveness of U and T, that

$$1 \geq \|x_1 - x_0\| \geq \|x_2 - x_1\| \geq \cdots \geq \|x_{N+1} - x_N\| \geq \varepsilon_0 \quad \cdots\cdots (2)$$

$$\| y_{n+1} - y_n \| \leq \| x_{n+1} - x_n \| \quad \cdots\cdots (3)$$

and, by their definitions and the definition of U,

$$x_{n+1} = \tfrac{1}{2} x_n + \tfrac{1}{2} y_n \quad \text{or} \quad y_n = 2x_{n+1} - x_n \cdots\cdots (4)$$

for $n = 0,1,\cdots,N$.

From (3) and (4) we obtain

$$\begin{aligned} \| x_n - x_{n-1} \| &\geq \| y_n - y_{n-1} \| \\ &= \| 2(x_{n+1} - x_n) - (x_n - x_{n-1}) \| \cdots\cdots (5). \end{aligned}$$

Now $[\varepsilon_0, 1]$ can be covered by 2^{M+1} subintervals each of length $1/2^{M+1}$, hence by (2) and the choice of N we can find a subinterval $I = [b, b + 1/2^{M+1}]$ of $[\varepsilon_0, 1]$ which contains at least M successive numbers of the form $\|x_{n+1} - x_n\|$.

That is, for some $K \leq N-M$ we have

$$b \leq \| x_{K+n+1} - x_{K+n} \| \leq b + \frac{1}{2^{M+1}}$$

for $n = 1, 2, \cdots, M$ .....(6)

Now choose $f \in X^*$ with $\|f\| = 1$ such that $f(x_{K+M+1} - x_{K+M}) = \|x_{K+M+1} - x_{K+M}\| \geq b$.

Then by (6) and this choice of f we have

$$2b - f(x_{K+M} - x_{K+M-1})$$

$$\leq f(2(x_{K+M+1} - x_{K+M})) - f(x_{K+m} - x_{K+M-1})$$

$$\leq \| 2(x_{K+M+1} - x_{K+m}) - (x_{K+M} - x_{K+M-1}) \|$$

$$\leq \| x_{K+M} - x_{K+M-1} \| \quad , \text{by } (5)$$

$$\leq b + \frac{1}{2^{m+1}} \quad , \text{by } (6).$$

So $f(x_{K+M} - x_{K+M-1}) \geq b - \frac{1}{2^{M+1}}$ ....(7·1)

Similarly,

$$2\left(b - \frac{1}{2^{m+1}}\right) - f(x_{K+m-1} - x_{K+m-2})$$

$$\leq f(2(x_{K+m} - x_{K+m-1}) - f(x_{K+M-1} - x_{K+M-2})$$

$$\leq \| x_{K+m-1} - x_{K+M-2} \|$$

$$\leq b + \frac{1}{2^{m+1}}$$

So $f(x_{K+m-1} - x_{K+m-2}) \geq b - \frac{1}{2^{M}} - \frac{1}{2^{M+1}}$

$> b - \frac{1}{2^{M-1}}$ .....(7.2)

Continuing we obtain in general

$$f(x_{K+M+1-n} - x_{K+M-n}) \geq b - \sum_{k=M+2-n}^{M+1} \frac{1}{2^{k}} > b - \frac{1}{2^{M+1-n}}$$

for $n = 0, 1, 2, \ldots, M-1$ .....(7·n)

From this epidemic of (7'o) we have:

$$
\begin{aligned}
f(x_{K+M+1}) &\geq f(x_{K+M}) + b \\
&\geq f(x_{K+M-1}) + 2b - \frac{1}{2^{M+1}} \\
&> \cdots \\
&> f(x_{K+M+1-n}) + nb - \sum_{k=M+1-n}^{M+1} \frac{1}{2^{k}} \\
&> \cdots \\
&> f(x_{K+}\ ) + Mb - \sum_{k=2}^{M+1} \frac{1}{2^{k}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\| x_{K+M+1} - x_{K+1} \| &\geq | f(x_{K+M+1} - x_{K+1}) | \\
&> Mb - 1
\end{aligned}
$$

and so

$\operatorname{diam}(C) > Mb - 1$ but $b > \varepsilon_0$ and $M > 2/\varepsilon$

so we have the contradiction that

$\operatorname{diam}(C) > 1$. ■

(1.15) Corollary: If D is $w(w^*)$-compact minimal invariant set for the mapping U of theorem (1.14), then for any $x \in D$ and $x_0 \in D$ we have

$$\lim_{n} \| x - U^{n} x_0 \| = \operatorname{diam}(D).$$

Proof. If this were not so we could find a subsequence such that

$$\lim_{k} \| x - U^{n_k} x_0 \| \lneq \operatorname{diam}(D).$$

Since $(U^{n_k} x_0)$ is an approximate fixed poin sequence for U this contradicts theorem (1.8). ■

Corollary (1·11) offers to endow minimal invariant sets with a richer structure than "mere" diametrality. "Unfortunately", as we will now show, this is in a certain sense not the case. We show that every diametral set contains a diametral sequence which is necessarily a diameterizing sequence for its closed convex hull.

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(1·16) THEOREM (Brodskii and Mil'man, 1948)

Let D be a closed bounded non-empty convex subset of X which is diametral. Then there exists a sequence (x_n) in D with

$$\lim_n \operatorname{dist}(x_{n+1}, \overline{co}\{x_k\}_{k=1}^n) = \operatorname{diam}(D).$$

Proof. We construct the sequence (x_n) inductively as follows.

Choose any point of D as x_1.

Now suppose $x_1, x_2, \dots, x_n$ have been selected. Let $b := \frac{1}{n}\sum_{k=1}^{n} x_k$, the bary-centre of $co\{x_k\}_{k=1}^n$, then since D is diametral we can find a point $x_{n+1} \in D$ satisfying

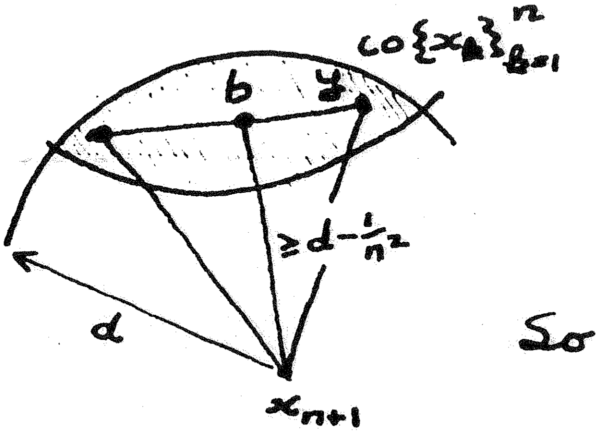
$\|b - x_{n+1}\| \geq d - \frac{1}{n^2}$, where $d := \operatorname{diam}(\overset{D}{c})$.

We show that the resulting sequence (x_n) has the required property.

To see this let $x \in co\{x_k\}_{k=1}^n$; that is

$x = \sum_1^n \alpha_k x_k$ for some $\alpha_k \geq 0$ with $\sum_1^n \alpha_k = 1$.

First observe that if $y \in co\{x_k\}_{k=1}^{n}$ and $\lambda \in (0,1]$ are such that $b = \lambda x + (1-\lambda) y$, then



$$d - \frac{1}{n^2} \leq \| b - x_{n+1} \|$$
$$\leq \lambda \| x - x_{n+1} \| + (1-\lambda) \| y - x$$
$$\leq \lambda \| x - x_{n+1} \| + d - \lambda d .$$

So, $d - \frac{1}{\lambda n^2} \leq \| x - x_{n+1} \|$ ..... (1).

Now, $$y = \frac{1}{1-\lambda} (b - \lambda x)$$
$$= \sum_{k=1}^{n} \frac{(\frac{1}{n} - \lambda \alpha_k)}{1-\lambda} x_k$$
is a convex combination of the x_k, and so in $co\{x_k\}_{k=1}^{n}$, provided $\lambda \alpha_k \leq \frac{1}{n}$ for $k = 1, 2, \cdots, n$ which is certainly true if we take $\lambda = \frac{1}{n}$. Thus (1) holds with $\lambda = \frac{1}{n}$ and so we have
$$d - \frac{1}{n} \leq dist(x_{n+1}, co\{x_k\}_{k=1}^{n})$$
establishing the claim. ∎

(1·17) Corollary: A compact convex set is necessarily non-diametral.

(1·18) REMARKS. 1) Since any subsequence of (x_n) in theorem (1·16) retains the same structure, if D is weakly compact we mass assume that (x_n) is weak convergent

Indeed by proposition (0·6) we may take (x_n) to be a weak null sequence.

2) It is possible to construct sequences with even richer structure than the sequence (x_n) of theorem (1·16).
For example; as a special case of vanDulst [84] we have:
If X contains a weakly compact convex diametral set, then there exists a weak null sequence (y_n) such that for every $k \in \mathbb{N}$ and $l \in \mathbb{N}$ we have

$$1 - \frac{1}{k} \leq \text{dist}\left(y_{k+l+1}, \overline{co}\{x_j\}_{j=k}^{k+l} \right) \leq 1 + \frac{1}{k}.$$