THE EXISTENCE OF
FIXED POINTS FOR NONEXPANSIVE MAPS

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0. STATEMENT OF THE PROBLEM; EXAMPLES.

The best known fixed point results in infinite dimensional spaces are undoubtedly the Banach contraction mapping principle and the Schauder-Tychonoff fixed point theorem.

(0.1) THEOREM (Banach contraction mapping principle): Let (X, d) be a complete metric space and let $T : X \to X$ be a strict contraction; that is, for some $k \in [0, 1)$, $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point x_0 in X. Further, for any point $x_1 \in X$ and $n \in \mathrm{N}$ we have

$$d(T^n x_1, x_0) \leq \frac{k^n}{1-k} d(x_1, Tx_1).$$

(0.2) THEOREM (Schauder-Tychonoff fixed point theorem): Let C be a compact convex subset of a locally convex topological vector space. Then every continuous mapping $T : C \to C$ has a fixed point.

In Theorem (0.1) a stringent form of continuity is imposed on the mapping T, while the assumption on the domain X is minimal for the existence of a fixed point.

On the contrary, in Theorem (0.2) a minimal condition is imposed on the mapping while the nature of the domain C is heavily constrained.

The questions with which we will be concerned are in a sense intermediate to these two results. More specifically, we will be interested in identifying Banach spaces X with one or other of the properties listed below.

A mapping $T : C \subseteq X \to C$ is **nonexpansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The fixed point property, FPP: X has the FPP if every nonexpansive self-mapping of a nonempty closed convex subset of X has a fixed point.

The weak fixed point property, w-FPP: X has the w-FPP if every nonexpansive self-mapping of a nonempty weak compact convex subset of X has a fixed point.

The weak* fixed point property, w^*-FPP: A dual space X^* has the w^*-FPP if every nonexpansive self-mapping of a nonempty weak*-compact convex subset of X^* has a fixed point.

One reason why deciding which spaces have the w (w^*)-FPP is both intriguing and difficult is that the continuity condition on the mapping is in a different (stronger) topology than the compactness of the domain.

For a reflexive space all three properties coincide. In general FPP $\Rightarrow$ w-FPP and for a dual space FPP $\Rightarrow$ w^*-FPP $\Rightarrow$ w-FPP. A natural advantage of the w^*-FPP is the ready supply of w^*-compact subsets guaranteed by the Banach-Alaoglu theorem.

SOME EXAMPLES

(0.3) c_0 fails the FPP. Let $C = B^+_{c_0} := \{(x_n) \in c_0 : 0 \le x_n \le 1, \text{ all } n\}$ and define T by

$$T(x_n) := (1 - x_1,\ x_1,\ x_2,\ \ldots).$$

Then for any $\underset{\sim}{x}, \underset{\sim}{y} \in c_0$ we have $\|T\underset{\sim}{x} - T\underset{\sim}{y}\| = \|\underset{\sim}{x} - \underset{\sim}{y}\|$, so T is nonexpansive and T maps C into C. On the other hand the only possible fixed point for T is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots) \notin c_0$.

As a general problem we have:

QUESTION. If X has the FPP is X necessarily reflexive? We remark that as a consequence of van Dulst and Pach [1981]; if every space finitely representable in X has the F.P.P. then X is superflexive. That is, "super-F.P.P." implies superreflexivity. Their result is particularly interesting in that the set C and fixed point free isometry on it are constructed from a "tree".

(0.4) [Lim, 1980] l_1 with the equivalent dual norm $\|\underset{\sim}{f}\|' := \|\underset{\sim}{f}^+\|_1 \vee \|\underset{\sim}{f}^-\|_1$ fails the w^*-FPP. We first show that $\|\cdot\|'$ is indeed an equivalent dual norm for l_1. To this end, for $\underset{\sim}{x} \in c_0$ define

$$\|\underset{\sim}{x}\| := \|\underset{\sim}{x}^+\|_\infty + \|\underset{\sim}{x}^-\|_\infty$$

Then $\|\cdot\|$ is an equivalent norm on c_0 satisfying $\|\underset{\sim}{x}\|_\infty \le \|\underset{\sim}{x}\| \le 2\|\underset{\sim}{x}\|_\infty$ and so it suffices to show that for $\underset{\sim}{f} \in l_1$ we have

$$\|\underset{\sim}{f}\|' = \sup\{\underset{\sim}{f}(\underset{\sim}{x}) : \underset{\sim}{x} \in c_0,\ \|\underset{\sim}{x}\| \le 1\}.$$

For $\underset{\sim}{x} \in c_0$ with $\|\underset{\sim}{x}\| \le 1$ let

$$y_i = \begin{cases} x_i & \text{if } f_i x_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

then $\| \underset{\sim}{y} \| \leq \| \underset{\sim}{x} \| \leq 1$ and

$$
\begin{aligned}
\underset{\sim}{f}(\underset{\sim}{x}) &:= \sum_{i=1}^{\infty} f_i x_i \\
&\leq \sum_{i=1}^{\infty} f_i y_i \\
&\leq \|\underset{\sim}{y}^+\|_\infty \|\underset{\sim}{f}^+\|_1 + \|\underset{\sim}{y}^-\|_\infty \|\underset{\sim}{f}^-\|_1 \\
&= \underbrace{\left(\frac{\|\underset{\sim}{y}^+\|_\infty}{\| \underset{\sim}{y} \|} \|\underset{\sim}{f}^+\|_1 + \frac{\|\underset{\sim}{y}^-\|_\infty}{\| \underset{\sim}{y} \|} \|\underset{\sim}{f}^-\|_1 \right)}_{\substack{\text{a convex combination of} \\ \|\underset{\sim}{f}^+\|_1 \text{ and } \|\underset{\sim}{f}^-\|_1}} \| \underset{\sim}{y} \| \\
&\leq (\|\underset{\sim}{f}^+\|_1 \vee \|\underset{\sim}{f}^-\|_1) \| \underset{\sim}{y} \| \\
&\leq \| \underset{\sim}{f} \|'.
\end{aligned}
$$

To see the reverse inequality note that $\|\underset{\sim}{f}^+\|_1$ (or $\| \underset{\sim}{f}^- \|_1$) can be approximated arbitrarily well by $\underset{\sim}{f}(\underset{\sim}{x})$ where the x_i are a suitable choice of 0 or 1 (0 or -1) and so $\| \underset{\sim}{x} \| \leq 1$.

Now let $C = \{\underset{\sim}{f} \in l_1 : f_i \geq 0, \| \underset{\sim}{f} \|' \leq 1\}$ and define T by

$$
T\underset{\sim}{f} := (1 - \sum_{i=1}^{\infty} f_i,\ f_1,\ f_2,\ \ldots).
$$

Then C is a weak*-compact convex subset of l_1 and it is readily verified that T is a fixed point free affine mapping of C into C. We conclude by showing that T is an isometry (hence certainly a non- expansive mapping) on C.

Given $\underset{\sim}{f}, \underset{\sim}{g} \in C$ let $P := \{i : f_i - g_i \geq 0\}$ and $N := \{i : f_i - g_i < 0\}$. In the case that $\sum_{i \in P} (f_i - g_i) \geq \sum_{i \in N} (g_i - f_i)$ we have

$$
\| \underset{\sim}{f} - \underset{\sim}{g} \|' = \sum_{i \in P} (f_i - g_i)
$$

and

$$\begin{aligned}
\|T\underset{\sim}{f} - T\underset{\sim}{g}\|' &= \Big\|\Big(\sum_{i=1}^{\infty}(g_i - f_i),\ f_1 - g_1,\ f_2 - g_2,\ \ldots\Big)\Big\|' \\
&= \Big\|\Big(\underbrace{\sum_{i\in N}(g_i - f_i) - \sum_{i\in P}(f_i - g_i)}_{\text{negative}},\ f_1 - g_1,\ f_2 - g_2,\ \ldots\Big)\Big\|' \\
&= \operatorname{Max}\Big\{\sum_{i\in P}(f_i - g_i),\ \sum_{i\in N}(g_i - f_i)\Big\} \\
&= \|\underset{\sim}{f} - \underset{\sim}{g}\|.
\end{aligned}$$

The equality follows similarly in the case when $\|\underset{\sim}{f} - \underset{\sim}{g}\|' = \sum_{i\in N}(g_i - f_i)$.

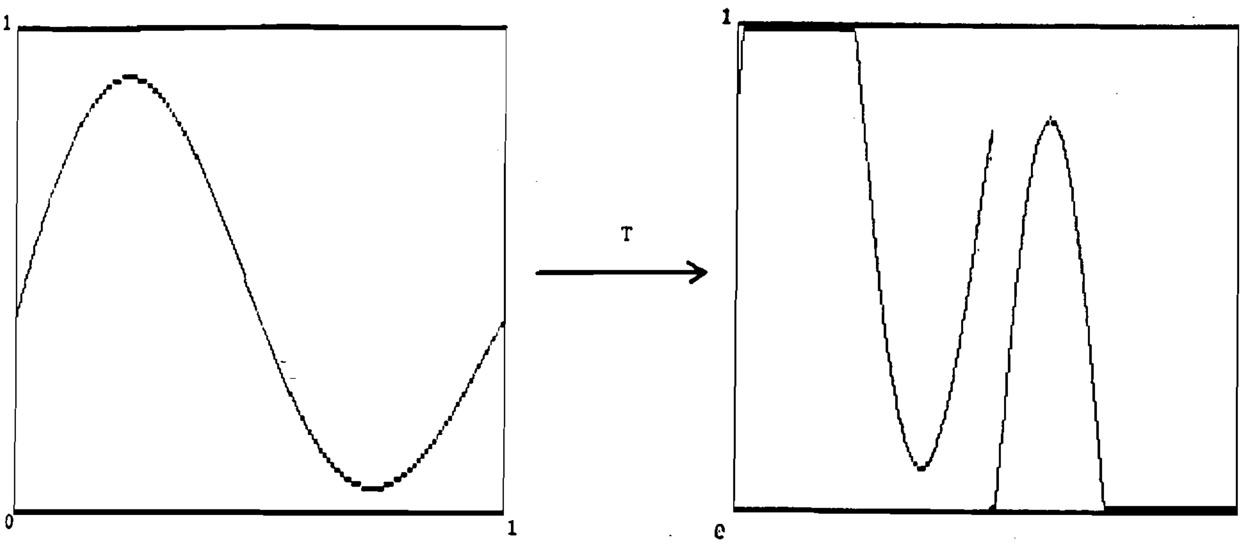
(0.5) $L_1(\mu)$ fails the w-FPP. Although the question had been raised more than twenty years earlier it was not until 1981 that Dale Alspach gave an example, drawn from ergodic theory, showing that not all Banach spaces enjoy the w-FPP.

(0.5.1) Alspach's example [Alspach, 1981] Here we take C to be the set

$$C := \{f \in L_1[0,1] : 0 \le f \le 1,\ \int_0^1 f = \tfrac{1}{2}\}$$

As the intersection of an order interval with a hyperplane in an order continuous Banach lattice, C *is weak compact.*

The mapping T is essentially the *baker transform* of ergodic theory illustrated below.



Formally, for $f \in C$

$$Tf(t) := \begin{cases} (2f(2t)) \wedge 1 & \text{for } 0 \le t \le \frac{1}{2}; \\ (2f(2t-1)-1) \vee 0 & \text{for } \frac{1}{2} < t \le 1. \end{cases}$$

It is clear from the above description that T is an isometry on C.

We now show that T is fixed point free.

Intuitively the idea is simple. First observe that the successive iterates of any point in C under T assume values closer to 0 or 1. Hence any fixed point for T must be a function which assumes only the values 0 or 1. By the "ergodic" nature of T it then follows that such a function must be either constantly 0 or constantly 1, and neither of these functions lie in C.

The details follow.

For any $f \in C$ we have $Tf(t) = 1$ if and only if either

$$0 \le t \le \tfrac{1}{2} \quad \text{and} \quad \tfrac{1}{2} \le f(2t) \le 1$$

or

$$\tfrac{1}{2} < t \le 1 \quad \text{and} \quad f(2t-1) = 1.$$

Now, suppose f is a fixed point for T then

$$\begin{aligned}
A &:= \{t : f(t) = 1\} \\
&= \{t : Tf(t) = 1\} \\
&= \{t : 0 \leq t \leq \tfrac{1}{2} \text{ and } \tfrac{1}{2} \leq f(2t) \leq 1\} \cup \{t : \tfrac{1}{2} < t \leq 1 \text{ and } f(2t-1) = 1\} \\
&= \{\tfrac{1}{2}t : \tfrac{1}{2} \leq f(t) \leq 1\} \cup \{\tfrac{1}{2} + \tfrac{1}{2}t : f(t) = 1\} \\
&= \tfrac{1}{2}\{t : \tfrac{1}{2} \leq f(t) < 1\} \cup \tfrac{1}{2}A \cup (\tfrac{1}{2} + \tfrac{1}{2}A)
\end{aligned}$$

Since the three sets in the above union are mutually disjoint and each of the last two sets has measure one half that of A it follows that:

$$B_1 := \{t : \tfrac{1}{2} \leq f(t) < 1\}$$

is a null set. But, then

$$\begin{aligned}
B_1 &= \{t : \tfrac{1}{2} \leq Tf(t) < 1\} \\
&\supset \{\tfrac{t}{2} : \tfrac{1}{4} \leq f(t) < \tfrac{1}{2}\}
\end{aligned}$$

and so $B_2 := \{t : \frac{1}{4} \leq f(t) < \frac{1}{2}\}$ is also a null set. Continuing in this way we have

$$B_n := \{t : \frac{1}{2^n} \leq f(t) < \frac{1}{2^{n-1}}\}$$

is a null set for $n = 1, 2, \ldots$, hence

$$\{t : 0 < f(t) < 1\} = \bigcup_{n=1}^{\infty} B_n$$

is null and

$$f \equiv \chi_A \quad (\text{where} \quad \text{meas}\,(\mathrm{A}) = \tfrac{1}{2}).$$

From the definition of T we have

$$T(\chi_A) = (\chi_{\frac{1}{2}A} + \chi_{(\frac{1}{2}+\frac{1}{2}A)})$$

so, up to sets of measure zero,

$$A = \tfrac{1}{2}A \cup (\tfrac{1}{2} + \tfrac{1}{2}A).$$

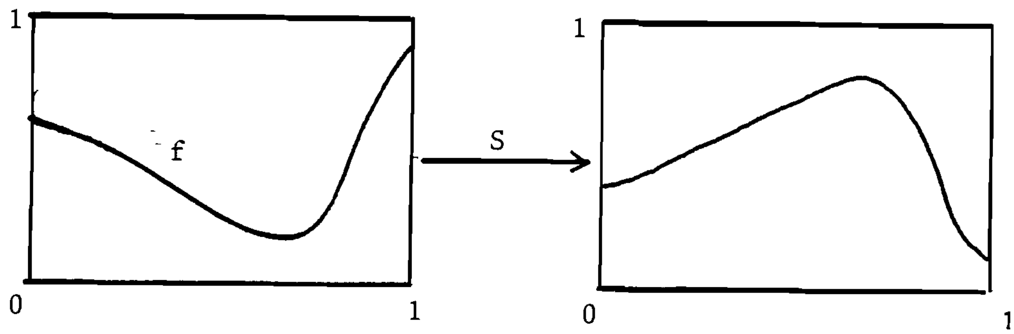
Continuing to iterate under T yields

$$\begin{aligned}
A &= \tfrac{1}{4}A \cup (\tfrac{1}{4} + \tfrac{1}{4}A) \cup (\tfrac{1}{2} + \tfrac{1}{4}A) \cup (\tfrac{3}{4} + \tfrac{1}{4}A) \\
A &= \tfrac{1}{8}A \cup (\tfrac{1}{8} + \tfrac{1}{8}A) \cup (\tfrac{1}{4} + \tfrac{1}{8}A) \cup \ldots
\end{aligned}$$

et hoc genus omne.

Thus the intersection of A with any dyadic interval (and hence any interval) has measure one half that of the interval, an impossibility for a set which is not of full measure.

(0.5.2) Sine's modification of the Alspach example Robert Sine [1981] gave the following modification to the example of (0.5.1) which allows us to take as the domain C of our fixed point free nonexpansive mapping the whole order interval of $0 \leq f \leq 1$.



For $f \in C := \{g : 0 \leq g \leq 1\}$ let $Sf := \chi_{[0,1]} - f$,
then S defines a mapping of C onto C with $\|Sf - Sg\| = \|f - g\|$ for all $f, g \in C$.

An argument similar to that for Alspach's example shows that the composition ST, where T is the baker transform of (0.5.1), is an isometry on the order interval $0 \leq f \leq 1$ with χ_A where $A = [0,1]$ or ϕ the only possible fixed points. However, the action of ST is to map each of these functions onto the other, hence *ST is fixed point free on the order interval* $0 \leq f \leq 1$.

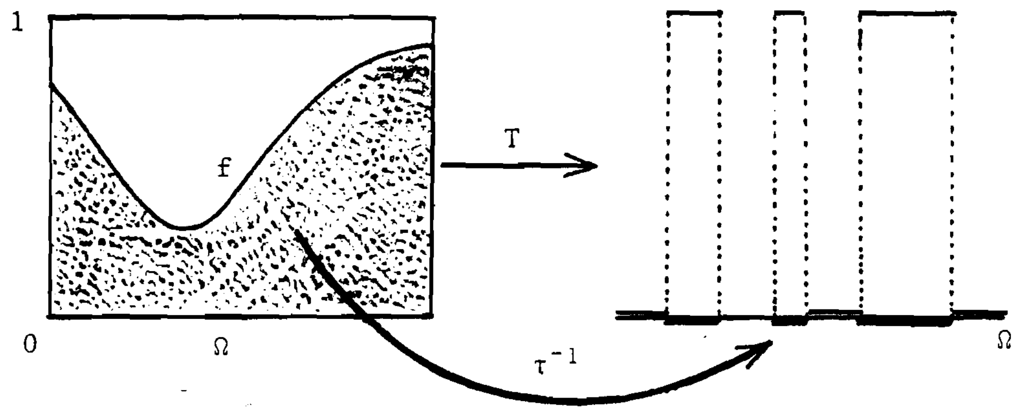
(0.5.3) Schechtman's construction. Gideon Schechtman [1982] gave a construction which leads to a greater variety of examples and is somewhat simpler than that of Alspach.

Suppose (Ω, Σ, μ) is a measure space for which there exists a measure preserving transformation $\tau : \Omega \to \Omega \times [0,1]$; that is, for any measurable $S \subseteq \Omega \times [0,1]$ we have $\mu(\tau^{-1}S) =$ meas(S) [Halmos, 1956]. Then if C is the weak compact convex set

$$C := \{f \in L_1(\mu) : 0 \leq f \leq 1 \quad \text{and} \quad \int_\Omega f = \tfrac{1}{2}\}$$

we can define a mapping $T : C \to C$ by

$$Tf := \chi_{\tau^{-1}\{(\omega,t)\, :\, 0 \leq t \leq f(\omega)\}}$$



Clearly T is an isometry on C and $f \in C$ is a fixed point for T if and only if $f = \chi_A$ where $A \in \Sigma$ is such that $\mu(A) = \frac{1}{2}$ and $\hat{\tau}(A) := \tau^{-1}(A \times [0,1]) = A$ *a.e.*

Thus if τ is further chosen so that $\hat{\tau}$ is "ergodic"; that is $\hat{\tau}(A) = A$ *a.e.* if and only if $A = \Omega$ or $A = \phi$, then T *is an example of a fixed point free nonexpansive mapping on* C.

Perhaps the simplest *example of an* (Ω, Σ, μ) *and* τ suitable for the above construction is the following.

Let $\Omega = [0,1]^{\aleph_0}$ with product Lebesgue measure and define τ by

$$\tau^{-1}\big((\omega_1, \omega_2, \ldots), t\big) := (t, \omega_1, \omega_2, \ldots).$$

Clearly τ is measure preserving, further if $A \neq \phi$ and $\hat{\tau}(A) = A$, then for any $(\omega_1, \omega_2, \ldots) \in A$ we see that $(t, \omega_1, \omega_2, \ldots) \in A$ for any $t \in [0,1]$. Iterating under $\hat{\tau}$ gives $(t_1, t_2, \ldots, t_n, \omega_1, \omega_2, \ldots) \in A$ for any $n \in \mathrm{N}$ and $t_1, t_2, \ldots, t_n \in [0,1]$, and so we have $A = \Omega$.

An alternative example with $\Omega = [0,1]$ is obtained by taking

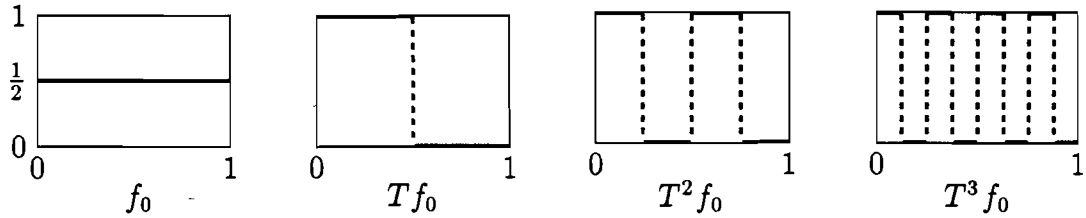
$$\tau^{-1}\Big(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}, \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}\Big) := \frac{\delta_1}{2} + \frac{\varepsilon_1}{2^2} + \frac{\delta_2}{2^3} + \frac{\varepsilon_2}{2^4} + \cdots,$$

where $\varepsilon_n, \delta_n \in \{0,1\}$ for $n = 1, 2, \ldots$ A good way to view this example is via the correspondence

$$[0,1] \longleftrightarrow \{0,1\}^{\aleph_0} : \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \longleftrightarrow (\varepsilon_1, \varepsilon_2, \ldots).$$

The measure of a set specified by prescribing precisely m of the ε_n's being $1/2^m$. It is then clear that the product of two such sets has measure $1/2^{m_1+m_2}$ where $m_1 + m_2$ is also the number of digits prescribed for points in the τ^{-1} image of the product. It follows that τ is measure preserving. The ergodicity is estabished by iterating under $\hat{\tau}$ and an argument similar to that used for the conclusion of Alspach's example.

(0.5.4) Remarks Schechtman's construction is both simpler and more versatile than that of Alspach and is of course also amenable to Sine's modification. None-the-less, the Alspach example has some advantages. The relatively simple action of the baker transform permits detailed calculations. For example, it is possible to determine the orbit f_0, Tf_0, $T^2 f_0$, $T^3 f_0$, ... of certain functions f_0 under T. If $f_0 = \frac{1}{2}\chi_{[0,1]}$ we obtain the iterates depicted below.



Here we see that the sequence $T^n f_0 = \frac{1}{2}(r_n + 1)$ is an orbit under T, where r_n is the n'th *Rademacher function.*

PROBLEMS:

These examples indicate an intimate connection between fixed point free isometries and ergodic transformations of the underlying measure space. In the true tradition of ergodic theory, *is the set of fixed point free isometries on the order interval* $[0 \leq f \leq 1]$ *residual in an appropriate sense, at least among isometries which map into the set of* $0, 1$*-valued functions?*

Clearly any space containing an isometric copy of $L_1(\mu)$ also fails the w-FPP. *What do the examples look like when translated into* l_∞, $C[0,1]$*?*

Examples 0.4 and 0.5 also suggest the following question. *If a space X fails the (w, w^*)-FPP does it necessarily fail with an isometry?*

All the examples presented in this section have been negative in nature, besides helping delineate the problem; this seems only fair since in the remainder of these notes we will be concerned with positive results. As a start in this direction we close the section with two simple observations.

(0.6) PROPOSITION: If X fails any of the fixed point properties, then given any point $x_0 \in X$ there exists a (w^*, w-compact) closed bounded convex set C_0 with $x_0 \in C_0$ and $\operatorname{diam}(C_0) := \sup_{x,y \in C_0} \|x - y\| = 1$, and there exists a fixed point free nonexpansive mapping $T_0 : C_0 \to C_0$.

Proof: If X fails the (w,w^*)-FPP then there exists a (w,w^*-compact) closed bounded convex subset C and a fixed point free nonexpansive mapping $T : C \to C$. C must contain more than one point (otherwise the solitary element would be fixed by T), hence $d =$ diam (C) > 0. Choose $x_1 \in C$ and let $C_0 := \frac{1}{d}(C - x_1) + x_0$ and define

$$T_0(x) := \frac{1}{d}\Big(T\big(d(x - x_0) + x_1\big) - x_1\Big) + x_0 \,.$$

A simple calculation now verifies the claim. ■

Proposition (0.6) will be used, often without comment, throughout the sequel to simplify calculations.

The next result should also be of use, though as far as I know it has rarely played a significant role in the theory.

Clearly a space has the w-FPP (or FPP) if and only if all of its closed subspaces do.

(0.7) PROPOSITION: The w-FPP and the FPP are separably determined.

Proof: If X fails the (w-)FPP then there exists a (w-compact) closed bounded convex subset C and a fixed point free nonexpansive mapping $T : C \to C$. Choose any point $c \in C$. Let $K_1 = \{c\}$ and inductively define K_n by

$$K_{n+1} = \overline{\text{co}}\ \{T(K_n) \cup K_n\}.$$

Let $\tilde{K} = \overline{\bigcup\limits_{n=1}^{\infty} K_n}$, then $\tilde{K}$ is a separable closed convex (and hence weak compact if C is) subset of C.

<u>Claim:</u> $\tilde{K}$ is invariant under T. Let $x \in \tilde{K}$. Given $\varepsilon > 0$ there exists $y \in K_n$ for some n with $\|x - y\| < \varepsilon$, but then $Ty \in K_{n+1} \subseteq \tilde{K}$ and $\|Tx - Ty\| \leq \varepsilon$, so $Tx \in \tilde{K}$ establishing the claim.

The result now follows by considering the set $\tilde{K}$ and T restricted to $\tilde{K}$, in the separable closed subspace spanned by $\tilde{K}$. ■

An analogous result for the w^*-FPP in dual spaces <u>would</u> be useful, but appears not to be known.

1. MINIMAL INVARIANT SETS

Given a *weak*(*weak**)-compact convex set C and a nonexpansive map $T : C \to C$ in a Banach space X, let $K = K(C,T)$ denote the class of nonempty *weak* (*weak**)-compact convex subsets of C which are invariant under T (S is *invariant under* T if $T(S) \subseteq S$). If K is partially ordered by inclusion the *weak* (*weak**)-compactness ensures that the intersection of any decreasing chain of sets is a lower bound for the chain in K. Thus we may apply Zorn's lemma to establish the existence of a minimal element of K. We shall refer to such a minimal element as a **minimal invariant subset** for T (Minimality within the class K being understood).

OBSERVATIONS

(1.1) If C is weak-compact and D is a minimal invariant subset for T then $D = \overline{\text{co}}T(D)$. (In the *weak** case we must take the w^*-closed convex hull.)

(1.2) If D is a weak-compact minimal invariant set, then D is separable. If this were not the case then proposition (0.7) would give the existence of a smaller invariant subset of D.

(1.3) X has the w (w^*) -FPP if and only if all minimal invariant subsets for nonexpansive mappings are "singleton" sets; that is, have only one element (or equivalently, have zero diameter).

The w (w^*)FPP has been established by indentifying specific properties of minimal invariant sets and then imposing conditions on the space which preclude the existence of sets, other than singelton sets, with these properties.

A useful tool has been the existence of *approximate fixed point sequences:* Given a nonexpansive map $T : C \to C$ on a closed bounded convex set C choose $x_0 \in C$ and for $\lambda \in [0,1)$ define $T_\lambda : C \to C$ by

$$T_\lambda(x) := \lambda Tx + (1-\lambda)x_0.$$

T_λ is a strict contraction and so by the Banach contraction mapping principle has a unique fixed point x_λ in C with

$$\begin{aligned}\|x_\lambda - Tx_\lambda\| &= (1-\lambda)\|x_0 - Tx_\lambda\| \\ &\leq (1-\lambda)\ \text{diam}\,(\text{C}).\end{aligned}$$

Thus $\|x_\lambda - Tx_\lambda\| \to 0$ as $\lambda \to 1$. In particular, letting $\lambda = (1-\frac{1}{n})$ we obtain a **sequence of approximate fixed points** of T; (x_n) with $\|x_n - Tx_n\| \to 0$.

(1.4) PROPOSITION: the set $\{x_\lambda : \lambda \in [0,1)\}$ is a connected arc in C.

Proof. We show the mapping $[0,1) \to C : \lambda \mapsto x_\lambda$ is continuous. Given $\lambda, \beta \in [0,1)$

$$\begin{aligned}\|x_\lambda - x_\beta\| &= \|\lambda T x_\lambda - \beta T x_\beta + (\beta - \lambda) x_0\| \\ &\leq \|\lambda T x_\lambda - \lambda T x_\beta\| + |\lambda - \beta| \|T x_\beta - x_0\| \\ &\leq \lambda \|x_\lambda - x_\beta\| + 2B|\lambda - \beta|,\end{aligned}$$

where B is a bound on the norms of elements in C. Thus,

$$\|x_\lambda - x_\beta\| \leq \frac{2B}{1-\lambda} |\lambda - \beta|.$$

∎

In particular putting $\lambda = 1 - \frac{1}{n}$ in the above proof we have

$$\|x_n - x_{n+1}\| \leq 2\frac{B}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Structure of minimal invariant sets

Let T be a nonexpansive mapping of a w (w^*)-compact convex subset and let D denote a w (w^*)-compact minimal invariant set for T.

(1.5) LEMMA: If $\psi : D \to \mathbf{R}$ is a *weak* (*weak**) lower semi-continuous convex mapping with $\psi(Tx) \leq \psi(x)$ for all $x \in D$, then ψ is constant on D.

Proof. Since D is $w(w^*)$-compact and ψ is $w(w^*)$-lower semi- contininuous, ψ achieves its minimum on D. Let $x_0 \in D$ be such that $\psi(x_0) = \inf \psi(D)$ and let $E = \{x \in D : \psi(x) = \psi(x_0)\}$, then E is a $w(w^*)$-closed convex set which is invariant under T. Thus, by minimality $E = D$. ∎

We will be particularly interested in three instances of such a ψ.

(a) $\psi(x) := \sup\{\|x - y\| : y \in D\}$

(b) $\psi(x) := \limsup_n \|x - x_n\|$,

where D is weak compact and

(i) (x_n) is a sequence of approximate fixed points for T in D,

or

(ii) (x_n) is an orbit of T in D; that is, $x_n = T^n x_0$ for some point $x_0 \in D$.

The function defined in (a) is the **radius of the set D about x**;

$$\operatorname{rad}(D,x) := \sup\{\|x-y\| : y \in D\}.$$

A point of D about which the radius is the diameter of D is termed a **diametral point** of D.

The **Chebyschev radius** of D is

$$\operatorname{rad}(D) := \inf_{x\in D} \operatorname{rad}(D,x)$$

and the (possibly empty) **Chebyschev centre** of D is

$$C(D) = \{x \in D : \operatorname{rad}(D,x) = \operatorname{rad}(D)\}.$$

$C(D)$ is convex if D is and if D is *weak* (or *weak**) compact then $C(D)$ is nonempty.

We will say a set D is **diametral** if $\operatorname{rad}(D) = \operatorname{diam}(D)$; that is, if every point of D is a diametral point. Clearly this happens if and ony if $D = C(D)$.

That diametral sets consisting of more than one point can exist may at first seem strange. However it is readily seen that the set

$$B_{c_0}^+ = \{(x_n) \in c_0 : 0 \le x_n \le 1, \quad \text{all} \quad n\}$$

is a diametral subset of c_0.

(1.6) THEOREM (Brodskii– Milman, 1948/Garkarvi, 1962/Kirk, 1965): If D is a *weak* (*weak**)-compact minimal invariant set for a nonexpansive mapping then D is diametral.

Proof. It suffices to verify that

$$\psi(x) := \sup\{\|x-y\| : y \in D\}$$

satisfies the hypotheses for lemma (1.5), as then ψ is a constant on D with value equal to

$$\sup_{x\in D} \psi(x) = \sup_{x\in D}\sup_{y\in D} \|x-y\| = \operatorname{diam}(D).$$

To complete the proof we first observe that

$$\psi(x) = \sup_{y \in co(T(D))} \|x-y\|. \quad \ldots(1)$$

In the weak case this follows immediately from observation (1.1); $\overline{\text{co}}(T(D)) = D$. In the weak*-case we have $D = \overline{\text{co}}^{w*}T(D)$, so given $\epsilon > 0$ there exists a $y_\epsilon \in D$ with $\psi(x) - \epsilon \le \|x - y_\epsilon\|$ and a net $y_\alpha \to^{w*} y_\epsilon$ with $y_\alpha \in \text{co}(T(D))$. Thus, since

$$\psi(x) - \epsilon \le \|x - y_\epsilon\| \le \liminf_\alpha \|x - y_\alpha\|$$

there exists a $y \in \operatorname{co}(T(D))$ with $\psi(x) - 2\epsilon \leq \|x - y\|$ and so in this case we also have (1).

From (1) it follows by standard convexity arguments that

$$\psi(x) = \sup_{y \in T(D)} \|x - y\|$$

from which it is readily seen that $\psi(Tx) \leq \psi(x)$, completing the proof. ∎

SOME EXAMPLES

(1.7) The domain $C := \{f \in L_1[0,1] : 0 \leq f \leq 1, \int f = \frac{1}{2}\}$ in the Alspach example (0.5.1) is not a minimal invariant set for the baker's transform. This follows since

$$\operatorname{diam}(C) = 1 \quad (1 \geq \operatorname{diam}(C) \geq \|\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}\|_1 = 1),$$

while for any $f \in C$ we have $-\frac{1}{2} \leq f - \frac{1}{2} \leq \frac{1}{2}$ hence

$$\|f - \frac{1}{2}\chi_{[0,1]}\|_1 = \int_0^1 |f - \frac{1}{2}| \leq \frac{1}{2}.$$

and so C is not diametral.

Indeed the author knows of no example of a non-trivial minimal invariant set for a nonexpansive map on a weak compact convex set.

(1.8) The domain C in Lim's example (0.4) is a w^*-compact minimal invariant subset. To see this note that for any $f = (f_m) \in C$ we have as successive iterates

$$Tf = (1 - \sum_1^\infty f_m, f_1, f_2, \ldots)$$

$$T^2 f = (0, 1 - \sum_1^\infty f_m, f_1, f_2, \ldots)$$

$$T^3 f = (0, 0, 1 - \sum_1^\infty f_m, f_1, f_2, \ldots)$$

etc. So $T^n f \rightharpoonup^{w^*} 0$.

Thus 0 belongs to any T invariant w^*-compact convex subset K of C. Hence the n'th basis vector, $e_n = T^n(0)$, is in K. It follows that $C = \overline{\operatorname{co}}\{e_n\} \subseteq K \subseteq C$, so $K = C$.

(1.9) THEOREM (Goebel, 1975/Karlovitz, 1976): If (x_n) is a sequence of approximate fixed points for the nonexpansive mapping T in the weak compact minimal invariant set D, then

$$\lim_n \|x - x_n\| = \operatorname{diam}(\mathrm{D}), \quad \text{for all} \quad \mathrm{x} \in \mathrm{D}.$$

We will call any sequence of points with this property a **diameterizing sequence** for D.

Proof. Let (y_n) be any sequence of approximate fixed points for T in D and define

$$\psi(x) := \limsup_n \|x - y_n\|.$$

By lemma (1.5) ψ is constant on D with value k say. Let (y_{nj}) be a subsequence (net) with $y_{n_j} \rightharpoonup^{w} y_0$ then

$$k \geq \limsup_j \|x - y_{nj}\| \geq \liminf_j \|x - y_{nj}\| \geq \|x - y_o\|.$$

Thus

$$k \geq \sup_{x \in D} \|x - y_0\| = \operatorname{diam}(\mathrm{D})$$

by Theorem (1.6).

Now taking as (y_n) any subsequence (x_{n_k}) of (x_n) we have

$$\limsup_k \|x - x_{n_k}\| = \operatorname{diam}(\mathrm{D})$$

for all x in D and so

$$\lim_n \|x - x_n\| = \operatorname{diam}(\mathrm{D}).$$

∎

(1.10) COROLLARY: Given any $\epsilon > 0$ and $x \in D$ there exists $\Lambda \in (0,1)$ such the the segment $\{x_\lambda : \lambda \in (\Lambda, 1)\}$ of the arc described in proposition (1.4) lies outside the ball of radius $(1-\epsilon)\operatorname{diam}(\mathrm{D})$ centred at x.

(1.11) COROLLARY: If K is any compact subset of D then

$$\lim_n \operatorname{dist}(x_n, K) = \operatorname{diam}(\mathrm{D}).$$

Proof. Assume not then there exists a subsequence (x_{n_k}) of (x_n) and points y_k of K with

$$\|x_{n_k} - y_k\| < (1 - \epsilon_0) \operatorname{diam}(\mathrm{D})$$
$$\text{for some} \quad \epsilon_0 > 0.$$

Passing to a further subsequences if necessary we may, by the compactness of K, assume that $y_k \to y_0$. But then,

$$\limsup_k \|x_{n_k} - y_0\| \leq (1 - \epsilon_0) \operatorname{diam}(\mathrm{D}),$$

contradicting theorem (1.9). ∎

(1.12) COROLLARY: The minimal invariant set D contains a sequence of approximate fixed points for T satisfying

$$\lim_n \operatorname{dist}(x_{n+1}, \ \operatorname{co}\{\mathrm{x_j}\}_{\mathrm{j}=1}^{\mathrm{n}}) = \ \operatorname{diam}(\ \operatorname{co}\{\mathrm{x_j}\}_{\mathrm{j}=1}^{\infty}).$$

We will call any such sequence a **diametral sequence.** Clearly the closed convex hull of a diametral sequence is a diametral set with the sequence as a diameterizing sequence.

Proof. Starting with any sequence of approximate fixed points proceed inductively to extract the subsequence (x_n) using $K = \ \operatorname{co}\{\mathrm{x_j}\}_{\mathrm{j}=1}^{\mathrm{n}}$ in Corollary (1.11).

(1.13) REMARK: The argument in the first part of the proof to Theorem (1.9) applied to the function

$$\psi(x) := \limsup_n \|x - T^n x_0\|$$

establishes a similar result for orbits, namely;

$$\limsup_n \|x - T^n x_0\| = \ \operatorname{diam}(\mathrm{D}).$$

for any x and $x_0 \in D$.

(1.14) COROLLARY: A non-trivial minimal invariant set D of the nonexpansive mapping T contains no periodic point of T.

Proof. Suppose $x_0 \in D$ is a periodic point of T; that is, for some $N \in \mathbf{N}$ $\mathbf{T^N x_0 = x_0}$. Let $x = \frac{1}{N}\sum_{n=1}^{N} T^n x_0$, then by (1.13) we have

$$\limsup_m \|x - T^m x_0\| = \ \operatorname{diam}(\mathrm{D}),$$

which is difficult to reconcile with the fact that $T^m x_0 \in \{T^n x_0\}_{n=1}^{N} \subset D$, unless $\operatorname{diam}(\mathrm{D}) = 0$. ∎

Before proceeding to new developments we pause to note an intriguing result of Edelstein and O'Brien [1978]. It shows that without loss of generality we can assume that the sequence in (1.13) is not only an orbit for T but also an approximate fixed point sequence.

(1.15) THEOREM: Let C be a closed bounded convex subset of X and let $T : C \to C$ be a nonexpansive mapping. Define $U : C \to C$ by

$$U(x) := \frac{1}{2}x + \frac{1}{2}Tx,$$

then $\|U^{n+1}x - U^n x\| \to 0$ uniformly for $x \in C$.

REMARKS.

(i) It follows that $(U^n x)$ is an approximate fixed point sequence for any $x \in C$.

(ii) Since T and U have precisely the same (possibly empty) set of fixed points, we may without loss of generality replace T by U when considering the FPP.

(iii) the conclusion remains valid if U is replaced by any of the nonexpansive maps

$$\lambda I + (1 - \lambda)T, \quad 0 < \lambda < 1.$$

Proof. Without loss of generality we take diam(C) = 1. Suppose the conclusion were not true, then for some $\epsilon_0 > 0$ and all $N_0 \in \mathbf{N}$ there exists an $N \geq N_0$ and $x \in C$ with

$$\|U^{N+1}x - U^N x\| \geq \epsilon_0 \ldots \ (1)$$

Choose $M \in \mathbf{N}$ with $M > 2/\epsilon_0$ and let N, x be such that (1) holds with $N > 2^{M+1}M$. Let $x_n = U^n x$ and $y_n = Tx_n$ for $n = 0, 1, \ldots, N$ then we have, by the nonexpansiveness of U and T, that

$$\begin{aligned} 1 \geq \|x_1 - x_0\| \geq \|x_2 - x_1\| \geq \ldots \geq \|x_{N+1} - x_N\| \geq \epsilon_0 \ldots \ (2) \\ \|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| \qquad \ldots \ (3) \end{aligned}$$

and, by their definitions and the definition of U, for $n = 0, 1, \ldots, N$.

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}y_n \quad \text{or} \quad y_n = 2x_{n+1} - x_n \ldots \ (4)$$

From (3) and (4) we obtain

$$\begin{aligned} \|x_n - x_{n-1}\| &\geq \|y_n - y_{n-1}\| \\ &= \|2(x_{n+1} - x_n) - (x_n - x_{n-1})\| \ldots \ (5). \end{aligned}$$

Now $[\epsilon_0, 1]$ can be covered by 2^{M-1} subintervals each of length $\frac{1}{2^{M-1}}$, hence by (2) and the choice of N we can find a subinterval $I = [b, b + \frac{1}{2^{M+1}}]$ of $[\epsilon_0, 1]$ which contains at least M successive numbers of the form $\|x_{n+1} - x_n\|$.

That is, for some $k \le N - M$ we have

$$b \le \|x_{k+n+1} - x_{k+n}\| \le b + \frac{1}{2^{m+1}}$$
$$\text{for} \quad n = 1, 2, \ldots, M \qquad \ldots (6)$$

Now choose $f \in X^*$ with $\|f\| = 1$ such that

$$f(x_{k+M+1} - x_{k+M}) = \|x_{k+M+1} - x_{k+M}\| \ge b.$$

Then by (6) and this choice of f we have

$$\begin{aligned}
&2b - f(x_{k+M} - x_{k+M-1}) \\
&\quad \le f(2(x_{k+M+1} - x_{k+M})) - f(x_{k+M} - x_{k+M-1}) \\
&\quad \le \|2(x_{k+M+1} - x_{k+M}) - (x_{k+M} - x_{k+M-1})\| \\
&\quad \le \|x_{k+M} - x_{k+M-1}\|, \quad \text{by (5)} \\
&\quad \le b + \frac{1}{2^{M+1}}, \quad \text{by (6)}.
\end{aligned}$$

So

$$f(x_{k+M} - x_{k+M-1}) \ge b - \frac{1}{2^{M+1}} \ldots (7.1)$$

Similarly

$$\begin{aligned}
&2(b - \frac{1}{2^{M+1}}) - f(x_{k+M-1} - x_{k+M-2}) \\
&\le f(2(x_{k+M} - x_{k+M-1}) - f(x_{k+M-1} - x_{k+M-2}) \\
&\le \|x_{k+M-1} - x_{k+M-2}\| \\
&\le b + \frac{1}{2^{M+1}}
\end{aligned}$$

So

$$\begin{aligned}
f(x_{k+M-1} - x_{k+M-2}) &\ge b - \frac{1}{2^M} - \frac{1}{2^{M+1}} \\
&> b - \frac{1}{2^{M-1}} \qquad \ldots \quad (7.2)
\end{aligned}$$

continuing we obtain in general

$$f(x_{k+M+1-n} - x_{k+M-n}) \ge b - \sum_{k=M+2-n}^{M+1} \frac{1}{2^k} > b - \frac{1}{2^{M+1-n}}$$
$$\text{for} \quad n = 0, 1, 2, \ \ldots, \ M - 1 \quad \ldots \quad (7.n)$$

From this epidemic of (7's) we have:

$$\begin{aligned}
f(x_{k+M+1}) &\geq f(x_{k+M}) + b \\
&\geq f(x_{k+M-1}) + 2b - \frac{1}{2^{M+1}} \\
&> \dots \\
&> f(x_{k+M+1-n}) + nb - \sum_{k=M+1-n}^{M+1} \frac{1}{2^k} \\
&> \dots \\
&> f(x_{k+1}) + Mb - \sum_{k=2}^{M+1} \frac{1}{2^k}
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{k+M+1} - x_{k+1}\| &\geq |f(x_{k+M+1} - x_{k+1})| \\
&> Mb - 1
\end{aligned}$$

and so

$$\text{diam}\,(\text{C}) > \text{Mb} - 1, \text{ but b} > \epsilon_0 \text{ and M} > 2/\epsilon_0.$$

So we have the contradiction that $\text{diam}\,(\text{C}) > 1$. ∎

(1.16) COROLLARY: If D is a w -compact minimal invariant set for the mapping U of Theorem (1.15), then for any $x \in D$ and $x_0 \in D$ we have

$$\lim_n \|x - U^n x_0\| = \text{diam}\,(\text{D}).$$

Proof. If this were not so we could find a subsequence such that

$$\lim_k \|x - U^{n_k} x_0\| < \text{diam}\,(\text{D}).$$

Since $(U^{n_k} x_0)$ is an approximate fixed point sequence for U this contradicts Theorem (1.9).

∎

Corollary (1.12) appears to endow minimal invariant sets with a richer structure than "mere" diametrality. Unfortunately, as we will now show, this is in a certain sense not the case. We show that every diametral set contains a diametral sequence which is necessarily a diameterizing sequence for its closed convex hull.

(1.17) THEOREM (Brodskii and Mil'man, 1948) Let D be a closed bounded convex subset of X which is diametral. Then there exists a sequence (x_n) in D with

$$\lim_n \text{dist}\,(x_{n+1},\ \text{co}\,\{x_k\}_{k=1}^n) = \text{diam}\,(D).$$

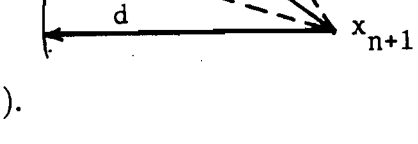
Proof. We construct the sequence (x_n) inductively as follows.

Choose any point of D as x_1. Now suppose $x_1, x_2, \ldots, x_n$ have been selected. Let $b := \frac{1}{n}\sum_{k=1}^n x_k$, the barry- centre of $\text{co}\,\{x_k\}_{k=1}^n$, then since D is diametral we can find a point $x_{n+1} \in D$ satisfying $\|b - x_{n+1}\| \geq d - \frac{1}{n^2}$, where $d :=$ diam (D). We show that the resulting sequence (x_n) has the required property.

To see this let $x \in \text{co}\,\{x_k\}_{k=1}^n$; that is $x = \sum_1^n \alpha_k x_k$ form some $\alpha_k \geq 0$ with $\sum_1^n \alpha_k = 1$.

First observe that if $y \in \text{co}\,\{x_k\}_{k=1}^n$ and $\lambda \in (0,1]$ are such that $b = \lambda x + (1-\lambda)y$, then

$$\begin{aligned} d - \frac{1}{n^2} &\leq \|b - x_{n+1}\| \\ &\leq \lambda\|x - x_{n+1}\| + (1-\lambda)\|y - x_{n+1}\| \\ &\leq \lambda\|x - x_{n+1}\| + d - \lambda d. \end{aligned}$$



So, $d - \frac{1}{\lambda n^2} \leq \|x - x_{n+1}\| \cdots (1)$.
Now

$$\begin{aligned} y &= \frac{1}{1-\lambda}(b - \lambda x) \\ &= \sum_{k=1}^n \frac{(\frac{1}{n} - \lambda\alpha_k)}{1-\lambda} x_k \end{aligned}$$

is a convex combination of the x_k, and so in $\text{co}\,\{x_k\}_{k=1}^n$, provided $\lambda\alpha_k \leq \frac{1}{n}$ for $k = 1, 2, \ldots, n$, which is certainly true if we take $\lambda = \frac{1}{n}$. Thus (1) holds with $\lambda = \frac{1}{n}$ and so we have

$$d - \frac{1}{n} \leq \text{dist}(x_{n+1},\ \text{co}\,\{x_k\}_{k=1}^n)$$

establishing the claim. ∎

(1.18) COROLLARY: A compact convex set is necessarily non-diametral.

Theorem (1.17) shows that a space contains a diametral set if and only if it contains a diametral sequence. A seemingly weaker characterization of this is the following, due to Landes [1984].

(1.19) PROPOSITION: X contains a non-trivial diametral sequence if and only if X contains a sequence (x_n) for which there exists a constant $c > 0$ with $\lim_n \|x_n - x\| = c$, for all $x \in \overline{\text{co}}\{x_n\}_{n=1}^{\infty}$.

Landes calls such a sequence *limit constant.*

Proof. ($\Rightarrow$) This is immediate from the definition of a diametral sequence.

($\Leftarrow$) Inductively construct a subsequence (y_k) of (x_n) as follows. Let $y_1 := x_1$. Suppose $y_1, y_2, \cdots, y_k$ have been selected, then $y_{k+1} := x_m$ where x_m is chosen so that

$$\|x_m - y_j\| \le (1 + 1/k)c, \text{ for all } j \le k.$$

Then for each k, $\|y_{k+1} - y_j\| \le (1 + 1/k)c$, for all $j \le k$.

Put $z_k := \frac{k}{k+1} y_{k+1} + \frac{1}{k+1} y_1 \in \text{co}\{x_n\}_{n=1}^{\infty}$.

We show that $C := \overline{\text{co}}\{z_k\}_{n=1}^{\infty}$ is diametral, with diameter c. The result then follows by Theorem (1.17).

Now, for $m > k$ we have

$$\begin{aligned}
\|z_m - z_k\| &= \|\frac{m}{m+1} y_{m+1} - \frac{k}{k+1} y_{k+1} - (\frac{1}{k+1} - \frac{1}{m+1}) y_1\| \\
&= \|\frac{k}{k+1}(y_{m+1} - y_{k+1}) + (\frac{1}{k+1} - \frac{1}{m+1})(y_{m+1} - y_1)\| \\
&\le (\frac{k}{k+1} + \frac{1}{k+1} - \frac{1}{m+1})(1 + \frac{1}{m})c \\
&= c.
\end{aligned}$$

Thus, $\text{diam}(C) \le c$.

On the other hand, since $C \subseteq \overline{\text{co}}\{x_n\}_{n=1}^{\infty}$ and $\|z_k - y_{k+1}\| \to 0$ as $k \to \infty$ we have

$$\lim_k \|z_k - x\| = c, \text{ for any } x \in C.$$

Hence C is diametral, with (z_k) as a diameterizing sequence. (In fact therefore, (z_k) contains a diameteral subsequence.) ∎

Note: In the condition of the above proposition, c is zero if and only if the sequence (x_n) is a constant sequence.

(1.20) REMARKS.

(1) Since any subsequence of (x_n) in Corollary 1.12 or Theorem (1.17) retains the same structure, if D is weak compact (sequentially weak*-compact) we may assume that (x_n) is weak (weak*) convergent. Indeed by proposition (0.6) we may take (x_n) to be a weak (weak*) null sequence. In the weak case we then have by Mazur's Theorem that $0 \in \overline{\mathrm{co}}\{x_n\}$ and so inparticular $\lim_n \|x_n\| = \mathrm{diam\,D} = \mathrm{diam}\,\overline{\mathrm{co}}\{\mathrm{x_n}\}$.

(2) It is possible to construct sequences with even richer structure than the sequence (x_n) of Theorem (1.17). *For example*; as a special case of van Dulst [84] we have: If X contains a weak compact convex diametral set, then there exists a weak null sequence (y_n) such that for every $k \in \mathrm{N}$ and $l \in \mathrm{N}$ we have

$$1 - \frac{1}{k} \leq \operatorname{dist}(y_{k+l+1}, \operatorname{co}\{x_j\}_{j=k}^{k+l}) \leq 1 + \frac{1}{k}.$$

2. NORMAL STRUCTURE

In this chapter we examine some of the basic geometric conditions on a Banach space which are sufficient to ensure that minimal invariant sets are singleton, and hence that the space has the (w) FPP. Most of the conditions considered represent gross overkills, being sufficient for the (w) FPP, but far from necessary.

(2.1) Definitions

We say that a Banach space X has:
normal structure if it contains no closed bounded convex diametral sets with more than one point;

w (w^*) **-normal structure** if it contains no w (w^*)-compact convex diametral sets with more than one point.

The **normal structure constant** of X is

$$N(X) := \sup_C \frac{\mathrm{rad}(C)}{\mathrm{diam}\,(\mathrm{C})},$$

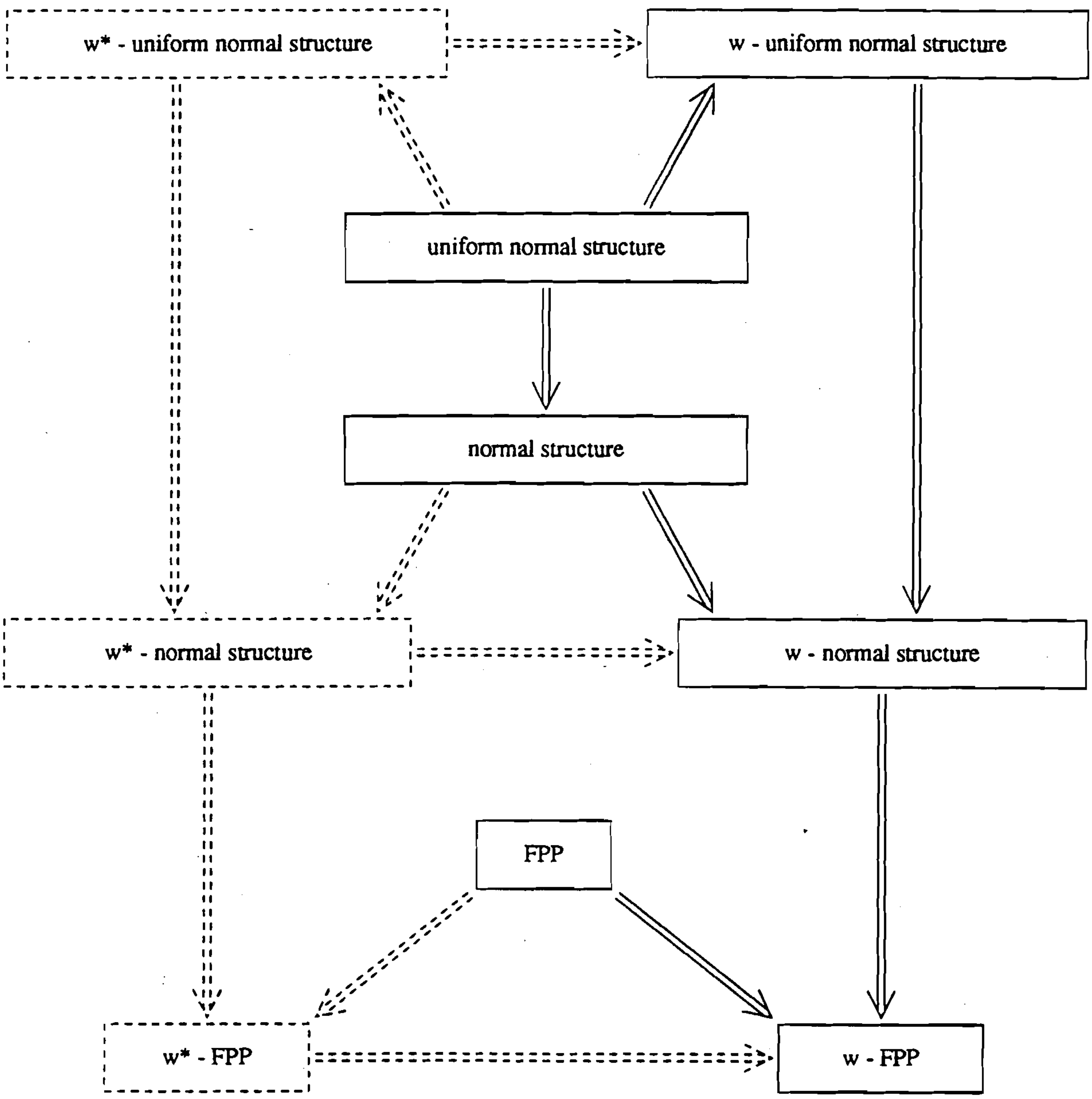
where the supremum is taken over all closed bounded convex sets C with more than one point.

If the admisible sets C are further required to be w (w^*) -compact we obtain the w (w^*)**-normal structure constant,** $N_w(X)$ $(N_{w^*}(X))$.

When the space X is clear from the context we will drop it from the notations above writing N for $N(X)$, and so on.

X has **uniform normal structure,** w (w^*)**-uniform normal structure** if we have respectively $N(X) < 1$, $N_{w\ (w^*)}(X) < 1$. With the aid of Theorem (1.5) we have the following **implications.**

(2.2)



Broken lines indicate concepts and implications which only apply when X is a dual space, of course many of these concepts coalesce when X is reflexive.

The **modulus of convexity** for the Banach space X is

$$\delta(\epsilon) := \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \ \|y\| \leq 1, \|x-y\| \geq \epsilon\}.$$

δ is an increasing function on $[0,2]$ with $\delta(0) = 0$.

We say X is **ϵ_0-inquadrate** if $\delta(\epsilon_0) > 0$, and **uniformly convex** if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. Equivalently, X is uniformly convex if and only if whenever (x_n) and (y_n) are such that

$$\|x_n\| \to 1, \ \|y_n\| \to 1 \quad \text{and} \quad \|\frac{x_n + y_n}{2}\| \to 1$$

we have $\|x_n - y_n\| \to 0$.

The spaces ℓ_p and $L_p(\mu)$ are uniformly convex for $1 < p < \infty$, with

$$\delta(\epsilon) = \begin{cases} 1 - (1 - (\frac{\epsilon}{2})^p)^{\frac{1}{p}} & \text{for } 2 \leq p < \infty \\ \zeta \ [= (p-1)\epsilon^2/8 + 0(\epsilon^2)], & \\ \text{where} & \\ (1 - \zeta + \frac{\epsilon}{2})^p + (1 - \zeta - \frac{\epsilon}{2})^p = 2, & \text{for } 1 \leq p \leq 2. \end{cases}$$

(2.3) PROPOSITION (Edelstein, 1963): Spaces which are ϵ_0-inquadrate for some $\epsilon_0 < 1$, in particular uniformly convex spaces, have uniform normal structure with a normal structure constant $N \leq 1 - \delta(1)$.

Proof. Let C be a closed convex subset of X with diam(C) $= 1$. For any $t \in [0,10$ choose $x_1, x_2 \in C$ with $\|x_1 - x_2\| \geq t$ and let $x_0 := \frac{1}{2}(x_1 + x_2)$. Then for any $x \in C$ we have

$$\|x - x_1\| \leq 1, \quad \|x - x_2\| \leq 1$$

and

$$\|(x - x_1) - (x - x_2)\| = \|x_1 - x_2\| \geq t.$$

It follows from the definition of δ that

$$\|x - x_0\| = \|\frac{(x - x_1) + (x - x_2)}{2}\| \leq 1 - \delta(t).$$

thus rad (C) $\leq \inf_{0 \leq t < 1} [1 - \delta(t)] = 1 - \delta(1)$. ■

(2.3.1) REMARK. In general proposition (2.3) does not give a sharp estimate for N. In the case of Hilbert space it yields $N \leq \sqrt{3}/2 \doteq 0.866$. In fact we have:

$$N(\ell_2) = \frac{1}{\sqrt{2}} \doteq 0.707.$$

Let C be a closed bounded convex subset in a Hilbert space. Choose $x_0 \in C$ so that rad (C) $= \sup_{x \in C} \|x_0 - x\|$. That is, x_0 is in the Chebyshev centre of C which is non-empty by the weak compactness. Let $x, y \in C$ then for any $\lambda \in [0,1]$ we have

$$\begin{aligned}\|\lambda x_0 + (1-\lambda)x - y\|^2 &= \|\lambda(x_0 - y) + (1-\lambda)(x-y)\|^2 \\ &= \lambda^2\|x_0 - y\|^2 + (1-\lambda)^2\|x-y\|^2 \\ &- 2\lambda(1-\lambda)(x_0 - y,\ x-y) \\ &= \lambda\|x_0 - y\|^2 + (1-\lambda)\|x-y\|^2 \\ &- \lambda(1-\lambda)\|x_0 - x\|^2 \\ &\text{(by the Polarization Identity).}\end{aligned}$$

Now, since $\lambda x_0 + (1-\lambda)x \in C$, taking the supremum over x then $y \in C$ we obtain

$$(\mathrm{rad} C)^2 \leq \lambda(\ \mathrm{rad}\ \ C)^2 + (1-\lambda)(\ \mathrm{diam}\, C)^2 - \lambda(1-\lambda)(\ \mathrm{rad}\, C)^2$$

or

$$\left(\frac{\mathrm{rad} C}{\mathrm{diam}\, C}\right)^2 \leq \frac{1}{1+\lambda} \quad \text{for all} \quad \lambda \in [0,1)$$

whence

$$N(\ell_2) \leq \frac{1}{\sqrt{2}}.$$

To establish equality, consider

$$C := \overline{\mathrm{co}}\{e_k\}_{k=1}^{\infty} \quad \text{for which} \quad \mathrm{diam}\, C = \sqrt{2}$$

while $\mathrm{rad}\, C = 1$ (with $0 \in C$ as centre). ∎

Lim [1983, 86] developed the above argument for ℓ_p with $1 < p < \infty$ to obtain upper bounds for $N(\ell_p)$, and hence as we shall see for $N(L_p)$ also.

For $2 < p < \infty$,

$$N(\ell_p) \leq \left(1 + \frac{1+\xi_0^{p-1}}{(1+\xi_o)^{p-1}}\right)^{-\frac{1}{p}}$$

where ξ_0 is the unique positive solution of $(p-2)\xi^{p-1} + (p-1)\xi^{p-2} = 1$.

The set $C := \overline{\mathrm{co}}\{e_n\}_{n=1}^{\infty}$ gives a lower bound of $N(\ell_p) \geq 2^{-\frac{1}{p}}$.

For $1 < p < 2$, the situation is more intricate. In this case Lim [86] obtained

$$N(\ell_p) \leq \gamma^{\frac{1}{p}},$$

where

$$\gamma := \inf_{0<\lambda<\frac{1}{2}} \frac{2(1-\lambda)}{(1-g(\lambda))^p + (1+g(\lambda))^p - 2\lambda}$$

with

$$g(\lambda) := \inf_{-1 \le x < 1} \min\{g(\lambda, x),\ g(1-\lambda,\ x)\}$$

and $g(\lambda, x)$ the unique non-negative solution of

$$|\lambda + (1-\lambda)x - g(\lambda, x)(1-x)|^p + |\lambda + (1-\lambda)x + g(\lambda, x)(1-x)|^p$$
$$= 2\lambda + 2(1-\lambda)|x|^p.$$

The best known lower bound for $1 < p < 2$ appears to be [Amir, 1982/83]:

$$N(\ell_p) \ge 2^{-\frac{1}{q}} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This is obtained by producing a sequence of sets $(C_n)_{n=1}^{\infty}$ with $\frac{\mathrm{rad}\,(C_n)}{\mathrm{diam}\,(C_n)} \to 2^{\frac{1}{p}-1}$. Such a sequence is obtained by defining

$$C_n := \mathrm{co}\,\{\underset{\sim}{x}_i^n\}_{i=1}^{2^{n-1}}$$

where

$$\underset{\sim}{x}_i^n := \sum_{j=1}^{2^{n-1}} a_{ij}^n \underset{\sim}{e}_j$$

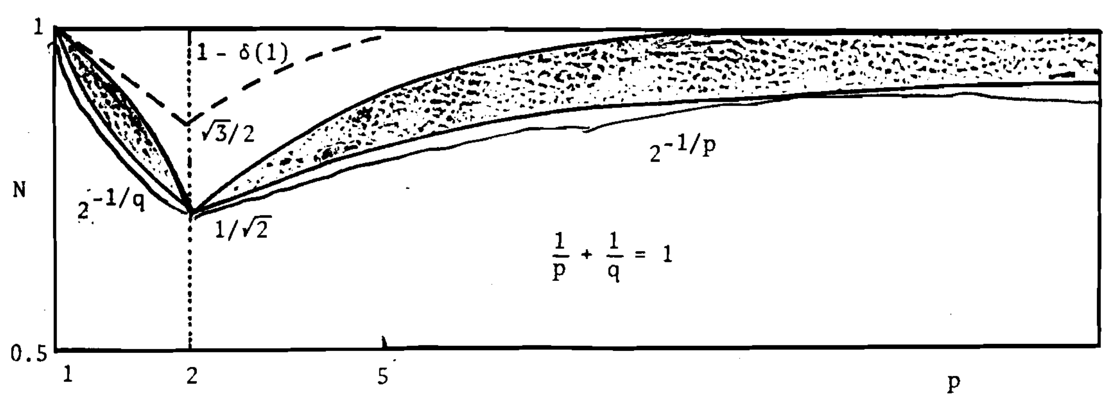
and

$$a_{ij}^n := \frac{1}{2}(1 - w_{i+1,j+1}^n).$$

Here $w_{i+1,j+1}^n$ is the $(i+1, j+1)$- entry of the $n'th$ Walsh matrix defined inductively by

$$W_0 = (1) \quad , \quad W_{n+1} = \begin{pmatrix} W_n & W_n \\ W_n & -W_n \end{pmatrix}.$$

These results are summarized graphically below.



QUESTION. Is a better estimate for N in terms of δ possible? In particular determine sharper (precise) estimates for $N(\ell_p), 1 < p < \infty, p \neq 2$.
Here the work of Bynum [1980], Maluta [1984] and in particular Amir [1982/83] and Amir and Franchetti [1982/83] are relevant.

Note: Our $N(X)$ is the reciprocal of Bynum's and is half the *self Jung constant* of X.

From corollary (1.18) we see that every finite dimensional Banach space has normal structure. Indeed it is an old result of Bohnenblust [1938] that **all finite dimensional Banach spaces have uniform normal structure.** To see this let X_n be any n-dimensional Banach space and let C be a closed bounded convex subset of diameter 1. Choose $r <$ rad C, then $\bigcap_{x\in C} B_r[x] \cap C = \phi$. By Helly's theorem there must exist points $x_0, x_1, \ldots, x_m \in C$ with $m \leq n$ such that

$$\bigcap_{i=0}^{m} B_r[x_i] \cap C = \phi.$$

In particular for some i_0 we must have

$$\|x_{i_0} - \sum_{i=0}^{m} \frac{1}{m+1} x_i\| > r$$

But,

$$\begin{aligned}\|x_{i_0} - \sum_{i=0}^{m} \frac{1}{m+1} x_i\| &= \frac{1}{m+1}\|\sum_{i=0}^{m}(x_{i_0} - x_i)\| \\ &\leq \frac{m}{m+1}\max_{i\neq i_o}\|x_{i_0} - x_i\| \\ &\leq \frac{m}{m+1} \leq \frac{n}{n+1}.\end{aligned}$$

Hence $r < \frac{n}{n+1}$ and so by the choice of r we have rad $\mathrm{C} \leq \frac{\mathrm{n}}{\mathrm{n}+1}$. Consequently $N(X_n) \leq \frac{n}{n+1}$.

To see that this upper bound is sharp let $C :=$ co$\{\mathrm{o}, \mathrm{e}_2 - \mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{n} - \mathrm{e}_1\}$. Then in the $(n-1)$-dimensional span of $e_2 - e_1, \ldots, e_n - e_1$ with the inherited l_∞^n norm it is readily calculated that $N(C) = 1 - \frac{1}{n}$. [See also Yost, 1982].

The following Corollary seems to have been noticed by many authors; Browder [1965], Edelstein [?], Göhde [?], Kirk [1965].

(2.4) COROLLARY: Spaces which are ϵ_0-inquadrate for some $\epsilon_0 < 1$, in particular uniformly convex spaces, have the FPP.

Proof. This is immediate from proposition (2.3), (2.2) and the fact that such spaces are reflexive (Mil'man-Pettis theorem). ∎

The dual (equivalently predual, by the reflexivity) of a Banach space X is uniformly convex if and only if the space is **uniformly smooth** (the norm is uniformly Fréchet differentiable on the unit sphere S_X). Baillon [1978-79] established the FPP for reflexive spaces whose **modulus of smoothness**

$$\varrho(\tau) := \sup\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\|) - 1 : x, y \in S_X\}$$

satisfies $\lim\limits_{\tau\downarrow 0} \varrho(\tau)/\tau < \frac{1}{2}$. In particular then, uniformly smooth Banach spaces have the FPP.

Barry Turett [1982] observed that X satisfies

$$\lim_{\tau\downarrow 0} \varrho(\tau)/\tau < \frac{1}{2}$$

if and only if X^* is ϵ_0-inquadrate for an $\epsilon_0 < 1$ [see also; Giles, Gregory and Sims, 1978]. Hence the assumption of reflexivity is redundant. Turett showed that spaces with an ϵ_0-inquadrate dual ($\epsilon_0 < 1$) have normal structure.

(2.5) LEMMA [Turret, 1982]: If X fails w-normal structure, then given $\epsilon > 0$ there exists $f, g \in S_{X^*}$ and $x \in S_X$ such that $\|f-g\|, f(x), g(x) > 1-\epsilon$. That is, the dual ball contains arbitrarily "thin" w^*-slices with diameter near one or more.

Proof. By Corollary 1.11 and Proposition 0.6 if X fails w-normal structure we can find $(x_n) \subset X$ with $x_n \rightharpoonup^w 0$ such that

$$\text{diam co}\{x_n\}_{n=1}^{\infty} = 1$$

and

$$\text{dist } (x_{n+1}, \text{ co}\{x_k\}_{k=1}^{n}) > 1 - \frac{1}{n}.$$

Observation. Without loss of generality we may take $x_1 = 0$; choose $x_2, x_3, \ldots$ as above. Since $x_n \rightharpoonup^w 0$, by Mazur's theorem

$$\text{dist } (0, \text{ co}\{x_k\}_{k=2}^{n}) \to 0 \quad \text{as} \quad n \to \infty$$

and so for n sufficiently large we have

$$\text{dist } (x_{n+1}, \text{ co}(\{0\} \cup \{x_k\}_{k=2}^{n})) \sim \text{ dist } (x_{n+1}, \text{ co}\{x_k\}_{k=2}^{n}).$$

Note, this argument also shows that $\|x_n\| \to 1$.

Now, since for each n, $B_{1-\frac{1}{n}}[x_{n+1}] \cap \operatorname{co}\{x_k\}_{k=1}^{n} = \phi$, we may apply the Eidelheit separation theorem to obtain a norm one functional f_{n+1} such that

$$1 \geq f_{n+1}(x_{n+1} - x_k) > 1 - \frac{1}{n}, \quad \text{for all} \quad k \leq n.$$

In particular $k = 1$ gives $f_{n+1}(x_{n+1}) > 1 - \frac{1}{n}$. Now choose $j_0 \in \mathbf{N}$ with $1 < j_0 < \frac{2}{\epsilon}$. Since $x_n \rightharpoonup^{w} 0$ there exists $n_0 > j_0$ so that $|f_{j_0}(x_{n_0})| < \epsilon/2$. But then,

$$\begin{aligned} -f_{j_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) &\geq -f_{j_0}(x_{n_0} - x_{j_0}), \quad \text{as} \quad \|x_{n_0} - x_{j_0}\| \leq 1 \\ &\geq 1 - \frac{1}{n_0} - \frac{\epsilon}{2} \\ &> 1 - \epsilon, \end{aligned}$$

while

$$\begin{aligned} f_{n_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) &\geq f_{n_0}(x_{n_0} - x_{j_0}) \\ &> 1 - \frac{1}{n_0} > 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} \|f_{n_0} - (-f_{j_0})\| &\geq (f_{n_0} + f_{j_0})(x_{n_0}) \\ &> 1 - \frac{1}{n_0} - \frac{\epsilon}{2} \\ &> 1 - \epsilon. \end{aligned}$$

The result now follows by taking

$$f = f_{n_0}, \; g = -f_{j_0} \quad \text{and} \quad x = \frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}. \quad \blacksquare$$

The last argument suggests a positive answer to the following may be possible.

QUESTION: If X is fully 2-rotund does X have the F.P.P. Recall X is *fully 2-rotund* [Fan and Glicksberg, 1958] if the sequence (x_n) is convergent whenever

$$\|x_n + x_m\| \to 2 \quad \text{as} \quad m, n \to \infty.$$

(2.5.1) COROLLARY: If X^* is ϵ_0-inquadrate for some $\epsilon_0 < 1$, then X has normal structure.

Proof: Since X is reflexive it suffices to note that if X^* is ϵ_0-inquadrate then choosing ϵ in lemma 2.5 so that $\epsilon < \min\{\delta(\epsilon_0), 1 - \epsilon_0\}$, if $f,\ g \in S_X^*$ and $x \in S_x$ are such that $f(x), g(x) > 1 - \epsilon$ then $\|\frac{f+g}{2}\| > 1 - \epsilon$ and so $\|f - g\| < \epsilon_0 < 1 - \epsilon$. $\blacksquare$

(2.6) THEOREM: If X^* is ϵ_0-inquadrate for some $\epsilon_0 < 1$, then X has uniform normal structure. In particular uniformly smooth spaces have uniform normal structure.

Proof. Suppose X fails to have uniform normal structure, then we can find a sequence (C_n) of diameter one subsets of X with rad $(C_n) \to 1$.

For any ultrafilter U on $\mathbf{N}$, let

$$C = (C_n)_U := \{(x_n)_U : x_n \in C_n\},$$

then C is a convex subset of the ultra power $(X)_U$ with diam C = 1 and rad (C) = 1. Thus $(X)_U$ fails to have normal structure. However, since X is superreflexive we have $(X)_U^* = (X^*)_U$ is ϵ_0-inquadrate [Sims, 82; §10 proposition 6] which contradicts Corollary 2.5.1. ∎

Proposition 2.3 and the last theorem suggest the following.

QUESTION: Is uniform normal structure a super-property?

The next proposition shows that the answer to this question would be "yes" if spaces with uniform normal structure were superreflexive, however this appears not to be known. What is known (see below) is that uniform normal structure implies reflexivity.

(2.7) PROPOSITION: The weak-normal structure constant is "finitely determined". That is, given any $\epsilon > 0$ there exists a finite subset F with

$$\frac{\text{rad (co(F))}}{\text{diam (co(F))}} \geq (1-\epsilon)N_w$$

Proof. Let C be any weakly-compact convex set with diam $(C) = 1$ and let $r <$ rad (C). Then

$$\bigcap_{x \in C} B_r[x] \cap C = \phi$$

(if x_0 were in this intersection then x_0 would be a point of C with $\|x - x_0\| \leq r$ for all $x \in C$; that is rad (C) $\leq$ r.)

Since each of the sets $B_r[x] \cap C$ is a weakly compact subset of C there exists a finite subset F of C with

$$\bigcap_{x \in F} B_r[x] \cap C = \phi.$$

But then

$$\bigcap_{x \in F} B_r[x] \cap \text{ co(F)} \subseteq \bigcap_{\text{x} \in \text{F}} \text{B}_\text{r}[\text{x}] \cap \text{C} = \phi$$

and it follws that $r \leq$ rad (co(F)). Thus

$$\frac{\text{rad(C)}}{\text{diam(C)}} \leq \sup\{\frac{\text{rad(co(F))}}{\text{diam(co(F))}} : F \subseteq C, \quad \text{F is finite}\}$$

estabishing the proposition. ∎

Remarks: 1) If X is of finite dimension n the appeal to weak compactness in the above argument may be replaced by an application of Helly's theorem showing that the cardinality of F need be no larger than $n+1$.

2) Dvoretsky's theorem sets a lower bound of $1/\sqrt{2}$ for the uniform normal structure constant of all infinite dimensional Banach spaces.

The following provides a useful tool for estimating N_w (N in the case of a reflexive space).

(2.7.1) COROLLARY [Amir, 1982-83]: Let $(X_\alpha)_{\alpha\in\Lambda}$ be a net of subspaces directed by inclusion with $X = \overline{\bigcup_{\alpha\in\Lambda} X_\alpha}$, then

$$N_w(X) = \sup_{\alpha\in\Lambda} N_w(X_\alpha) = \lim_\alpha N_w(X_\alpha).$$

Proof. By (2.7) it is enough to consider sets of the form $C = \operatorname{co}\{x_1, x_2, \ldots, x_n\}$. Now given $\epsilon > 0$ we can find $y_1, y_2, \ldots, y_n \in X_\alpha$, for some α, with $\|x_i - y_i\| < \epsilon/2 (i = 1, 2, \ldots, n)$. Let $K = \operatorname{co}\{y_1, \ldots, y_n\} \subset X_\alpha$, then

$$N_w(X_\alpha) \geq \frac{\operatorname{rad} K}{\operatorname{diam} K} \geq \frac{\operatorname{rad} C - \epsilon}{\operatorname{diam} C + \epsilon}$$

and the result follows. ∎

(2.7.2) COROLLARY: For $1 < p < \infty$ we have $N(L_p(\mu)) = N(l_p) = \lim_n N(l_p^n)$.

Proof. Let $P := \Omega_1, \Omega_2, \ldots \Omega_n$ be a measurable partition of the measure space $(\Omega, \sum, \mu)$ with $\mu(\Omega_i) > 0$ $(i = 1, 2, \ldots, n)$ and let $X_P =< \chi_{\Omega_i} >_{i=1}^n$, the subspace spanned by the characteristic functions χ_{Ω_i}. Then, X_P is isometric to l_p^n and (X_P) is clearly a net directed by inclusion whose union is dense in $L_p(\mu)$. ∎

That spaces with uniform normal structure are reflexive was proved by Bae [1983]. Independently it would seem Maluta [1984] observed that the result is an immediate consequence of an earlier result of Mil'man and Mil'man [1965] which in turn represents a mild strengthening of a contemporaneous result by R.C. James [1964, see for example Beauzamy, 1982]. Since the result of Mil'man and Mil'man is of independent interest we choose to develop it here, although the proof is somewhat more intricate and longer than is necessary for our purpose (see Remark 2 at the end of the Proof).

(2.8) THEOREM [Mil'man and Mil'man, 1965]: If X is a non-reflexive Banach space then for every $\epsilon > 0$ there exists a sequence of unit vectors (x_n) such that for all $m \in \mathrm{N}$ if $y \in \operatorname{co}\{x_1, \ldots, x_m\}$ and $y' \in \operatorname{co}\{x_k\}_{k=m+1}^\infty$ then we have

$$1 - \epsilon \leq \|y - y'\| \leq 1 + \epsilon.$$

Taking $C = \overline{\text{co}}\{x_k\}_{k=1}^{\infty}$ we have $\operatorname{diam} C \leq 1+\epsilon$ while $\operatorname{rad} C \geq 1-\epsilon$ and so we obtain

(2.8.1) COROLLARY: If X has uniform normal structure then X is reflexive.

Proof (of Theorem 2.8). Since X is non-reflexive it contains a non-reflexive separable subspace E. We will carry out the construction inside E. Since B_E is not weakly compact we can find a nested family F of non-empty closed bounded convex subsets

$$K_1 \subset K_2 \subset K_3 \subset \dots$$

with

$$\bigcap_{n=1}^{\infty} K_n = \phi$$

Further $d(F) := \lim_n \operatorname{diam} \mathrm{K_n}$ exists and is strictly greater than zero (otherwise Cantor's intersection theorem would apply to give a non- empty intersection).

Similarly the sequence $(\operatorname{dist}\ (\mathrm{x}, \mathrm{K_n}))_{\mathrm{n}=1}^{\infty}$ is increasing and bounded above for each $x \in E$, so

$$r(x, F) := \lim_n \operatorname{dist}\ (\mathrm{x}, \mathrm{K_n})$$

exists, and can be regarded as the distance from x to the *vacuum* of the nested family F, in particular $r(x, F) > 0$.

The proof now proceeds through a series of steps.

STEP 1) Given $F = (K_n)_{n=1}^{\infty}$ as above we can find a nested family $F' = (K'_n)_{n=1}^{\infty}$ of closed convex sets which is **subordinate** to F in the sense that $K'_n \subseteq K_n$ for all n, and such that for each $x \in E$

$$\begin{aligned} r(x, F) = r(x, F') : &= \lim_n \operatorname{dist}\ (\mathrm{x}, \mathrm{K'_n}) \\ &= \lim_n \|x - x_n\| \end{aligned}$$

for any sequence (x_n) with $x_n \in K'_n$. We will say that F' is a **closely - flattened family.**.

Proof of Step 1. Let (y_n) be a dense sequence in the separable space E.

Let $F_1 = (K_n^1)_{n=1}^{\infty}$ where

$$K_n^1 = K_n \cap B_{(1+\frac{1}{n})r(y_1, F)}[y_1].$$

F_1 is a nested family of <u>non-empty</u> closed convex sets subordinate to F. Further for $x_n \in K_n^1$,

$$\begin{aligned} \operatorname{dist}(\mathrm{y_1}, \mathrm{K_n}) &\leq \operatorname{dist}(\mathrm{y_1}, \mathrm{K_n^1}) \\ &\leq \|y_1 - x_n\| \\ &\leq (1+\frac{1}{n})r(y_n, F), \end{aligned}$$

so

$$\lim_n \|y_1 - x_n\| = r(y_1, F) = r(y_1, F_1).$$

Continuing in this way we construct a sequence $F_m = (K_n^m)$ of nested families of decreasing subordination so that

$$\lim_n \|y_m - x_n\| = r(y_m, F_m) = r(y_m, F).$$

for any $x_n \in K_n^m$.

Let $F' = (K_m^m)_{m=1}^{\infty}$, then $K_m^m \subseteq K_m^{m-1} \subseteq \ldots \subseteq K_m$ so F' is subordinate to F.

Further, for any n and $x_m \in K_m^m$ where $m \geq n$, we have

$$\begin{aligned}\mathrm{dist}\,(\mathrm{y_n}, \mathrm{K_m}) &\leq \mathrm{dist}\,(\mathrm{y_n}, \mathrm{K_m^m}) \\ &\leq \|y_n - x_m\| \\ &\leq (1 + \frac{1}{m}) r(y_n, F)\end{aligned}$$

as $K_m^m \subseteq K_m^{m-1} \subseteq \ldots \subseteq K_m^n \subseteq B_{(1+\frac{1}{m})r(y_n,F)}[y_n]$. So letting $m \to \infty$ we have

$$r(y_n, F) = r(y_n, F') = \lim_m \|y_n - x_m\|.$$

That F' is the desired family now follows since $\{y_k\}$ is dense in E and $|\mathrm{dist}\,(\mathrm{x}, \mathrm{C}) - \mathrm{dist}\,(\mathrm{y}, \mathrm{C})| \leq \|\mathrm{x} - \mathrm{y}\|$ for any set C.

NOTE. The sequence of numbers

$$\Big(\inf_{x \in K_n} r(x, F)\Big)_{n=1}^{\infty}$$

is increasing for any nested family $F = (K_n)$ and so converges.

Let

$$r(F) := \lim_n \inf_{x \in K_n} r(x, F)$$

Clearly, $r(F) \leq r(F')$ if F' is subordinate to F and $r(F) \leq d(F)$.

STEP 2) For F' as in Step 1) we can find a nested family F'' subordinate to F' for which

$$0 < d(F'') = r(F'') = r(F') = r(F)$$

Proof of Step 2). Choose $x_n \in K_n'(:= K_n^n)$ so that $\inf_{x \in K_n'} r(x, F') > r(x_n, F') - \frac{1}{n}$ then

$$r(F') = \lim_{n \to \infty} r(x_n, F')$$

Further, by the flattening

$$r(x_n, F') = \lim_{m\to\infty} \|x_n - x_{n+m}\|$$

Now let $m_1 = 1$ and choose m_2 so that for $m > m_2$ we have $|r(x_{m_1}, F') - \|x_{m_1} - x_m\|| < 1$. Then choose m_3 so that for $m > m_3$

$$|r(x_{m_2}, F') - \|x_{m_2} - x_m\|| < \frac{1}{2}$$

etc.

In this way we obtain integers $m_1 < m_2 < \ldots$ so that

$$|r(x_{m_k}, F') - \|x_{m_k}, x_m\|| < \frac{1}{k} \quad \text{for} \quad m > m_{k+1}.$$

In particular then,

$$|r(x_{m_k}, F') - \|x_{m_k}, x_{m_r}\|| < \frac{1}{k} \quad \text{for} \quad r > k$$

and so

$$\begin{aligned} r(F') &= \lim_k r(x_{m_k}, F') \\ &= \lim_{k\to\infty, r>k} \|x_{m_k} - x_{m_r}\| = \lim_{r,k\to\infty_{r\neq k}} \|x_{m_k} - x_{m_r}\|. \end{aligned}$$

Now let $K_n'' = \overline{\text{co}}\{x_{m_k}\}_{k=n+1}^{\infty}$ and let $F'' = (K_n'')_{n=1}^{\infty}$.

Then

$$\begin{aligned} d(F'') &= \lim_n \ \text{diam}\,(\text{K}_\text{n}'') \\ &= \limsup_{nr,k>n,_{r\neq k}} \|x_{m_k} - x_{m_r}\| \\ &= r(F') \leq r(F'') \leq d(F'') \end{aligned}$$

establishing the result.

STEP 3) For the nested family $F'' = (K_n'')$ of Step 2, if $x_n \in K_n''$, given $\epsilon_1 > 0$ we can find a subsequence $y_k := x_{n_k}$ so that for any sequence $(n_m) \subseteq \mathbf{N}$ with $n_m \geq m$ and for any $u_m \in \text{co}\,\{\text{y}_\text{k}\}_{\text{k=m}}^{\text{n}_\text{m}}$ and $u_m' \in \text{co}\,\{\text{y}_\text{k}\}_{\text{k}=\text{n}_{\text{m}+1}}^{\infty}$ we have:

(i) For all m, $d(F'') - \epsilon_1 \leq \|u_m - u_m'\|$ and

(ii) $\limsup_n \|u_m - u_m'\| \leq \lim_m \ \text{diam}\,(\{\text{y}_\text{k}\}_{\text{k=m}}^{\infty}) \leq \text{d}(\text{F}'')$.

Proof of Step 3). Since $d(F'') = r(F'') := \liminf_{n} \inf_{x\in K_n''} r(x, F'')$ there exists n_1 so that

$$d(F'') - \frac{\epsilon_1}{2} \leq \inf_{x\in K_{n_1}''} r(x, F'').$$

Further, $r(x, F'') := \lim_n \operatorname{dist}(\mathrm{x}, \mathrm{F''})$ so there exists n_2 such that

$$r(x_{n_1}, F'') \leq \operatorname{dist}(\mathrm{x_{n_1}}, \mathrm{k''_{n_2}}) + \frac{\epsilon_1}{2}.$$

Thus for any element $u' \in K_{n_2}$, in particular any element of $\operatorname{co}\{\mathrm{x_k}\}_{\mathrm{k=n_2}}^{\infty}$, we have

$$d(F'') - \epsilon_1 \leq \|x_{n_1} - u'\|.$$

Now let $u \in \operatorname{co}\{\mathrm{x_{n_1}}, \mathrm{x_{n_2}}\}$ then, as above, there exists $m_3(u)$ so that

$$d(F'') - \frac{\epsilon_1}{2} \leq \operatorname{dist}(\mathrm{u}, \mathrm{K''_{m_3(u)}}).$$

Claim There exists n_3 so that for all$u \in \operatorname{co}\{\mathrm{x_{n_1}}, \mathrm{x_{n_2}}\}$ we have

$$d(F'') - \epsilon_1 \leq \operatorname{dist}(\mathrm{u}, \mathrm{K''_{n_3}}).$$

Assume not, then there exists a sequence (m_k) with $m_k \to \infty$ and points $u_k \in \operatorname{co}\{\mathrm{x_{n_1}}, \mathrm{x_{n_2}}\}$ so that $\operatorname{dist}(\mathrm{u_k}, \mathrm{K''_{m_k}}) < \mathrm{d(F'')} - \epsilon_1$. Since $\operatorname{co}\{\mathrm{x_{n_1}}, \mathrm{x_{n_2}}\}$ is compact there exists a subsequence $u_{k_j} \to u$, but then for $m_{k_j} > m_3(u)$ we have

$$\begin{aligned} \operatorname{dist}(\mathrm{u_{k_j}}, \mathrm{K''_{m_{k_j}}}) &\geq \operatorname{dist}(\mathrm{u_{k_j}}, \mathrm{K''_{m_{3(u)}}}) \\ &\longrightarrow \operatorname{dist}(\mathrm{u}, \mathrm{K''_{m_{3(u)}}}) \geq \mathrm{d(F'')} - \frac{\epsilon_1}{2} \end{aligned}$$

which is impossible.

Continuing in this way we obtain a sequence $n_1 < n_2 < \ldots$ such that for any $u \in \operatorname{co}\{\mathrm{x_{n_k}}\}_{\mathrm{k=1}}^{\mathrm{m}}$ $\mathrm{d(F'')} - \epsilon_1 \leq \operatorname{dist}(\mathrm{u}, \mathrm{K''_{n_{m+1}}})$.

In particular, $d(F'') - \epsilon_1 \leq \|u - u'\|$ for any $u' \in \operatorname{co}\{\mathrm{x_{n_k}}\}_{\mathrm{k=m+1}}^{\infty}$, establishing (i).

Now since $u_m, u'_m \in \{y_k\}_{k=m}^{\infty} \subseteq K'''_m := \operatorname{co}\{\mathrm{y_k}\}_{\mathrm{k=m}}^{\infty}$ we have

$$\begin{aligned} \limsup_n \|u_m - u'_m\| &\leq \lim_m \operatorname{diam}(\{\mathrm{y_k}\}_{\mathrm{k=m}}^{\infty}) \\ &= \lim_m \operatorname{diam}(\mathrm{K'''_m}) \\ &= d(F''') \leq d(F'') \end{aligned}$$

as $F''' := (K'''_n)$ is subordinate to F'', and so we have (ii).

STEP 4) (the last). The sequence (x_n) of the theorem is constructed from the sequence (y_k) of Step 3 as follows.

From step 2) for any $\epsilon_2 > 0$ we can find n_0 so that for any m,

$$d - \epsilon_1 \leq \|u - u'\| \leq d + \epsilon_2,$$

where

$$u \in \operatorname{co}\{y_k\}_{k=n_0}^{n_0+m}, \; u' \in \operatorname{co}\{y_k\}_{k=n_0+m+1}^{\infty}, \quad \text{and} \quad d := d(F'') > 0.$$

Let $z_k := (y_{n_o+k} - y_{n_0})/d$ then

$$1 - \frac{\epsilon_1}{d} \leq \|w - w'\| \leq 1 + \frac{\epsilon_2}{d}$$

for

$$w \in \operatorname{co}\{z_k\}_{k=1}^{m}, w' \in \overline{\operatorname{co}}\{z_k\}_{k=m+1}^{\infty}.$$

Further, from above with $m = 1$ we have
$1 - \frac{\epsilon_1}{d} \leq \|z_k\| \leq 1 + \frac{\epsilon_2}{d}$.

Since ϵ_1 and ϵ_2 are arbitrary it is clear that for any $\epsilon > 0$ we can choose them so that $x_k := z_k/\|z_k\|$ satisfies the claim of the theorem. ■

REMARKS:

(1) The converse of Theorem (2.8) is also true (and easier to show): If there exists a sequence $(x_n) \subset S_X$ satisfying the condition in Theorem (2.8) for some $\epsilon \in [0, 1]$, then X is non-reflexive.

(2) We can deduce Corollary (2.8.1) from Step 2 of the above proof, this is essentially Bae's Proof: From Step 2) we have

$$0 < d := \lim_n \operatorname{diam}(K_n'') = \lim_n \inf_{x \in K_n''} \lim_m \operatorname{dist}(x, K_m).$$

Thus given any $\epsilon > 0$ we have for n sufficiently large;

$$|d - \operatorname{diam}(K_n'')| < \epsilon$$

while

$$\begin{aligned} \operatorname{rad}(K_n'') &:= \inf_{x \in K_n''} \sup_{y \in K_n''} \|x - y\| \\ &\geq \inf_{x \in K_n''} \operatorname{dist}(x, K_m''), \quad \text{all} \quad m > n \\ &\geq d - \epsilon. \end{aligned}$$

Hence

$$\frac{\operatorname{rad}(K_n'')}{\operatorname{diam}(K_n'')} \geq \frac{d - \epsilon}{d + \epsilon}$$

can be chosen arbitrarily close to 1.

The converse to corollary (2.8.1) is spectacularly false. Indeed reflexivity is far from sufficient even for normal struture.

There exist reflexive spaces lacking normal structure. We begin with the following.

(2.9) REMARK: Uniform normal structure is stable under small perturbations of the space. This follows from the readily verified inequality : For two spaces X and Y we have

$$\frac{1}{d(X,Y)}N(X) \leq N(Y) \leq d(X,Y)N(X),$$

where $d(X,Y)$ is the **Banach-Mazur distance** between X and Y;

$$d(X,Y) := \text{infimum} \quad \|U^{-1}\| \, \|U\|.$$

where the infinum is taken over all linear isomorphisms U of X onto Y.

In particular then, if $|||\cdot|||$ is an equivalent norm of X satisfying

$$m\|x\| \leq |||x||| \leq M\|x\|, \quad \text{for all} \quad x \in X$$

we have

$$\frac{m}{M}N(X, \|\cdot\|) \leq N(X, |||\cdot|||) \leq \frac{M}{m}N(X, \|\cdot\|).$$

As a consequence we have the following.

(2.9.1) EXAMPLE. For $\alpha \in (0,1]$ let X_α denote ℓ_2 with norm $\|x\|_\alpha := (\alpha\|x\|_2) \vee \|x\|_\infty$.

$\|\cdot\|_\alpha$ is an equivalent norm for ℓ_2, indeed

$$\alpha\|x\|_2 \leq \|x\|_\alpha \leq \|x\|_2.$$

Hence, X_α **is reflexive** $(0 < \alpha \leq 1)$ **and by the above remark** X_α **has uniform normal structure for** $\alpha > \frac{1}{\sqrt{2}}$. In contrast, **for** $0 < \alpha \leq \frac{1}{\sqrt{2}}$ X_α **fails to have even normal structure:** $C := \overline{\text{co}}\{e_n\}_{n=1}^{\infty}$ has diameter $\sqrt{2}$ in the 2-norm and diameter 1 in the ∞-norm. So, for $\alpha \leq \frac{1}{\sqrt{2}}$, C has diameter 1 in X_α. Now, for any $x \in C$ we have

$$\|x - e_n\|_\alpha \geq \|x - e_n\|_\infty \to 1,$$

That is; C is diametral and so X_α fails to have normal structure.

The space $X_{\frac{1}{2}}$ was introduced by R.C. James explicitly for the purpose of this example, and has played an important role. Karlovitz [1976 (b)] demonstrated the FPP for $X_{\frac{1}{\sqrt{2}}}$. Indeed as we will subsequently see X_α has the FPP for all $\alpha > 0$. This will provide us with the first of many examples which demonstrate that normal structure, while sufficient, is not necessary for the w-FPP (FPP in reflexive spaces.).

A localized version of uniform convexity was introduced by Lovaglia [1955]. Say the Banach space X is **locally uniformly convex** if given x with $\|x\| = 1$ and a sequence (y_n) with $\|y_n\| = 1$ such that $\|\frac{x+y_n}{2}\| \to 1$ we have that $\|x - y_n\| \to 0$.

We will see that local uniform convexity, even with the additional assumption of reflexivity, is not sufficient to ensure normal structure.

(2.9.2) EXAMPLE [Smith and Turett, 1982]: A reflexive locally uniformly convex space which lacks normal structure.

Let us begin by recalling Day's [1955] locally uniformly convex renorming of c_o given by

$$\|x\|_D := \|Dx\|_2.$$

Here $D: c_o \to \ell_2$ is the mapping

$$Dx(n) = \begin{cases} x_{n_k}/2^k & \text{if} \quad n = n_k \quad \text{for some } k \\ 0 & \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $(n_k)_{k=1}^{\infty}$ is an enumeration of the support of x so that $|x_{n_k}| \geq |x_{n_{k+1}}|$. For details see Rainwater [1969]. The subadditivity for $\|\cdot\|_D$ being the main difficulty.

Now define $T: \ell_2 \to c_o$ by

$$Tx := (\frac{1}{2}\|x\|_2, x_1, x_2, x_2, x_3, x_3, x_3, \ldots), \qquad \text{where } x = (x_1, x_2, x_3, \ldots),$$

then

$$\|x\| := \|Tx\|_D$$

defines an equivalent locally uniformly convex norm on ℓ_2 satisfying

$$\frac{1}{4}\|x\|_2 \leq \|x\| \leq \frac{1}{\sqrt{3}}\|x\|_2$$

(Day's norm on c_o satisfies $\frac{1}{2}\|x\|_\infty \leq \|x\|_D \leq \frac{1}{\sqrt{3}}\|x\|_\infty$).

To see that $(\ell_2, \|\cdot\|)$ fails normal structure, let (surprise, surprise!)

$$C := \overline{\text{co}}\{e_n\}_{n=1}^{\infty}$$

Then, for $n < m$ we have:

$$\|e_n - e_m\| = \|\frac{1}{\sqrt{2}}, 0, \ldots, \underbrace{1, 1, \ldots, 1}_{n}, 0, \ldots, \underbrace{-1, -1, \ldots, -1}_{m}, 0, \ldots\|_D$$

$$= \|\frac{1}{2^{m+n+1}\sqrt{2}}, 0, \ldots, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^m}, 0, \ldots, -\frac{1}{2^{m+1}}, \ldots, -\frac{1}{2^{m+n}}, 0, \ldots, \|_2$$

$$\leq (\sum_{k=1}^{\infty} \frac{1}{4^k})^{\frac{1}{2}} = \frac{1}{\sqrt{3}},$$

so $\operatorname{diam} C \le \frac{1}{\sqrt{3}}$.

On the other hand for $x_1, x_2, \ldots, x_n \ge 0$ with $\sum_1^n x_k = 1$ we have

$$\begin{aligned}
&\|\sum_1^n x_k e_k - e_{n+1}\| \\
&= \|\underbrace{\frac{1}{2}\|\sum x_k e_k - e_{n+1}\|_2}_{\le\frac{1}{\sqrt{2}}}, x_1, x_2, x_2, \ldots, \underbrace{x_n, \ldots, x_n}_{n}, \underbrace{-1, \ldots, -1}_{n+1}, 0, \ldots \|_D \\
&\ge \sum_{k=1}^{n+1} \frac{1}{4^k}
\end{aligned}$$

(as there are at least $n+1$ entries of absolute value one and all other entries have smaller absolute values). Thus $\operatorname{rad} C \ge \sup_n \sum_{k=1}^{n+1} \frac{1}{4}k = \frac{1}{\sqrt{3}}$, and so we may conclude that C is diametral.

A convexity condition weaker than uniform convexity which does imply normal structure was introduced by Garkavi [1962]: X is **uniformly convex in every direction** (U.C.E.D.) if whenever (x_n) and (y_n) are sequences in B_X for which $\|x_n + y_n\| \to 2$ and for some $z \ne 0, x_n - y_n = \lambda_n z$ we then have $\lambda_n \to 0$. Geometrically this means that for each "direction" $z \ne 0$, the collection of all chords of the unit ball which are parallel to z and whose lengths are bounded away from zero have midpoints which lie uniformly deep inside the ball.

(2.10) THEOREM [Garkavi, 1962]: X is U.C.E.D. if and only if the Chebyshev centre of each closed bounded convex subset of X consists of at most one point. In particular, such spaces have normal structure.

Proof ($\Longrightarrow$) Let C be a closed bounded convex subset of X and suppose $x_1, x_2 \in \mathcal{C}(C)$, clearly $x_o = \frac{1}{2}(x_1 + x_2) \in \mathcal{C}(C)$ also. Choose $(y_n) \in C$ so that

$$\|y_n - x_o\| \to r = \operatorname{rad}(C).$$

Let $u_n = \frac{1}{r}(x_1 - y_n)$ and $v_n = \frac{1}{r}(x_2 - y_n)$ then we have

$$\begin{aligned}
\|u_n\|, \|v_n\| \le 1, \quad \|u_n + v_n\| &= \frac{2}{r}\|x_o - y_n\| \\
&\to 2
\end{aligned}$$

and so, since X is U.C.E.D. and $u_n - v_n = \frac{1}{r}(x_1 - x_2)$ for all n we conclude that $x_1 = x_2$.

($\Longleftarrow$) Suppose X is not U.C.E.D. Then there exists $z \neq 0$ and $(x_n), (y_n) \subset B_X$ with $\|x_n + y_n\| \to 2$ and $x_n - y_n = \lambda_n z$ where $\lambda_n \geq \lambda > 0$.

Let

$$C := \overline{\mathrm{co}}\{\lambda z/2 \pm \frac{1}{2}(x_n + y_n) : n = 1, 2, \ldots\}.$$

Clearly $\mathrm{rad}(C) = 1$ with $0 \in \mathcal{C}(C)$, further

$$\begin{aligned} \mathrm{rad}(C, \frac{\lambda z}{2}) &= \sup_n \|\frac{\lambda z}{2} \pm \frac{x_n + y_n}{2}\| \\ &= \sup_n \|(\frac{\lambda}{2\lambda_n} \pm \frac{1}{2})x_n - (\frac{\lambda}{2\lambda_n} \mp \frac{1}{2})y_n\| \\ &\leq (\frac{1}{2} \pm \frac{\lambda}{2\lambda_n}) + (\frac{1}{2} \mp \frac{\lambda}{2\lambda_n}) = 1 \end{aligned}$$

so $\frac{\lambda z}{2} \in \mathcal{C}(C)$. ∎

A detailed study of U.C.E.D. was made by Day, James and Swaminathan [1971]. They obtained several equivalent formulations of the property, investigated the stability of U.C.E.D. under products and obtained a number of results concerning spaces which admit an equivalent U.C.E.D. norm. We will be interested in a special case, previously obtained by Zizler [ ].

(2.11) THEOREM: Every separable Banach space admits an equivalent U.C.E.D. norm.

Proof. Let $(x_n) \subset S_X$ be a dense sequence in the unit sphere of the separable Banach space X. Choose $f_n \in S_{X^*}$ such that $f_n(x_n) = 1$. Then (f_n) is a strictly norming subset of X^* for X, so the mapping

$$T : X \to \ell_2 : x \mapsto (f_n(x)/2^n)_{n=1}^{\infty}$$

is continuous, $1 - 1$ and linear. Hence

$$|||x||| := (\|x\|^2 + \|Tx\|_2^2)^{\frac{1}{2}}$$

is an equivalent norm on X. We show it is U.C.E.D. Let $(x_n), (y_n)$ be such that $x_n - y_n = \lambda_n z$ with $z \neq 0$ and for which $|||y_n||| \leq 1, |||x_n|| = |||y_n + \lambda_n z||| \leq 1$ while $|||x_n + y_n||| = |||2y_n + \lambda_n z||| \to 2$. Then we must have

$$2|||y_n|||^2 + 2|||y_n + \lambda_n z|||^2 - |||2y_n + \lambda_n z|||^2 \to 0$$

or

$$\begin{aligned}(2\|y_n\|^2 + 2\|y_n + \lambda_n z\|^2 - \|2y_n + \lambda_n z\|^2) \\ + (2\|Ty_n\|_2^2 + 2\|T(y_n + \lambda_n z)\|_2^2 - \|T(2y_n + \lambda_n z)\|_2^2) \\ = (2\|y_n\|^2 + 2\|y_n + \lambda_n z\|^2 - \|2y_n + \lambda_n z\|^2) + \|T(\lambda_n z)\|_2^2 \\ \to 0,\end{aligned}$$

and so, since the first term is positive (convexity of the function $\frac{1}{2}\|\cdot\|^2$), we have $\lambda_n\|Tz\|_2 \to 0$. Since T is $1-1$ it follows that $\lambda_n \to 0$ and $(X, |||\cdot|||)$ is U.C.E.D.

∎

(2.11.1) COROLLARY: There exist non-reflexive spaces which are U.C.E.D. In particular there exist non-reflexive spaces with normal structure. (2.11.2) REMARKS:

(1) A more specific example of the last corollary is obtained by taking $X = \ell_1$ and noting that the argument in the proof of theorem (2.11) with $T = I$ shows that $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}$ is an equivalent U.C.E.D. norm on ℓ_1. Hence $(\ell_1, \|\cdot\|)$ has normal structure, however, it fails to have the F.P.P. Let $C := \{x \in \ell_1 : \|x\|_1 = 1,\ x_k \geq 0 \text{ all } k\}$, then the right shift $T: x \mapsto (0, x_1, x_2, \ldots)$ is a fixed point free isometry on C in $(\ell_1, \|\cdot\|)$. It does however have the w-F.P.P. As far as I know this example is due to E.Lami-Dozo.

(2) In case X is the dual of a separable space the proof of theorem (2.11) is readily adpated to obtain an equivalent **dual** U.C.E.D. norm.

(3) An alternative equivalent renorming of separable spaces (Separable dual spaces) which ensures w-normal structure (w^*-normal structure) is given by van Dulst [1982].

Van Dulst [1982] also shows that every Banach space can be equivalently renormed to fail normal structure.

(2.12) THEOREM: Every Banach space X admits an equivalent norm $|||\cdot|||$ so that $(X, |||\cdot|||)$ fails to have normal structure.

Proof. Let (b_n) be a normalized basic sequence in X (see Beauzamy [1982] Ch. II§1) with coefficient functionals c_n extended to X. That is; $c_n \in X^*, c_n(b_m) = \delta_{mn}$ and $\|c_n\| \leq K$ for some $K > 0$ and all n.

Let

$$x_n := b_o + 2b_n$$

and

$$f_n := c_o - c_n \quad \text{for} \quad n = 1, 2, \ldots.$$

Then

$$f_n(x_m) = \begin{cases} 1 & \text{if } n \neq m \\ -1 & \text{if } n = m. \end{cases}$$

Define

$$|||x||| = (\sup_n |f_n(x)|) \vee (\frac{1}{3}\|x\|),$$

then $\frac{1}{3}\|x\| \leq |||x||| \leq K\|x\|$, so $|||\cdot|||$ is an equivalent norm on X. Further, since $\|x_n\| \leq 3$, we have $|||x_n||| = 1$, and so $1 \geq |||f_n||| \geq |f_n(x_n)| = 1$.

Now let $C = \text{co}\{x_n\}_{n=1}^{\infty}$, clearly $\text{diam}(C) \leq 2$ in $(X.|||\cdot|||)$, while for any $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ we have

$$|||\sum_{i=1}^{n} \lambda_i x_i - x_{n+1}||| \geq f_{n+1}(\sum_{i=1}^{n} \lambda_i x_i - x_{n+1}) = 2$$

Thus C is diametral with (x_n) a diameterizing sequence. ∎

We conclude this chapter with a few observations on **normal structure as a Banach space property.** As we shall see our knowledge of the situation is far from complete.

Stability under isomorphisms, or lack there-of, has already be considered; theorem 2.11, theorem 2.12. and remark 2.9.

Subspaces and quotients. Clearly (uniform) normal structure is inherited by subspaces.

M. Smith and B. Turett [1988a] have constructed an example showing that normal structure need not be inherited by a quotient. This leaves open the following question.

QUESTION: Is uniform normal structure inherited by quotients, or at least do such quotients enjoy normal structure?

Smith and Turett [1988b] have also shown that uniform normal structure is not a self-dual property and does not imply any degree of k-*uniform rotundity.*

Stability under substitution. The oldest result in this direction is due to Belluce, Kirk, and Steiner [1968].

(2.13) PROPOSITION: For Banach spaces X_1 and X_2 let $X := (X_1 \oplus X_2)_\infty$ and for $i = 1, 2$ let $P_i : X :\to X_i : (x(1), x(2)) \mapsto x(i)$ be the natural coordinate projection. Then if C is a closed bounded convex subset of X and $C_i = P_i(C)$ we have

$$\frac{\mathrm{rad}\,(\mathrm{C})}{\mathrm{diam}\,(\mathrm{C})} \le \frac{1}{2}[1 + \frac{\mathrm{rad}\,(\mathrm{C}_1)}{\mathrm{diam}\,(\mathrm{C}_1)} \vee \frac{\mathrm{rad}\,(\mathrm{C}_2)}{\mathrm{diam}\,(\mathrm{C}_2)}].$$

Proof For $\epsilon > 0$ and $i = 1, 2$ choose $y_i \in C_i$ such that $\|y_i - z\| \le \mathrm{rad}\,(\mathrm{C_i}) + \epsilon$ for all $z \in C_i$. Select $x_i \in P_i^{-1}(y_i)$ and set $x_o = \frac{1}{2}(x_1 + x_2)$. Then for any $x \in C$ we have

$$\|x - x_o\| = \max_{i=1,2}\{\|x(i) - x_o(i)\|_i\}$$

Now

$$\begin{aligned} &\|x(1) - x_o(1)\|_1 \\ &\quad = \|x(1) - \frac{1}{2}(x_1(1) + x_2(1))\|_1 \\ &\quad = \|x(1) - \frac{1}{2}(y_1 + x_2(1))\|_1 \\ &\quad \le \frac{1}{2}[\|x(1) - y_1\|_1 + \|x(1) - x_2(1)\|_1] \\ &\quad \le \frac{1}{2}[\,\mathrm{rad}\,(\mathrm{C}_1) + \epsilon + \mathrm{diam}\,(\mathrm{C}_1)] \end{aligned}$$

Similarly

$$\begin{aligned} &\|x(2) - x_o(2)\|_2 \\ &\quad \le \frac{1}{2}[\,\mathrm{diam}\,(\mathrm{C}_2) + \mathrm{rad}(\mathrm{C}_2) + \epsilon]. \end{aligned}$$

Since ϵ is arbitrary it follows that

$$\begin{aligned} \mathrm{rad}\,(\mathrm{C}) &\le \mathrm{rad}\,(\mathrm{C}, \mathrm{x_o}) \\ &\le \frac{1}{2}[\,\mathrm{diam}\,(\mathrm{C}) + \mathrm{rad}\,(\mathrm{C}_1) \vee \mathrm{rad}\,(\mathrm{C}_2)] \end{aligned}$$

The conclusion now follows since,

$$\begin{aligned} \mathrm{diam}\,(\mathrm{C_i}) &\le \mathrm{diam}\,(\mathrm{C}) \\ &\le \|(\,\mathrm{diam}\,(\mathrm{C}_1),\ \mathrm{diam}\,(\mathrm{C}_2))\|_\infty \end{aligned}$$

so $\mathrm{diam}\,(\mathrm{C}) = \max\{\,\mathrm{diam}\,(\mathrm{C}_1),\ \mathrm{diam}\,(\mathrm{C}_2)\}$. ∎

(2.13.1) COROLLARY: For X_1, X_2 and X as in proposition (2.13) we have,

$$N(X) \le \frac{1}{2}[1 + N(X_1) \vee N(X_2)].$$

So a finite ℓ_∞ – sum of spaces each of which has uniform normal structure, also has uniform normal structure.

(2.13.2) COROLLARY [Belluce, Kirk and Steiner, 1968]: A finite ℓ_∞ – sum of spaces, each of which has normal structure, also has normal structure.

This last corollary has been extended by Landes [1984, See also, the comprehensive survey of Kirk, 1983]. He obtains very general conditions on the substitution space's norm for a finite sum of spaces with normal structure to also have normal structure. The following is another particular case of Landes' work, which together with the above corollaries is adequate for most purposes.

(2.14) THEOREM [Landes, 1984]: Let F denote a uniformly convex Banach space of functions with countable support from Γ into $\mathbf{R}$ such that if $f \in F$ and $g : \Gamma \to \mathbf{R}$ satisfies $|g(\gamma)| \leq |f(\gamma)|$, then $g \in F$ and $\|g\|_F \leq \|f\|_F$. It will sometimes be convenient to identify $f \in F$ with the indexed family $(f(\gamma))_{\gamma \in \Gamma}$.

If $(X_\gamma)_{\gamma\in\Gamma}$ is a family of Banach spaces each of which has normal structure, then the substitution space F, consisting of those functions f $: \Gamma \to \bigcup_\Gamma X_\gamma$ with f $(\gamma) \in X_\gamma$ and for which the function $(\|\mathbf{f}\ (\gamma)\|_\gamma)_{\gamma\in\Gamma}$ is in F, also has normal structure with respect to the substitution norm

$$\|\mathbf{f}\ \|_{\mathbf{F}} := \|(\|\mathbf{f}(\gamma)\|_\gamma)_{\gamma\in\Gamma}\|_F.$$

Proof Suppose F fails normal structure, then we can find a diametral sequence $(\mathbf{f}_n)$ with $d :=$ diam co$\{\mathrm{f_n}\}_{\mathrm{n=1}}^\infty > 0$. By the *diagonal* extraction of a subsequence we may assume that $z_k(\gamma) := \lim_n \|\mathbf{f}_n(\gamma) - \mathbf{f}_k(\gamma)\|_\gamma$ exists for each k and γ.

Now let $\mathbf{f} \in$ co$\{\mathrm{f_n}\}_{\mathrm{n=1}}^\infty$, then

$$\lim_n \|(\|\mathbf{f}_n(\gamma) - \mathbf{f}_k(\gamma)\|_\gamma)_{\gamma\in\Gamma}\|_F := \lim_n \|\mathbf{f}_n - \mathbf{f}_k\|_F = d,$$
$$\lim_n \|(\mathbf{f}_n(\gamma) - \mathbf{f}(\gamma)\|_\gamma)_{\gamma\in\Gamma}\|_F := \lim_n \|\mathbf{f}_n - \mathbf{f}\|_{\mathbf{F}} = d$$

and

$$\|\frac{1}{2}[(\|\mathbf{f}_n(\gamma) - \mathbf{f}_k(\gamma)\|_\gamma)_{\gamma\in\Gamma} + (\|\mathbf{f}_n(\gamma) - \mathbf{f}(\gamma)\|_\gamma)_{\gamma\in\Gamma}\|_F$$
$$\geq \|(\|\mathbf{f}_n(\gamma) - \frac{1}{2}[\mathbf{f}_k(\gamma) + \mathbf{f}(\gamma)]\|_\gamma)_{\gamma\in\Gamma}\|_F$$
$$:= \|\mathbf{f}_n - \frac{1}{2}[\mathbf{f}_k + \mathbf{f}]\|_{\mathbf{F}} \to d.$$

So, by the uniform convexity of F we have

$$(\|\mathbf{f}_n(\gamma) - \mathbf{f}_k(\gamma)\|_\gamma - \|\mathbf{f}_n(\gamma) - \mathbf{f}(\gamma)\|_\gamma)_{\gamma\in\Gamma} \to 0$$

and so

$$\lim_n \|\mathbf{f}_n(\gamma) - \mathbf{f}(\gamma)\|_\gamma = z_k(\gamma) \text{ for all } \mathbf{f} \in \operatorname{co}\{\mathbf{f}_n\}_{n=1}^\infty.$$

It follows that for each $\gamma \in \Gamma$

$$\lim_n \|\mathbf{f}_n(\gamma) - \mathbf{f}(\gamma)\|_\gamma$$

is constant for all $\mathbf{f} \in \operatorname{co}\{\mathbf{f}_n\}_{n=1}^\infty$.

Since X_γ has normal structure it follows from proposition (1.19) that the constant must be zero. But, then $(\mathbf{f}_n(\gamma))_{n=1}^\infty$ is a constant sequence (see that remark after proposition 1.19) and hence so too is $(\mathbf{f}_n)$, contradicting $d > 0$. ■

In the case when Γ is a finite set, F is finite dimensional and so uniform convexity and strict convexity coincide. Hence we have the following.

(2.14.1) COROLLARY: If $X_1, X_2, \ldots, X_n$ are Banach spaces each of which has normal structure and if $\|\cdot\|_c$ is a strictly convex norm on $\mathbf{R}^n$, then $X_1 \oplus X_2 \oplus \ldots \oplus X_n$ has normal structure with respect to the norm;

$$\|(x_1, x_2, \ldots, x_n)\| := \|(\|x_1\|_1, \ldots, \|x_n\|_n)\|_c.$$

QUESTION: If in theorem (2.14) there exists $k < 1$ so that $N(X_\gamma) \le k$ for all $\gamma \in \Gamma$ does it follow that $\mathbf{F}$ has uniform normal structure?

Landes' results have recently been extended by Smith and Turett [1987]. They show that for $1 < p < \infty$ the Bochner $L_p -$ space, $L_p(\mu, X)$, has normal structure exactly when X has normal structure.

3-space properties for (uniform) normal structure. Again no really satisfactory answer seems to be known. A very partial result in this direction is the following.

(2.15) PROPOSITION [Giles, Sims and Swaminathan, 1985]: Let M be a complemented subspace of X such that M has finite co-dimension in X (or more generaly M has a complement that is a Schur space). Then X has (weak) normal structure if M has uniform normal structure.

Proof Since M has uniform normal structure it is reflexive (Corollary 2.8.1) and so, since M is of finite co-dimension, X too is reflective.

Now suppose X does not have (weak) normal structure, then, by Remark 1.18 (1), X contains a weakly null sequence (x_n) satisfying:

$$\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n) \to 1.$$

Let P be the linear projection from X onto M with $I - P$ a projection onto the (Schur) complement of X. Since $x_n \rightharpoonup^w 0$ we therefore have $\|x_n - Px_n\| \to 0$.

Choose $\epsilon > 0$ so that $\epsilon < \frac{1-N}{4(1+N)}$ where $N := N(M)$, then there exists n_o so that

$$\|x_n - Px_n\| \leq \epsilon \quad \text{for all} \quad n \geq n_o.$$

Let $C := \mathrm{co}\,\{\mathrm{x_n}\}_{\mathrm{n=n_0}}^{\infty}$, then since $\|x - Px\| \leq \epsilon$ for all $x \in C$ we have

$$\begin{aligned} \mathrm{diam\,P(C)} &\leq \mathrm{diam\,C} + 2\epsilon \\ &= 1 + 2\epsilon. \end{aligned}$$

From the uniform normal structure of M we can find a point $x_o \in C$ such that

$$\|Px_o - Px\| \leq N(1 + 2\epsilon) \quad \text{for all} \quad x \in C.$$

But then, for each $x \in C$ we have

$$\begin{aligned} \|x - x_o\| &\leq \|x - Px\| + \|Px - Px_o\| + \|Px_o - x_o\| \\ &\leq 2\epsilon + N(1 + 2\epsilon) \\ &< \frac{1}{2}(1 + N), \text{by the choice of } \epsilon \\ &< 1 \end{aligned}$$

contradicting the diametrality of C. ∎

3. FURTHER CONDITIONS FOR NORMAL STRUCTURE AND THE FPP.

(3.1) THE CONDITION OF OPIAL.

A Banach space [dual space] X satisfies the **weak [weak*] Opial condition** if whenever (x_n) converges weakly [weak*] to x_∞ and $x_0 \neq x_\infty$ we have

$$\liminf_n \|x_n - x_\infty\| < \liminf_n \|x_n - x_0\|.$$

If equality is allowed in the above inequality we will say X satisfies the **non-strict weak [weak*] Opial condition.**

In the *weak** case it is natural to allow (x_n) to be a net. Otherwise we would need to restrict attention to spaces with a *weak** -sequentially compact ball. [For example; the dual of a separable space, or more generally the dual of any smoothable space – Sullivan and Hagler, 1979].

Remark: Arguing by the extraction of appropriate subsequences readily establishes the equivalence of Opial conditions with seemingly weaker ones.

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The weak Opial condition was introduced by Zdjisław Opial [1967] to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a non-expansive map on a closed convex subset to a fixed point.

A more extensive examination of the condition was made by Gossez and Lami-Dozo [1972]. In particular they prove the following.

(3.1.1) THEOREM: If X is a Banach space satisfying the weak Opial condition then X has weak normal structure, in particular then X has the w-F.P.P.

Proof Suppose X fails to have w-normal structure then by Remark (1.20 (1)) X contains a nontrivial weakly null sequence (x_n) satisfying

$$\lim_n \|x_n\| = \operatorname{diam} \overline{\operatorname{co}}\{x_k\}_{k=1}^\infty = \lim_n \|x_n - x\|,$$

which contradicts the weak Opial condition. ∎

REMARKS:

(1) As observed by Tingley, in order to deduce weak normal structure it is sufficient to require that X satisfy a weaker condition than that of Opial, namely;

if $\|x_n\| \to 1$ and $x_n \to^w 0$ then

$$\limsup_n \|x_n - x\| > 1, \quad \text{for some } x \in \overline{\text{co}}\{x_k\}_{k=1}^{\infty}.$$

He raises the question of whether or not this last condition is equivalent to weak normal structure.

(2) The use of Mazur's result in the above proof leaves open the following.

QUESTION Does a dual space with the *weak Opial condition have w^*-normal structure?**

None the less we do have the following result, proved indirectly by Karlovitz [1976].

(3.1.2) PROPOSITION If X is a dual space satisfying the *weak** Opial condition, then X has the w^*-FPP.

Proof [van Dulst, 1982]. Let C be any w^*-compact convex subset of X and let $T : C \to C$ be a non-expansive map. Choose $(x_n) \subset C$ so that $\|x_n - Tx_n\| \to 0$ (an approximate fixed point sequence for T). Passing to a subnet if necessary we may assume that $x_n \xrightarrow{w^*} x_\infty$. Then

$$\begin{aligned} &\liminf_n \|Tx_\infty - x_n\| \\ &= \liminf_n \|Tx_\infty - Tx_n\| \\ &\le \liminf_n \|x_\infty - x_n\| \end{aligned}$$

contradicting the *weak** Opial conndition unless $Tx_\infty = x_\infty$. ∎

REMARK: An analogous argument to that in the proof of the above proposition gives a direct proof of the w-FPP for spaces satisfying the weak Opial condition. Indeed these arguments establish the following stronger result. **In a space satisfying the weak (weak*) Opial condition any weak (weak*) limit of an approximate fixed point sequence for a nonexpansive map T is a fixed point of T.**

QUESTION: What can be said of spaces satisfying the non-strict Opial conditions?

As we shall see in the next chapter the weak (weak*) Opial conditions are natural generalizations of the type of orthogonality conditions recently shown to be sufficient for a Banach lattice to have the $w(w^*)$–FPP.

We remark that $L_1[0,1]$ fails even the non-strict weak Opial condition. To see this, let f_n be the function obtained by extending periodically to all of $[0,1]$ the function defined on $[0, \frac{1}{n}]$ by

$$f_n(t) = \begin{cases} 2 & \text{for } 0 \le t \le \frac{1}{3n} \\ -1 & \text{for } \frac{1}{3n} < t \le \frac{1}{n} \end{cases}$$

It is readily seen that $f_n \underset{w}{\rightharpoonup} 0$, $\|f_n\|_1 = \frac{4}{3}$, for all n, while $\|f_n - f_o\|_1 = 1$ where $f_0(t) \equiv -1$. Thus the weak non-strict Opial condition is violated.

In order to facilitate the investigation of specific spaces we consider the relationship between the Opial conditions and other properties of the space. Karlovitz [1976] established intimate relationships between the weak [and *weak**] Opial condition and "approximate symmetry" in the Birkhoff-James notion of orthogonality, we however will persue a different line. Recall, the **duality map** for a Banach space X is

$$D: X \to 2^{X^*}: x \mapsto \{f \in X^*: f(x) = \|f\|^2 = \|x\|^2\}$$

By an (extended) **support mapping** for X we shall understand a mapping of the form

$$x \mapsto \mu(\|x\|)f$$

where μ is a strictly increasing continuous gauge function with $\mu(0) = 0$ and f is a selection from $D(\frac{x}{\|x\|})$.

Opial [1967] observed that a uniformly convex space which admits a weak to weak* sequentially continuous support mapping satisfies the weak Opial condition. That uniform convexity need not imply the existence of such a support mapping had been observed by Browder [1966] and Hayes and Sims (in connnection with operator numerical ranges). Indeed $L_4[0,1]$ does not have a weak to weak (equal to *weak**) continuous support mapping. Opial extended this to $L_p[0,1]$ for all $p \neq 2$. In fact, Fixman and Rao have characterized $L_p(\Omega, \sum, \mu)$ spaces with a weak to weak continuous support mapping as those spaces for which every $A \in \sum$ with $0 < \mu(A) < \infty$ contains an atom.

The early results were substantially improved by Gossez and Lami-Dozo [1972]. They showed that the assumption of uniform convexity is unnecessary:

A Banach space [dual space] with a weak [*weak] to *weak** sequentially continuous support mapping satisfies the weak [*weak**] Opial condition.** This condition is not however necessary: For $1 < p < q < \infty$ the space $(\ell_p \oplus \ell_q)_2$ satisfies the weak Opial condition, but [Bruck, 1969] no support maping is weak to weak continuous.

The result of Gossez and Lami-Dozo is an immediate corollary of

(3.1.3) THEOREM [Sims, 1984]: The Banach space [dual space with a w-sequentially compact ball] X satisfies the weak [*weak] Opial condition if and only if whenever (x_n) converges weakly [*weak**] to a non-zero limit x_∞ there is a $\delta > 0$ so that eventually $D(x_n)x_\infty \subset [\delta, \infty)$.**

Proof. ($\Longrightarrow$) Assume this were not the case, by pasing to a subsequence and scalling we can find (x_n) converging weakly [*weak**] to x_∞ with $1 = \|x_n\| \geq \|x_\infty\| > 0$ and $f_n \in D(x_n)$ such that $\lim_n f_n(x_\infty) \leq 0$.

But

$$\begin{aligned}1 &= \liminf_n \|x_n - 0\| \\ &> \liminf_n \|x_n - x_\infty\| \\ &\geq \liminf_n f_n(x_n - x_\infty) \\ &= \liminf_n (1 - f_n(x_\infty)) \\ &= 1 - \lim_n f_n(x_\infty)\end{aligned}$$

whence $\lim_n f_n(x_\infty) > 0$, a contradiction.

($\Leftarrow$) [a modification of the proof in Gossez and Lami-Dozo 1972.]

Using the integral representation for the convex function $t \mapsto \frac{1}{2}\|x+ty\|^2$ [Roberts and Varberg, 1973, 12 Theorem A] we have

$$\frac{1}{2}\|x+y\|^2 = \frac{1}{2}\|x\|^2 + \int_0^1 g^+(x+ty; y)dt$$

where

$$g^+(u;y) := \lim_{h \to 0+} \frac{\frac{1}{2}\|u+hy\|^2 - \frac{1}{2}\|u\|^2}{h}.$$

to establish the weak [*weak**] Opial condition it suffices to show that if (y_n) converges weakly [*weak**] to $y_\infty \neq 0$, then

$$\liminf_n \frac{1}{2}\|y_n\|^2 > \liminf_n \frac{1}{2}\|y_n - y_\infty\|^2.$$

Now,

$$\frac{1}{2}\|y_n\|^2 = \frac{1}{2}\|y_n - y_\infty\|^2 + \int_0^1 g^+(y_n - y_\infty + ty_\infty; y_\infty)dt$$

so

$$\begin{aligned}\liminf_n \frac{1}{2}\|y_n\|^2 \geq{}& \liminf_n \frac{1}{2}\|y_n - y_\infty\|^2 \\ &+ \liminf_n \int_0^1 g^+(y_n - y_\infty + ty_\infty; y_\infty)dt\end{aligned}$$

By Fatou's lemma [see for example, Halmos 1950] it is therefore sufficient to prove that

$$\liminf_n g^+(y_n - y_\infty + ty_\infty; y_\infty) > 0$$

for each $t \in (0,1)$.

But, by a well known characterization of the upper Gateaux derivative [see for example, Barbu and Precupanu, 1978, §2.1 Example 2° and Proposition 2.3] we have

$$g^+(y_n - y_\infty + ty_\infty; y_\infty) = \mathrm{Max}\{f(y_\infty): f \in D(y_n - y_\infty + ty_\infty)\}$$

and so since $y_n - y_\infty + ty_\infty$ converges weakly [*weak**] to $ty_\infty \neq 0$ we have for n sufficiently large and some $\delta > 0$ that

$$f(ty_\infty) > \delta \quad \text{for all} \quad f \in D(y_n - y_\infty + ty_\infty)$$

That is, for n sufficiently large (depending on t)

$$g^+(y_n - y_\infty + ty_\infty; y_\infty) \geq \frac{\delta}{t} > 0. \quad \blacksquare$$

(3.1.3.a) REMARK: From the ($\Rightarrow$) part of the above proof we see it is only necesary that X satisfy: **If (x_n) converges weakly [*weak**] to $x_\infty \neq 0$ we have $\liminf_n \max\{f(x_\infty): f \in D(x_n)\} > 0$. Thus for a space satisfying this we also have**

$$\liminf_n \min\{f(x_\infty): f \in D(x_n)\} > 0.$$

Precisely what this means in terms of the geometry of support mappings is unclear.

(3.1.4) EXAMPLES.

(1) **For $1 \leq p < \infty$ the space ℓ_p satisfies the weak Opial condition.** For $p = 1$ this follows by the Schur property. For $p > 1$ the duality map is single valued and given by

$$x = (x(1), x(2), \ldots) \mapsto \|x\|_p^{2-p}(|x(1)|^{p-1}\text{sgn}x(1), \ldots)$$

Thus, if $x_n \underset{w}{\rightharpoonup} x_\infty \neq 0$ we have

$$\|x_n\|_p^{p-2} f_n \underset{w}{\rightharpoonup} \|x_\infty\|_p^{p-2} f_\infty$$

where $f_i = D(x_i)$. Consequently, since $\liminf_n \|x_n\|_p \geq \|x_\infty\|_p$, we have

$$\liminf_n f_n(x_\infty) \geq \|x_\infty\|_p^2 > 0.$$

Further, if $p = 1$ and $x_n \rightharpoonup^{w^*} x_\infty \neq 0$, choosing $f_n \in D(x_n)$ so that

$$f_n(i) = |x_n(i)|\text{sgn } x_n(i)$$

we see that:

$$\text{given } \epsilon > 0 \text{ there exist } n_o \text{ so that } \sum_{i=n_o}^{\infty} |x_\infty(i)| < \epsilon$$

and so

$$f_n(x_\infty) \geq \sum_{i=1}^{n_0} f_n(i)x_\infty(i) - \|f_n\|\epsilon.$$

Since $\{\|f_n\|\}_{n=1}^{\infty}$ is bounded and $x_n(i) \to x_\infty(i)$ it follows that

$$\liminf_n f_n(x_\infty) \geq \|x_\infty\|^2$$

Thus $\lim_n f_n(x_\infty) = \|x_\infty\|^2 = f_\infty(x_\infty) > 0$, and so ℓ_1 **satisfies the** ***weak**** **Opial condition.** In particular, ℓ_p **has the** w**-FPP.**

(2) **For $p \neq 2$ the space $L_p[0,1]$ fails to satisfy the weak Opial condition.** The case $p=1$ has already been considered. The same example works in $L_p[0,1]$ for all $p \neq 2$. Indeed for the sequence f_n defined previously we have for any number $c \in [-2,1]$

$$\|f_n + c\|_p^p = \frac{1}{3}(2+c)^p + \frac{2}{3}(1-c)^p$$

is a minimum at c_o satisfying

$$\frac{2+c_o}{1-c_o} = 2^{\frac{1}{p-1}}.$$

That is;

$$\begin{aligned} p = 1 &\Rightarrow c_o = 1 \\ 1 < p < 2 &\Rightarrow 0 < c_o < 1 \\ p = 2 &\Rightarrow c_o = 0 \\ p > 2 &\Rightarrow -\frac{1}{2} \leq c_o < 0. \end{aligned}$$

In particular then for $p \neq 2$

$$\|f_n + c_o\|_p < \|f_n\| \qquad (\text{as } c_o \neq 0)$$

and so the space fails to satisfy the weak (non-strict) Opial condition.

(3) **The space $X_\alpha := (\ell_2, \|\cdot\|_\alpha)$ where $\|x_\alpha\| := \alpha\|x\|_2 \vee \|x\|_\infty$ has the non-strict weak Opial condition for $0 < \alpha \leq 1$, but fails to satisfy the weak Opial condition for any such α.**

To see that X_α has the non-strict Opial condition suppose, without loss of generality, that $x_n \rightharpoonup^w 0$ and that $\liminf_n \|x_n\| = 1$. By passing to a subsequence we may assume that either $\|x_n\|_2 \longrightarrow \frac{1}{\alpha}$ or $\|x_n\|_\infty \longrightarrow 1$. In the first case, for $x \neq 0$, $\liminf_n \|x_n - x\|_2 > \frac{1}{\alpha}$, so $\liminf_n \|x_n - x\|_\alpha > 1$. In the second case, if there exists an $x := (x(i))$ such that $\liminf_n \|x_n - x\| = 1 - \delta$ for some $\delta > 0$, then since $\lim_n x_n(i) = 0$ and so only finitely many of the x_n's can *nearly* achieve their norms on the i'th coordinate it follows that for infinitely many i we must have $|x(i)|$ *near to* δ making it difficult for x to live in ℓ_2.

Taking $(x_n) = (e_n)$, the sequence of standard basis vectors and $x := \frac{1-\alpha}{1+\alpha} \cdot \frac{\sqrt{3}}{2} \cdot (1, \frac{1}{2}, \frac{1}{4}, \cdots)$ shows that the space X_α fails to have the weak Opial condition, as we have;

$$\|e_n - x\|_\infty \uparrow 1 \quad \text{and} \quad \|e_n - x\|_2 \leq 1 + \frac{1-\alpha}{1+\alpha} < 1/\alpha,$$

and so $\liminf_n \|e_n\|_\alpha = 1 = \liminf_n \|e_n - x\|_\alpha$.

(3.1.5) REMARKS:

(1) **Relationship to U.C.E.D.** The uniformly convex space $L_4[0,1]$ of example (2) above amply demonstrates that: U.C.E.D. is not sufficient for the weak Opial condition to be satisfied.

On the other hand ℓ_1 has the weak Opial condition but is not even strictly convex, so weak Opial $\not\Rightarrow$ U.C.E.D.

Thus U.C.E.D. and the weak [*weak**] Opial conditions are effectively independent conditions sufficient to ensure weak [*weak**] normal structure.

(2) van Dulst [1982] has shown that every separable Banach space admits an equivalent norm with respect to which the weak Opial condition is satisfied.

He also gives an equivalent renorming for any separable dual space with respect to which the space satisfies the *weak** Opial condition. (This was the basis for Remark (2.11.2) (3).

(3.2) UNIFORM KADEC - KLEE CONDITIONS.

The material of this section is a development of ideas in van Dulst-Sims [1983], which are based on notions introduced by Huff [1980].

Recall a Banach space has the **property of Kadec-Klee** (also known as Property H, and perhaps more properly termed the Radon-Riesz property) if whenever $x_n \rightharpoonup^w x$ and $\|x_n\| \to \|x\|$ we have $\|x_n - x\| \to 0$.

This may be reformulated as stating: **every weakly compact subset of the unit sphere;**

$$S_X := \{x \in X : \|x\| = 1\},$$

is norm compact.

Define the **measure of compactness** of a subset S by

$$\gamma(S) := \sup_{(x_n) \subseteq S} \inf_{m \neq n} \|x_m - x_n\|$$

The supremum being taken over all infinite sequences of points in S.

REMARKS:

(1) γ is equivalent to the "usual" measure of compactness;

$$K(S) := \inf\{\epsilon > 0 : S \text{ has a finite } \epsilon - \text{ cover }\},$$

indeed $K(S) \leq \gamma(S) \leq 2K(S)$.

(2) γ enjoys the following properties.

(a) $\gamma(S) = 0$ if and only if S is (norm) compact.

(b) If $S_1 \subseteq S_2$ then $\gamma(S_1) \leq \gamma(S_2)$

(c) $\gamma(S_1 \cup S_2) = \max\{\gamma(S_1), \gamma(S_2)\}$

(d) [Kuratowski] If $S_1 \supseteq S_2 \supseteq \ldots \supseteq S_n \supseteq \ldots$ is a nested sequence of closed non-empy sets with $\gamma(S_n) \to 0$, then

(i) $K = \bigcap_{n=1}^{\infty} S_n$ is non-empty and compact,

(ii) For any $\epsilon > 0$ and n sufficiently large $S_n \subseteq K + \epsilon B_X$.

(3) By theorem (1.16) a diametral set D has $\gamma(D) =$ diam (D).

(4) In terms of γ the Kadec-Klee property becomes: If S is a weakly compact subset of B_X with $\gamma(S) > 0$, then $\operatorname{dist}(S, 0) < 1$.

Given $\epsilon \in (0,1)$ we shall say that S is **ϵ-Uniformly Kadec-Klee (ϵ-UKK)** if there exists $\delta > 0$ so that whenever S is a weakly compact subset of B_X with $\gamma(S) > \epsilon$ we have $\operatorname{dist}(S, 0) \leq 1 - \delta$ or equivalently $S \cap (1-\delta)B_X \neq \phi$.

REMARK: ϵ-UKK may be compared to the notion of ϵ-inquadrate, where if S is *metrically big* (diam S $> \epsilon$) we have $S \cap (1-\delta)B_X \neq \phi$. Here the same conclusion follows if S is *topologically large* ($\gamma(S) > \epsilon$).

Note: our ϵ-UKK is the w-UKK of van Dulst-Sims [83].

(3.2.1) PROPOSITION: For a Banach space X t.f.a.e.

(i) X is ϵ-UKK

(ii) whenever $C \subseteq B_X$ is a weak compact <u>convex</u> set with $\gamma(C) > \epsilon$ we have $C \cap (1-\delta)B_X \neq \phi$.

(iii) whenever $(x_n) \subset B_X$ has $sep(x_n) := \inf_{m \neq n} \|x_n - x_m\| > \epsilon$ **and** $x_n \underset{w}{\rightharpoonup} x$ **we have** $\|x\| \leq 1 - \delta$.

Proof. Clearly, (i)$\Rightarrow$ (ii) and (iii)$\Rightarrow$ (i). *(ii) $\Rightarrow$ (iii).* Suppose there exists $(x_n) \subset B_X$ with $\text{sep}(x_n) > \epsilon, x_n \underset{w}{\rightharpoonup} x$ and $\|x\| > 1 - \delta$. Let f be a norm one linear functional which strictly separates x from $(1 - \delta)B_X$ and let n_o be such that for $n \geq n_o$ we have

$$f(x_n) \geq \frac{1}{2}(f(x) + 1 - \delta).$$

Then, $C = \overline{\text{co}}\{x_n\}_{n=n_o}^{\infty}$ is a weak compact convex set with $\gamma(C) > \epsilon$ and $f(y) \geq \frac{1}{2}(f(x) + 1 - \delta) > 1 - \delta$ for all $y \in C$. That is $C \cap (1 - \delta)B_X = \phi$. ∎

If X is ϵ-UKK for all $\epsilon \in (0, 1)$ we say X is **UKK.** Huff's original definition of this notion is the equivalent reformulation resulting from proposition (3.2.1) (iii).

(3.2.2) THEOREM [van Dulst-Sims, 83]: If X is $\epsilon - UKK$, then X has w-normal structure. In particular X has the w-FPP.

Proof. Suppose X contains a weak compact convex diametral set containing more than one point. Then, by Theorem (1.18) and the ensuing remark (1.20), there exists $(x_n) \subset X$ with $x_n \underset{w}{\rightharpoonup} 0$ and $\lim_n \text{dist}(x_{n+1}, \text{co}\{x_k\}_{k=1}^{n}) = \text{diam}\,\text{co}\{x_k\}_{k=1}^{n} = 1$. Since $0 \in \ \text{co}\,\{\text{x}_\text{k}\}_{\text{k}=1}^{\infty}$, by Mazur, it follows that $\|x_n\| \to 1$. Let δ be as in the definition of ϵ-UKK and choose n_o so that for $n \geq n_o$ we have $\text{dist}(x_{n+1}, \text{co}\{x_k\}_{k=1}^{n}) > \epsilon$ and $\|x_n\| > 1 - \delta$. Let $y_n = x_{n_0+n} - x_{n_o}$, then $\|y_n\| \leq 1, \text{sep}(y_n) > \epsilon$ and $y_n \underset{w}{\rightharpoonup} -x_{n_o}$. But $\|x_{n_o}\| > 1 - \delta$, contradicting ϵ-UKK by part (iii) of the previous proposition.

The above theorem may be strengthened as follows.

(3.2.3) THEOREM [van Dulst-Sims, 83]: Let X be UKK and let C be a weak compact convex set. Then, the Chebyshev centre of $C, \mathcal{C}(C)$, is norm compact (convex and non-empty).

Proof. Suppose $\mathcal{C}(C)$ is not compact, then it contains a sequence (x_n) with $\text{sep}(x_n) \geq \epsilon_o$, for some $\epsilon_o > 0$. By passing to a subsequence we may assume $x_n \underset{w}{\rightharpoonup} x$. Let δ be as in the definition of $\frac{\epsilon_o}{\text{rad}\,\text{C}} - UKK$, fix $y \in C$, and let $y_n := (x_n - y)/\ \text{rad}\,(\text{C})$. Then $\|y_n\| \leq 1, \text{sep}(y_n) \geq \epsilon_o/\ \text{rad}\,\text{C}$ and $y_n \underset{w}{\rightharpoonup} (x - y)/\ \text{rad}\,(\text{C})$, so by UKK $\|x - y\| \leq (1 - \delta)\,\text{rad}\,\text{C}$. Since y is arbitrary this gives $\text{rad}\,\text{C} \leq (1 - \delta)\,\text{rad}\,\text{C}$, a clear contradiction. ∎

(3.2.4) EXAMPLES.

(1) Vacuously every finite dimensional space and every Shur space has UKK. In particular ℓ_1 has UKK. Indeed any ℓ_1 sum of finite dimensional spaces has UKK [Huff, 80]. This

shows that in general UKK does not imply U.C.E.D. The next example establishes the essential independence of these two properties even in the presence of reflexivty.

(2) By theorem (2.11) the space

$$X := (\ell_2 \oplus \ell_3 \oplus \ldots \oplus \ell_n \oplus \ldots)_2$$

is a reflexive space which can be given an equivalent U.C.E.D. norm. However, as we now show, it admits no equivalent UKK norm [Huff, 1980].

For any Banach space X and $\epsilon > 0$ define for $S \subseteq X$

$$\beta_\epsilon(S) := \{x : \text{there exists } (x_n) \subseteq S \text{ with } \operatorname{sep}(x_n) > \epsilon \text{ and } x_n \underset{w}{\rightharpoonup} x\}$$

Claim If X has an equivalent UKK norm, then for each $\epsilon > 0$ there exists n_o such that $\beta_\epsilon^{n_o}(B_X) = \phi$.

Since the conclusion is isomorphically invariant we might as well assume that the UKK norm is the given one. Now let δ be that associated with ϵ in the definition of UKK, then by (3.2.1)(iii) $\beta_\epsilon(B_X) \subseteq (1-\delta)B_X$, iterating $\beta_\epsilon^n(B_X) \subseteq (1-\delta)^n B_X$. Choosing n_o so that $(1-\delta)^{n_o - 1} < \frac{\epsilon}{2}$ we see from the definition of $\beta_\epsilon(S)$ that $\beta_\epsilon^{n_o}(B_X)$ must be empty, as $\beta_\epsilon^{n_o - 1}(B_X)$ has diameter less than ϵ and so cannot contain any sequence with a separation constant of ϵ or more.

To see that $X = (\ell_2 \oplus \ldots \oplus \ell_2 \oplus \ldots)_2$ cannot be equivalently renormed to be UKK it suffices, in the light of the above claim, to show

$$\beta_{\frac{1}{2}}^{2^p}(B_{\ell_p}) \neq \phi.$$

Let $e_{n_1}, \ldots, e_{n_{2^p - 1}}$ be any $2^p - 1$ basis vectors in ℓ_p, then for

$$m > \max\{n_1, \ldots, n_{2^p - 1}\}$$

$$y_m = \frac{1}{2}(e_{n_1} + \ldots + e_{n_{2^p - 1}} + e_m)$$

is such that

$$\|y_m\|_p = 1, \quad \|y_m - y_n\|_p = \frac{1}{2}\|e_m - e_n\|_p > \frac{1}{2}$$

and $y_m \underset{w}{\rightharpoonup} \frac{1}{2}(e_{n_1} + \ldots + e_{n_{2^p - 1}})$.

Thus one half the sum of any $2^p - 1$ basis vectors is in $\beta_{\frac{1}{2}}(B_{\ell_p})$.

An identical calculation yields that one-half the sum of any $2^p - 2$ basis vectors is in $\beta_{\frac{1}{2}}^2(B_{\ell_p})$.

Continuing in this way we eventually arrive at $\frac{1}{2}e_n \in \beta_{\frac{1}{2}}^{2^p-1}(B_{\ell_p})$ for any n and so $0 = w - \lim\limits_n \frac{1}{2}e_n \in \beta_{\frac{1}{2}}^{2^p}(B_{\ell_p})$.

S. Swarminathan [Private Communication] using similar arguments, combined with a result of Rudin [1955], has shown that the space $H^2(\mu)$ cannot be equivalently renormed to be UKK.

(3) The space $L_4[0,1]$ shows that UKK need not imply the weak Opial condition. The previous example shows that the converse implication may also fail:

$$X := (\ell_2 \oplus \ldots \oplus \ell_n \oplus \ldots)_2$$

has the weak-Opial condition.

To see this, let $x_n = (x_n^{(k)})$, $x_n^{(k)} \in \ell_k$, converge weakly to $x_o \neq 0$. Let $f_n^{(k)}$ be the unique (by smoothness) element of $D(x_n^{(k)})$, then $f_n := (f_n^{(k)})$ is the unique element of $D(x_n)$. Choose k_o such that $x_o^{(k_o)} \neq 0$, then $x_n^{(k_o)} \xrightarrow[w]{} x_o^{(k_o)}$ and so by theorem (3.1.3) for ℓ_{k_o} we can find n_o and $\delta > 0$ so that for $n > n_o$ we have $f_n^{(k_o)}(x_o^{(k_o)}) > \delta$.

Now we can find a finite subset N of $\mathbf{N}$ so that $k_o \in N$,

$$(\sup_n \|f_n\|) \cdot (\sum_{k \notin N} \|x_o^{(k)}\|^2)^{\frac{1}{2}} < \frac{\delta}{2}$$

and, again by theorem (3.1.3), there is an $n_1 \geq n_o$ so that $n \geq n_1$ implies

$$f_n^{(k)}(x_o^{(k)}) \geq 0, \quad k \in N.$$

It follows that for $n > n_1$

$$\begin{aligned} f_n(x_o) &= \sum f_n^{(k)}(x_o^{(k)}) \\ &\geq \sum_{k \in N} f_n^{(k)}(x_o^{(k)}) - \frac{\delta}{2} \\ &\geq f_n^{(k_o)}(x_o^{(k_o)}) - \frac{\delta}{2} \\ &\geq \frac{\delta}{2}. \end{aligned}$$

That X has the weak Opial condition now follows from (3.1.3).

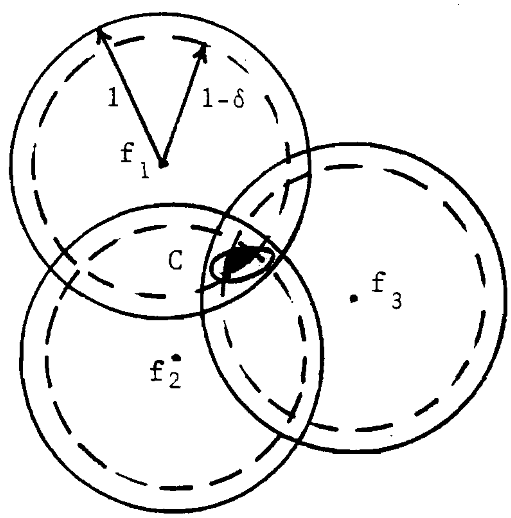
We now turn to **weak* -case**, when X^* is a dual space.

In view of the previous discussion it seems natural to say for $\epsilon > 0$ that X^* is $\epsilon - UKK^*$ if there exists $\delta > 0$ so that whenever C is a w^*-compact convex subset of B_{X^*} with $\gamma(C) > \epsilon$ we have $C \cap (1-\delta)B_{X^*} \neq \emptyset$. X^* is UKK^* if it is $\epsilon - UKK^*$ for all $\epsilon > 0$.

The appeal to Mazur's theorem in (3.2.2) precludes a similar argument for the w^*-case, non-the-less the conclusions remain valid. To see this we make use of the following.

(3.2.5) LEMMA. **Let X^* be an ϵ–UKK^* dual space and let δ be that associated with ϵ by the definition of $\epsilon - UKK^*$. If C is a w^*-convex subset of X with $\gamma(C) > \epsilon$ and if $f_1, f_2, \ldots, f_n$ are points of X^* with $C \subseteq B_1[f_i]$ for $i = 1, 2, \ldots, n$ then we have**

$$C \cap \bigcap_{i=1}^{n} B_{1-\delta}[f_i] \neq \phi.$$



PROOF. By the definition of $\epsilon - UKK^*$ the result is true for $n = 1$. Suppose the result were to fail, then there is a largest $n(\geq 1)$ for which it is true. Denote this largest n by n_o, then there exists a w-compact convex set $C \subseteq X^*$ with $\delta(C) > \epsilon$ and points $f_0, f_1, \ldots, f_{n_0}$ with $C \subseteq B_1[f_i]$ for $i = 0, 1, \ldots, n$ but for which

$$C \cap \bigcap_{i=0}^{n_0} B_{1-\delta}[f_i] = \phi.$$

Let $C_0 := C \cap \bigcap_{i=1}^{n_0} B_{1-\delta}[f_i]$ then, by the definition of $n_0, C_0 \neq \phi$. Further $C_0 \cap B_{1-\delta}[f_0] = \phi$ so there exists a w^*-continuous linear functional x and real number k with

$$\inf x(C_0) > k > \sup x(B_{1-\delta}[f_0]).$$

Let

$$C_1 := \{f \in C : x(f) \geq k\}$$

and let

$$C_2 := \{f \in C : x(f) \le k\}.$$

Then,

$$C_1 \subseteq C \subseteq B_1[f_o]$$

while

$$C_1 \cap B_{1-\delta}[f_o] = \phi.$$

Since C_1 is w^*-compact and convex it follows from $\epsilon - UKK^*$ that $\gamma(C_1) \le \epsilon$ and so since $C = C_1 \cup C_2$ we must have (by remark 2(c)) that $\gamma(C_2) > \epsilon$. But then, C_2 is a w^*-compact convex set with $\gamma(C_2) > \epsilon$ such that

$$C_2 \cap \bigcap_{i=1}^{n_0} B_{1-\delta}[f_i] \subseteq C_2 \cap C_0 = \phi$$

contradicting our choice of n_o as the largest value for which the implication held. ∎

(3.2.6) THEOREM [van Dulst-Sims, 83]: If X^* is a dual space with the $\epsilon - UKK^*$ property for some $\epsilon \in (0,1)$, then X^* has w^*-normal structure and hence in particular X^* has the w^*-FPP.

PROOF. Suppose not, then we can find a diametral w^*-compact convex subset K of X with $\operatorname{diam} \mathrm{K} = 1$. Then $\gamma(K) = 1 > \epsilon$ and for each $x \in K$, $K \subset B_1[x]$. Let $E_x = K \cap B_{1-\delta}[x]$, then E_x is a w^*-compact subset of K which is non-empty by the $\epsilon - UKK^*$ property. Further the above lemma ensures that the family E_x has the finite intersection property, and so by the w^*-compactness of K there exists $x_0 \in \bigcap_{x \in K} E_x$, but then for any $x \in K$ we have $x_0 \in E_x \subset B_{1-\delta}[x]$. So $\|x - x_0\| < 1 - \delta$, contradicting the diametrically of K. ∎

Indeed the stronger analogue of theorem (3.2.3) is true.

(3.2.7) THEOREM [van Dulst, Sims, 83]: Let X^* be a UKK^* dual space and C be a *weak compact convex subset of X^*. Then, $\mathcal{C}(C)$ is norm-compact.**

Proof. Suppose this were not the case, then we can find a *weak** compact convex subset of X^* with $\operatorname{diam} \mathrm{C} = 1$ and $\gamma(\mathcal{C}(C)) > \epsilon_0$ for some $\epsilon_0 > 0$. From the definition of $\operatorname{rad} \mathrm{C}$ it follows that

$$\mathcal{C}(C) \subseteq B_1[x] \quad \text{for each} \quad x \in C.$$

Let δ correspond with ϵ_0 in the definition of UKK^* then

$$E_x := \mathcal{C}(C) \cap B_{1-\delta}[x]$$

is a non-empty *weak** compact convex subset of C for each $x \in C$. The argument now proceeds along the same lines as those of the last part of the proof for theorem (3.2.6). ∎

We now consider necessary and sufficient conditions for a dual space to be $\epsilon - UKK^*$. Some conditions will be sufficient, others necessary.

Our first result shows that for the w^*-compact convex sets in the definition of $\epsilon - UKK^*$ it is sufficient to consider "w^*-slices of B_{X^*}".

(3.2.8) LEMMA: X^* is $\epsilon - UKK^*$ if and only if there exists a $k \in (0,1)$ such that for every norm one w^*–continuous linear functional f on X^* the slice of the dual unit ball

$$S[f,k] \;:=\; \{x \in X^* : \|x\| \leq 1 \textbf{ and } f(x) \geq 1\}$$

has $\gamma(S[f,k]) \leq \epsilon$.

PROOF. ($\Longrightarrow$) is obvious, since for any $k > 1-\delta$, where δ is given in the definition of $\epsilon - UKK^*$, we have that $S[f,k]$ is a nonempty w^*– closed convex subset of the ball which is disjoint from $B_{1-\delta}[0]$.

($\Longleftarrow$) Let K be a w^*– compact convex subset of the dual unit ball with $\gamma(K) > \epsilon$ and suppose that $K \cap B_{(1-\delta)}[0] = \emptyset$, where $\delta = 1-k$. Then we may separate K from $B_{1-\delta}[0]$ by a w^*– continuous linear functional f to obtain $\inf f(K) > \sup f(B_{1-\delta}[0]) = 1-\delta = k$. Thus $K \subseteq S[f,k]$ and so $\gamma(S[f,k]) \geq \gamma(K) > \epsilon$, contradicting our hypothesis. ∎

Recall the duality map

$$D: x \mapsto D(x) := \{f \in X^* : f(x) = \|f\|^2 = \|x\|^2\}$$

is **norm to norm upper semi-continuous** if given $\epsilon > 0$ and $x \in S_X$ there exists $\delta > 0$ such that for all $y \in S_X$ with $\|x-y\| < \delta$ we have $D(y) \subseteq D(x) + B_\epsilon[0]$. D is **uniformly norm to norm upper semi-continuous** if there exists a common δ for all $x \in S_X$.

(3.2.9) LEMMA: **D is norm to norm upper semi-continuous if and only if for each $\epsilon > 0$ and $x \in S_X$ there exists a $k \in (0,1)$ so that the slice $S[x,k] := \{f \in B_{X^*} : f(x) \geq k\} \subseteq D(x) + B_\epsilon[0]$. (The continuity is uniform if and only if k may be chosen independent of $x \in S_X$.)**

PROOF. ($\Longleftarrow$) Suppose $D(x) + B_\epsilon[0]$ contains the slice $S[x,k]$ determined by x. Then, setting $\delta := 1 - k$, for $y \in B_\delta[x] \cap S_X$ we have

$$|f(x) - 1| = |f(x) - f(y)| < \delta$$

for all $f \in D(y)$. That is, $D(y) \subseteq S[x,k]$ and the result follows.

($\Longrightarrow$) Suppose D is norm to norm upper semi-continuous then there exists $\delta_1 > 0$ so that $D(y) \subseteq D(x) + B_{\frac{\epsilon}{2}}[0]$ whenever $y \in B_{\delta_1}(x) \cap S_X$.

Let $\delta = \min\{\delta_1, \frac{\epsilon}{2}\}$ and let $k = 1 - \frac{\delta^2}{4}$. Then, for $f \in S[x,k]$ we have $|f(x) - 1| \leq \frac{\delta^2}{4}$, so by the Bishop-Phelps Bollobás theorem, there exists $y \in S(x)$ and $g \in D(y)$ such that $\|x - y\| < \delta$ and $\|f - g\| < \delta$. But then, $D(y) \subseteq D(x) + B_{\frac{\epsilon}{2}}[0]$ and so

$$\begin{aligned} f \in g + B_{\frac{\epsilon}{2}}[0] &\subseteq D(y) + B_{\frac{\epsilon}{2}}[0] \\ &\subseteq D(x) + B_\epsilon[0]. \end{aligned}$$

That is $S[x,k] \subseteq D(x) + B_\epsilon[0]$. ∎

(3.2.10) COROLLARY: **For the conditions listed below we have (i)$\Longrightarrow$ (ii)$\Longrightarrow$ (iii).**

(i) (a) D is norm to norm uniformly upper semi-continuous.

and

(b) For each $x \in S_X$, $D(x)$ is norm compact.

(ii) X^* has UKK^*.

(iii) (a) D is norm to norm upper semi-continuous

and

(b) For each $x \in S_X$, $D(x)$ is norm compact.

PROOF. (i) $\Longrightarrow$ (ii) Given any $\epsilon > 0$, from (3.2.9) there exists $k \in (0,1)$ such that for all $x \in S$ we have $S[x,k] \subseteq D(x) + B_\epsilon[0]$. From this and (i) (b) it follows easily that $\gamma(S[x,k]) < \epsilon$ and hence (ii) follows by (3.2.8).

(ii)$\Longrightarrow$ (iii) From (ii) via (3.2.8) we have $\gamma(S[x, 1 - \frac{1}{n}]) \to 0$ as $n \to \infty$, hence by (d) of Remark (1).

$D(x) = \bigcap_{n=1}^{\infty} S[x, 1 - \frac{1}{n}]$ is norm compact (giving (iii) (b)). Further for any $\epsilon > 0$ we have for n sufficiently large that

$$S[x, 1 - \frac{1}{n}] \subseteq D(x) + B_\epsilon[0]$$

from which (iii) (a) follows by (3.2.9). ∎

(3.2.11) EXAMPLES

The most obvious example is

(1) ℓ_1 **has** UKK^*: we prove this using the characterization given in (3.2.8). First observe: In ℓ_1 let $f = (f_i)_{i=1}^{\infty} \in S[x, k]$ where $x = (x_i) \in c_0$ with $\|x\|_\infty = 1$. That is; $\sum_{i=1}^{\infty} |f_i| \leq 1$ and $\sum_{i=1}^{\infty} x_i f_i \geq k$ where $x_i \to 0$ and $\max |x_i| = 1$.

Given any $\epsilon > 0$, let M be such that $|x_i| < \epsilon$ for $i > M$, then

$$\begin{aligned}
\sum_{i=M+1}^{\infty} |f_i| &\leq 1 - \sum_{i=1}^{M} |f_i| \\
&\leq 1 - \sum_{i=1}^{M} x_i f_i \\
&\leq 1 - (k - \sum_{i=M+1}^{\infty} x_i f_i) \\
&\leq (1 - k) + \epsilon.
\end{aligned}$$

Now, let $f_n = (f_i^n)$ be a sequence in $S[x, k]$ converging *weak** to $f \in S[x, k]$.

Then $|f_i^n - f_i^m| \to 0$ as $m, n \to \infty$ and

$$\begin{aligned}
\|f_n - f_m\| &= \sum_{i=1}^{M} |f_i^n - f_i^m| + \sum_{i=M+1}^{\infty} |f_i^n - f_i^m| \\
&\leq \sum_{i=1}^{M} |f_i^n - f_i^m| + 2(1 - k + \epsilon).
\end{aligned}$$

Thus

$$\begin{aligned}
\inf_{m \neq n} \|f_n - f_m\| &\leq \limsup_{m,n} \|f_n - f_m\| \\
&\leq 2(1 - k + \epsilon).
\end{aligned}$$

Since ϵ was arbitrary it follows that

$$\gamma(S[x,k]) \leq 2(1-k).$$

(2) An easy calculation establishes that if $(X, \|\cdot\|_1)$ has the $\epsilon - UKK$ [$\epsilon - UKK^*$] property (with corresponding δ), and $\|\cdot\|_2$ is an equivalent norm [dual norm] on X with $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$, then $(X, \|\cdot\|_2)$ has the $\epsilon' - UKK$ [$\epsilon' - UKK^*$] property where $\epsilon' = \frac{M}{m}\epsilon$ (with corresponding $\delta' = 1 - \frac{M}{m}(1-\delta)$), provided ϵ and $1-\delta$ are both less than $\frac{m}{M}$.

In particular it follows that the space $X_\alpha := (\ell_2, \|\cdot\|_\alpha)$ where $\|x_\alpha\| := \alpha\|x\|_2 \vee \|x\|_\infty$ is $\epsilon - UKK$ (for any $\epsilon > 2\sqrt{1-\alpha^2}$) provided $1 \geq \alpha > \frac{2}{\sqrt{5}}$. Infact X_α is $\epsilon - UKK$ for $1 \geq \alpha > \frac{1}{\sqrt{2}}$.

REMARKS :

(1) The space ℓ_1 shows that the implication (ii) $\Longrightarrow$ (i) of corollary (3.2.10) is not generally valid. The space $X = (\ell_2 \oplus \ell_3 \oplus \ldots \ell_n \oplus \ldots)_2$ provides a counter example to (iii) $\Longrightarrow$ (ii) of the same corollary.

(2) The conclusions of theorems (3.2.6) and (3.2.7) for ℓ_1 were first proved by Lim [1980]. Indeed for this space he obtains a stronger conclusion than that of (3.2.7), namely that the asymptotic centres with respect to a non-empty subset of ℓ_1 are norm compact. For C a non-empty subset of a Banach space X and $(A_\alpha : \alpha \in \Delta)$ a decreasing net of bounded non-empty subsets of X, let

$$r(x) := \inf_\alpha \operatorname{rad}(\mathrm{A}_\alpha, \mathrm{x}) = \lim_\alpha \operatorname{rad}(\mathrm{A}_\alpha, \mathrm{x}),$$

$$r := \inf_{x \in C} r(x)$$

and

$$A := \{x \in C : r(x) = r\}.$$

A is the **asymptotic centre** of $(A_\alpha : \alpha \in \Delta)$ with respect to C. If $A_\alpha \equiv C$ we obtain the Chebyshev centre of C, $\mathcal{C}(C)$. We say X has w (w^*)**-asymptotic normal structure** if for every $w(w^*)$ compact convex subset K of X containing more than one point the asymptotic centre of any decreasing net of non-empty subsets of K with respect to K is a proper subset of K. Since K is diametral if and only if $\mathcal{C}(K) = K$ we see that $w(w^*)$-asymptotic normal structure implies $w(w^*)$ normal structure. In 1974 Lim proved the equivalence of w-asymptotic normal structure and w-normal structure, however no such equivalence seems known in the w^*-case. None-the-less Lim's 1980 result verifies that ℓ_1 has w^*-asymptotic normal structure. This suggests;

QUESTION: Does UKK^* imply w^*-asymptotic normal structure?

(3) We note that from corollary (3.2.10) and the results of section 3 in Giles, Gregory and Sims [1978] we have that X^* has the Radon-Nikodym property whenever X^* has UKK^*. Further X is reflexive whenever X^{**} has UKK^*. We also remark that a result of Lima [1981] establishes a connection between UKK^* and approximation theory (more precisely, the theory, of M-ideals).

(4) Lau and Mah [1986] consider the space $\Gamma(H)$, the trace-class of operators on a Hilbert space with the trace norm (which may be identified with the dual $K(H)^*$ of the ideal of compact operators on H). They show $\Gamma(H)$ has the w^*quasi-normal structure introduced by Soardi [1972]: a dual space has **w-quasi-normal structure** if for every w^*compact convex subset C with more than one point, there exists $x \in C$ so that $\|x - y\| <$ diam C, for all $y \in C$. Unfortunately this is not enough to establish the w^*FPP for $\Gamma(H)$. Based on the analogy with $\ell_1(\Gamma)$ Lau and Mah ask whether $\Gamma(H)$ has UKK* and hence w^*-normal structure and the w^*-FPP. Arazy [1981] had already established the w^*-Kadec-Klee property for $\Gamma(H)$. Building from Arazy's argument, Chris Lennard [1986] has answered the question of Lau and Mah in the affirmative, showing that $\Gamma(H)$ **has UKK*.**

(5) D. Van Dulst and V. de Valk have made a through investigation of the various Kadec-Klee properties considered in this section for Orlicz sequence spaces. For example, they show that for an Orlicz function M, satisfying the Δ_2-*condition* is equivalent to h_M or ℓ_M having any of the properties KK*, UKK*, or ϵ-UKK* for some $\epsilon \in (0,1)$. [See Lindenstrauss and Tzafriri, 1977, for definitions and notation.] They also establish the following interesting result for subsititution spaces. Let M be an Orlicz function satisfying the Δ_2-condition and let (X_n) be a family of UKK spaces. Then the substitution space.

$$(X_1 \oplus X_2 \oplus \ldots X_n \oplus \ldots)_{h_M}$$

has weak normal structure.

(3.3) THE ASYMPTOTIC NORMAL STRUCTURE OF BAILLON AND SCHÖNEBERG.

In 1981 Baillon and Schöneberg introduced a weakening of normal structure which they called asymptotic normal structure (not to be confused with the asymptotic normal structure of Lin, *cf.* Remark 2 at the end of (3.2)): The Banach space X (dual space) has $w(w^*)$**-ANS** if whenever C is a non-trivial weak ($weak^*$) compact convex subset of X and $(x_n) \subset C$ is a sequence satisfysing $\|x_{n+1} - x_n\| \to 0$, then there exists an $x \in C$ such that $\liminf_n \|x_n - x\| <$ diam C.

Clearly $w(w^*)$-normal structure implies $w(w^*) - ANS$.

Since an approximate fixed point sequence (x_n) in a $w(w^*)$-compact convex minimal invariant set for a non-expansive mapping satisfies $\|x_{n+1} - x_n\| \to 0$ (Proposition 1.3) but $\lim\limits_n \|x - x_n\| =$ diam C (Theorem 1.8), we have:-

(3.3.1) PROPOSITION: $w(w^*)$**-ANS implies the** $w(w^*)$**-FPP.**

(3.3.2) EXAMPLES

(1) Baillon and Schöneberg [1981] show that the reflexive space

$$X_\alpha := (\ell_2, \|\cdot\|_\alpha), \quad \text{where} \quad \|x\|_\alpha := \alpha\|x\|_2 \vee \|x\|_\infty$$

has ANS if and only if $\alpha > \frac{1}{2}$, while it has normal structure if and only if $\alpha > \frac{1}{\sqrt{2}}$. Thereby establishing that **ANS is genuinely weaker than normal structure.** They also establish the FPP for $X_{\frac{1}{2}}$. Since we will obtain the FPP for all $\alpha > 0$ in Chapter 4, we will not persue the details here.

(2) That ANS is far from necessary for the FPP in reflexive spaces is further illustrated by Bynum [1980 and ???] who shows that the reflexive space

$$\ell_{p,\infty} = (\ell_p, |||\cdot|||), \quad \text{where} \quad |||x||| = \|x^+\|_p \vee \|x^-\|_p,$$

lacks ANS for all p, though it has the FPP for $1 < p < \infty$.

BIBLIOGRAPHY

Alspach, Dale E. [**1981**], *A fixed point free nonexpansive map*, Proc. A.M.S., 82, pp423-424.

Amir, Dan [**1982/83**], *On Jung's constant and related constants in normed linear spaces*, Longhorn Notes - Uni of Texas, Texas Functional Analysis Seminar, pp143-159.

Amir, D. & Franchetti, C. [**1982/83**], *The radius ratio and convexity properties in normed linear spaces*, Longhorn Note - Uni of Texas, Texas Functional Analysis Seminar, pp73-94.

Arazy, J.A. [**1981**], *More on convergence in unitary matrix spaces*, Proc. Amer. Math. Sco. 83, pp44-48.

Bae, Jong Sook, [**1983**], *Studies on generalized nonexpansive maps*, Ph.D. Thesis, Seoul National University.

Baillon, J.B. [**1978-79**], *Quelques aspects del la theorie des points fixes dans les espaces de Banach I*, Seminaire d'Analyse fonctionnelle, Ecole Polytechnique.

Baillon, J.B. & Schöneberg, R. [**1981**], *Asymptotic normal structure and fixed points of non-expansive mappings*, Proc. Amer. Math. Soc., 81, pp257-264.

Barbu, V. & Precupanu, Th. [**1978**], *Convexity and optimization in Banach spaces*, Sisthoff & Noordhoff (Int.Pub.).

Beauzamy, Bernard [**1982**], *Introduction to Banach spaces and their geometry*, North-Holland, Mathematics Studies, 68, pp308.

Brodskii, M.S. and Mil'man, D.P. [**1948**], *On the center of a convex set*, Dokl. Akad. Nauk. SSSR 59, pp837–840.

Bynum, W.L. [**1980**], *Normed Structure Coefficients for Banach spaces*, Pacific J. of Math. 86, pp427–436.

Bynum, W.L. [], *An example of a reflexive Banach space lacking asymptotic normal structure*, Submitted (see Bynum 1980 reference 4).

Danzer, L., Grunbaum, B. & Klee, V. [**1963**], *Helly's theorem and its relatives*, Amer. Math. Soc. 7, pp101-180.

Day, J., James, R.C. & Swaminathan, S. [**1971**], *Normed linear spaces that are uniformly convex in every direction*, Can. J. Math. Vol.XXIII, pp1051-1059.

van Dulst, D. [**1984**], *Some more Banach spaces with normal structure*, Math. Anal. and Applic. Vol 104, pp285-292.

van Dulst, D. [**1982**], *Equivalent norms and the fixed point property for non-expansive* *mappings*, J. Lond. Math. Soc., 25, pp139-144.

van Dulst, D. & Pach, A.J. [**1981**], *On flatness and some ergodic super-properties of Banach spaces*, Indagationes Mathematical , Vol 43, Fasciculus, 2, pp 153-164.

van Dulst, D. & Sims, B. [**1983**], *Fixed points of nonexpansive mappings and Chebyshev centres in Banach spaces with norms of type KK*, Banach space theory and its applications, Springer Verlag – Lecture Notes in Maths., 991, pp35-43.

van Dulst, D. & de Valk, V. [**1986**], *(KK)-properties, normal structures and fixed points of nonexpansive mappings in Orlicz sequence spaces*, preprint.

Edelstein, M. & O'Brien, R.C. [**1978**], *Nonexpansive mappings, asymptotic regularity and successive approximations*, J. Lond. Math. Soc. (2)17, pp547-554.

Edelstein, M. [**1963**], *A theorem on fixed points under isometries*, Amer. Math. Monthly, 70, pp298-300.

Elton, J., Lin, P.-K., Odell, E. & Szarek, S. [**1983**], *Remarks on the fixed point problem for nonexpansive maps*, Contemporary Mathematics, Vol 18, pp87-120.

Garkavi, A.L. [**1962**], *the best possible net and the best possible cross-section of a set in a normed space*, Izv. Akad. Nauk. SSSR Ser. Math. 26, pp87-106. [Amer. Math. Soc. Translns. Ser 2, 39(1964), pp111- 131.]

Giles, J.R., Gregory, D.A. & Sims, B. [**1978a**], *Characterization of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc., 18, pp105-123.

Giles, J.R., Gregory, D.A. & Sims, B. [**1978b**], *Geometrical Implications of upper semi-continuity of the duality mapping on a Banach space*, Pacific J. of Math., 79(1), pp99-109.

Giles, J.R., Sims, B. & Swarminathan, S. [**1985**], *A geometrically aberrant Banach space with normal structure*, Bull. Austral. Math. Soc., 31, pp75-81.

Goebel, K. [**1975**], *On the structure of minimal invariant sets for nonexpansive mappings*, Ann. Univ. Mariae Curie-Sktodowska Lublin, Vol xxix, 9, Sec. A, pp73-77.

Gossez, J.-P. & Lami-Dozo, E. [**1972**], *Some geometric properties related to the fixed point theory for nonexpansive mapping*, Pacific J. of Math., 40, pp565-573.

Halmos Paul R., [**1950**], Measure Theory, Van Nostrand.

Huff, R. [**1980**], *Banach spaces which are nearly uniformly convex*, Rocky Mountain Journal of Math. 10, pp743–749.

James, R.C. [**1964**], *Weak compactness and reflexivity*, Israël J. of Maths. 2, pp542-550.

Karlovitz, L.A. [**1976(a)**, *Existence of fixed points of nonexpansive mappings in a space without normal structure*, Pacific J. Math., 66, pp153-159.

Karlovitz, L.A. [**1976(b)**], *On non-expansive mappings*, Proc. Amer. Math. Soc., 55, pp321-325.

Kirk, W.A. [**1965**], *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, 72, pp1004-1006.

Kirk, W.A. [**1983**], *Fixed point theory for non-expansive mappings II*, Contemporary Mathematics, Vol. 18, pp121-140.

Klee, V.K. [**1960**], *Circumspheres and inner products*, Math. Scand., 8, pp363-370.

Landes, T. [**1984**], *Permanance properties of normal structure*, Pacific J. of Math., Vol. 110, No.1, pp125-143.

Lau, A.T.-M. & Mah, P.F. [**1986**] *quasi-normal structures for certain classes of operators on a Hilbert space*, Pacific J. of Math., 121, pp109-117.

Lennard, C. [**1986**], *C_1 is uniformly Kadec-Klee*, preprint.

Lim, T.C. [], *Fixed point theorems for uniformly Lischitzian mappings in L^p-spaces*, J. Nonlinear Anal. Theory Method and Appl.

Lim, T.C. [**1986**], *Some L^p inequalities and their applications to fixed point theorems of uniformly Lipschitzian mappings*, Proc. of Symp. in P. Maths. Vol 45, Part 2, pp119-125.

Lim, T.-C. [**1983**], *On normal structure coefficients and bounded sequence coefficient*, Proc. Amer. Math. Soc., 88, pp262-264.

Lim, T.-C. [**1980**], *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. of Math., Vol. 90, pp135-143.

Lim, T.-C. [**1974**], *Characterizations of normal structure*, Proc. Amer. Math. Soc., 43, pp313-319.

Lima, Å. [**1981**], *On M-ideals and best approximation*,

Lovaglia, [**1955**], *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc., 78, pp225-238.

Maluta, E. [**1984**], *Uniformly normal structure and related coefficients*, Pacific J. Mathy., 111, pp357-369.

Maurey, B. [**1980/81**], *Points fixes des contractions sur un convexe forme de L^1*, Seminarie d'Analyse Functionnelle 80-81 Exposé VIII, Ecole Polytechnique, Palaiseau.

Mil'man, D.F. & Mil'Man, V.D. [**1965**] *The Geometry of nested families with empty* *intersection. Structure of the unit sphere of a non-reflexive space*, American Math. Socl. Transl., (2), Vol. 85.

Opial, Z.O. [**1967**], *Weak convergence of the sequence of successive approximations for non-expansive mappings*, Bull. Amer. M. Soc., 73, pp591-597.

Rainwater, J. [**1969**], *Local uniform convexity of Day's norm on* $c_o(\Gamma)$, Proc. A.M.S. 22, pp335-339.

Roberts, A.W. & Varberg, D.,E. [**1973**], *Convex Functions*, Academic Press.

Routledge, N. [**1952**], *A result in Hilbert space*, Quart. J. Math. 3, pp12-18.

Rudin, W. [**1955**], *Analytic functions of class* H_p, Trans. Amer. Math. Soc., 78, pp46-66.

Schechtman, G. [**1982**], *On commuting families of nonexpansive operators*, 84, pp373-376.

Sims, B. [**1982**], *Ultra - techniques in Banach space theory*, Queens' Papers in Pure and Applied Mathematics, Vol. 60.

Sims, B. [**1984**], *A support map characterization of the Opial conditions*,

Sine, R. [**1981**], *Remarks on the example of Alspach*, preprint.

Smith, M.A. & Turett, B. [], *A reflexive LUR space that lacks normal structure.*

Smith, M.A. & Turett, B. [**1988a**], *some examples concerning normal structure and uniform normal structure in Banach spaces*, to appear J. Aust. Math. Soc.

Smith, M.A. & Turett, B. [**1988b**], *Normal structure in Bochner* L_p*-spaces*, preprint

Soardi, P.M. [**1972**], *Struttura quasi normale e teoremi di punto unito*, Rend. 1st Math. Univ. Trieste, 4, pp105-114.

Swaminathan, S. [**1983**], *Normal structure in Banach spaces and its generalisations*, Contemporary Mathematics, Vol. 18, pp201-215.

Turett, B. [**1982**], *A dual view of a theorem of Baillon*, Proc. Conf. on Nonlinear Anal. and Applications, St. Johns Newfoundalnd, 1981. Lec. Notes in Math. 80, Marcel Dekker, inc. N.Y., pp279- 286.

Yost, D. [**1982**], *Average distances in compact connected spaces*, Bull. Aust. Math. Socl., Vol. 26, pp331-342.

Zizler, V. [], *On some rotundity and smoothness properties of Banach spaces*, Dissertations Math. Rozprawy Mat.

	uced	normal Structure	w-shrinking ball	ε; w-shrinking ball	w-strict Opial	"⊥" w-approximately Symmetric	w-Opial	w-normal Structure	w-FPP		w*-shrinking ball	ε; w*-shrinking ball	w*-strict Opial	"⊥" w*-approximately Symmetric	w*-Opial	w*-normal Structure	w*-FPP
uced	✓	✓	X	X	X	X	X	✓	✓		X	X	X	X	X	✓	✓
normal Structure	X	✓	X	X	X	X	X	✓	✓		X	X	X	X	X	✓	✓
w-shrinking ball	X	X	✓	✓	X	X	X	✓	✓								
ε; w-shrinking ball	X	X	X	✓	X	X	X	✓	✓								
w-Strict Opial	X	X	X	X	✓	✓	✓	✓	✓								
"⊥" w-approximately symmetric	X	X	X	X	.	✓	✓	.	.								
w-Opial	X	X	X	X	X	X	✓	X	.								
w-normal Structure	X	X	X	X	X	X	X	✓	✓								
w-FPP	X	X	X	X	X	X	X	X	✓								
w*-shrinking ball	X	X	✓										X	X	X	✓	✓
ε; w*-shrinking ball	X	X	X	✓							X	✓	X	X	X	✓	✓
w*-Strict Opial	X	X			✓						X	X	✓	✓	✓	.	✓
"⊥" w*-approximately Symmetric	X	X				✓					X	X	.	✓	✓	.	.
w*-Opial	X	X					✓				X	X	X	X	✓	X	X
w*-normal Structure	X	X						✓			X	X	X	X	X	✓	✓
w*-FPP	X	X							✓		X	X	X	X	X	X	✓

When a row and column intersect in a "√" ("X") the row property implies (fails to imply) the column property. A "." signifies the implication is undecided.

PROPERTY

SPACE	ucod	normal Stucture	w-shrinking ball	ε; w-shrinking ball	w-strict Opial	w-Opial	w-normal Structure	w-FPP		w*-shrinking ball	ε; w*-shrinking ball	w*-strict Opial	w*-Opial	w*-normal Structure	w*-FPP
$\ell_p(1<p<\infty)$	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓
$L_p[0,1](1<p<\infty)$	✓	✓	✓	X	X	✓	✓	✓		✓	✓	X	X	✓	✓
$\ell_2\ \frac{1}{2}\|x\|_2 \vee \|x\|_\infty$	X	X	X	X	X	✓	X	✓		X	X	X	✓	X	✓
$\ell_2\ \alpha\|x\|_2 \vee \|x\|_\infty$ (α near 1)	X	✓	X	✓	X	✓	✓	✓		X	✓	X	✓	✓	✓
$\ell_2\ \|x\|_2+(1+\alpha)\|x\|_\infty$ (α small)	X	·	✓	✓	✓	✓	✓	✓		·	✓	X	·	✓	✓
$\ell_1\ \frac{1}{3}\|x\|_1 \vee \|x(1)e-x\|_\infty$	X	X	✓	✓	✓	✓	✓	✓		·	·	X	X	·	·
$\ell_1\ \|x^+\| \vee \|x^-\|$	X	X	✓	✓	✓	✓	✓	✓		X	X	X	✓	X	X
$(\ell_2 \oplus \ell_3 \oplus \ldots)_2$	✓	✓	X	X	✓	✓	✓	✓		X	X	✓	✓	✓	✓
with Van Dulst's renorming	·	✓	X	·	✓	✓	✓	✓		X	·	✓	✓	✓	✓
$\mathcal{L}_1[0,1]$	X	X	X	X	X	X	X	X		not defined					
c_0	X	X	X	X	X	X	X	✓		not defined					
ℓ_1	X	X	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓
ℓ_∞	X	X	X	X	X	X	X	X		X	X	X	X	X	X