K 24·2 – 24·3

__Experiment__ situation or process, contrived or otherwise, each __trial__ (repetition) of which leads to an observable __outcome__.

__Eg__

__Experiment__	__trial__	__outcome__
$n^{\underline{o}}$ showing after throwing a dice	a particular throw	particular $n^{\underline{o}}$ / $n^{\underline{o}}$ showing on top face (1,2,…,6)
where ball ~~stops after release a ball~~	particular release	distance "d" it travels before ~~stopping~~ ($n^{\underline{o}} \geq 0$)
what happens to light wh. ~~switching~~ the light switch is switched	a particular flick of the switch	resulting state of the light (~~on/off~~) on → off, off → on, on → on, off → off
what happens wh. ~~dropping~~ a brick is dropped	particular drop	state of the brick (falls / ~~fails to fall~~)

Interested in case when outcome of any given trial is uncertain – dependent on random / chance effects.

__Sample Space__ of a given experiment S is the set of all possible outcomes.

An __event__ E is a subset of S.

Say E occurs if outcome $\in E$.

Company sent to received ~~message~~

Parties transmission

same (= no transerror)

differ (= transmission

sent message → encode → channel → decode → received

Often nature of events we are interested in determine what is the approp sample space

expand & hence

Eg 2 H's out of 3 tosses.

Complementing events $\bar{A} = S \setminus A$

Compound event

A and B $\cap$

mutually exclusive events

A or B $\cup$

---

likelyhood

Probability of events is a measure of the likelihood of that event occurring

certainty $\to$ 1 so $P(S) = 1$ (long term frequency)

impossible $\to 0$ so $P(\emptyset) = 0$

$0 \leq P(A) \leq 1$ with

$A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$

A_i mutually exclusive

May not be possible to assign prob. measure to all events, those for which $P(A)$ is defined form a σ-algebra.

In general

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Eg   Finite sample spaces in which outcomes are equally likely;

$$S = \{ O_1, O_2, \ldots, O_N \}$$

$$P(O_i) = P(O_j) \quad \text{all } i, j \in \{1, 2, \ldots, N\}$$

Since outcomes are mutually exclusive

$$P(O_1) + \cdots + P(O_N) = N\, P(O_i) = 1$$

ie   prob. of any outcome is $\frac{1}{N}$.

(often asserted from physical symmetries)

Toss of "fair" dice; For $i = 1, 2, \ldots, 6$

$$P(i \text{ showing}) = \frac{1}{6} \qquad SS = \{ 3, 1, 5, 6, 2, 4 \}$$

Double throw of dice (or 2 dice)

$$SS = \{ 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad \ldots \quad 66 \}$$

$$\Pr(i, j \text{ showing}) = \frac{1}{36}$$

Condnl prob A, B events

$P(A|B)$ = prob. A will occur given that B has

$$= \frac{P(A \cap B)}{P(B)}$$

A indept of B if knowing B has occurred does not alter prob. of A occurring

i.e. $P(A|B) = P(A)$

$\Leftrightarrow \quad P(A \cap B) = P(A)\,P(B)$ – multn rule for indept events

$\Leftrightarrow \quad P(B) = \dfrac{P(B \cap A)}{P(A)} = P(B|A)$

i.e. B indept of A

So simply say A & B are (statistically independent) $\Leftrightarrow$ A & $\overline{B}$ indept.

E.G. drawing card from pack of 52

~~A~~ card is Ace

B = card is red i.e. ♡ or ◇

A & B are independent

Sometimes assert independence on physical grounds

i.e. outcome on 1st (throw of) dice indept of outcome of 2nd (throw of) dice.

n^o of tosses of fair coin until a H

$SS = \{1, 2, 3, 4, \dots\}$

outcome		Prob
H	1	$\frac{1}{2}$
TH	2	$\frac{1}{4}$ = Prob (T first then H)
TTH	3	$\frac{1}{8}$ = Prob(T) × prob(H)
⋮		
	n	$\frac{1}{2^n}$
⋮		

$\sum = 1$ so H will eventually occur.

Random (stochastic) variable

$X : \Omega \to \mathbb{R}$

$X : S \to \mathbb{R}$

Prob distribution f^n of random variable X

is $F(x) = \text{Prob}(X \leq x)$

$= \text{prob}\left(\{ o : X(o) \leq x \}\right)$

$\Rightarrow \text{Prob}(a < \cancel{X} \leq b) = F(b) - F(a)$

X <u>descrete</u> RV values

$t_1 < t_2 < \dots < t_n \dots$

$\text{Prob}(t_{i-1} < X \leq t_i) = \text{prob}(X = t_i) = p(t_i)$

only possible value in the rang

prob. f^n of distribution of RV X

$F(x) = \sum_{t_i \leq x} p(t_i)$

$$\boxed{F \geq 0 \quad F \nearrow 1}$$

F differentiable, with $f = F'$

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

density f^n for distribution of RV X

$f(t) = \sum_i p(t_i)\,\delta_{t_i}(t)$ with

EXAMPLES

1) X = sum of n^{os} showing when 2 dice are tossed

S

	1	2	3	4		7
	1 1	1 2	1 3	1 4	1 5	1 6
	2 1	2 2	2 3	2 4	2 5	2 6
	3 1	3 2	3 3	3 4	3 5	3 6
	4 1	4 2	4 3	4 4	4 5	4 6
	5 1	5 2	5 3	5 4	5 5	5 6
	6 1	6 2	6 3	6 4	6 5	6 6

11   12

- Discrete (finite SS)

X	$p(x)$	$F(x)$
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36 = 1
	Σ 1	

2) X = n^{o} of throws of fair coin until H tossed

X	$P(x)$	$F(x)$
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$3/4$
3	$\frac{1}{8}$	$7/8$
4	$\frac{1}{16}$	$15/16$
$\vdots$		
n	$\frac{1}{2^n}$	$\frac{2^n-1}{2^n}$ (sum of G P)
$\vdots$		
	$\Sigma = 1$	

inf SS - Discrete

3) X = Time for a radioactive nucleus to disintergrate (<u>continuous</u>)

$$P\left(\text{not decayed by time } T\right) = \lim_{n\to\infty}\left(1-\frac{p}{n}\right)^{nT}$$

[prob. will decay if not already in interval of $1/n$ is $p \times \frac{1}{n}$]

$$= e^{-pT}$$

$t \geq 0$

So $F(t) = P(X \leq t)$

<u>distribution fn</u>

$= P\left(\text{decayed by time } T\right) = 1 - e^{-pT}$

<u>density fn</u>

$$f(t) = F'(t) = pe^{-pT}$$

check $\int_0^\infty f(t) = 1$

2) & 3) are "waiting time problems"

<u>Expectation</u>

Expected value of a random variable X is

$$E(X) = \sum_i x_i p(x_i) \quad \text{- discrete case}$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{- cont case}$$

Eg 1) X = nº showing when dice is tossed

x	$p(x)$
1	$1/6$
2	"
3	"
4	"
5	"
6	"

$$E(X) = \frac{1}{6}(1+2+3+4+5+6)$$
$$= \frac{21}{6} = 3\tfrac{1}{2}$$

(= amnt you'd win on avg per toss if won what was shown
Note need not = possible outcome)

For $f(t) = pe^{-pt}$

$$E(T) = \int_0^\infty \underset{u}{t}\, \underset{v'}{pe^{-pt}}\, dt$$

$$= \frac{1}{p}$$

$$p\left(\left[\frac{te^{-pt}}{-p}\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\,dt\right)$$
$$= p\left(0 + \frac{1}{p}\left[-\frac{1}{p}e^{-pt}\right]_0^\infty\right)$$
$$= p\left(0 + \frac{1}{p^2}\right)$$
$$= p$$

For any fn $g: \mathbb{R} \to \mathbb{R}$

$$E(g(X)) := \begin{cases} \sum_i g(x_i)\,p(x_i) \\ \int_{-\infty}^{\infty} g(x)\,f(x)\,dx \end{cases}$$

taking $g(x) = x^r$ $\quad r = 0, 1, 2, \ldots$
we obtain the
r th moment of X about the origin O

$$\mu_r^o = \int_{-\infty}^{\infty} x^r f(x)\,dx = E(X^r)$$

$$\mu_0^o = 1$$

$$\mu_1^o = E(X)$$ usually denoted by μ and referred to as the mean of the distribution

Taking $g(x) = (x-c)^r$ yields moments about c.
Of particular importance are (higher) moments about the mean μ.

$$\mu_0^\mu = 1$$

$$\mu_1^\mu = \int_{-\infty}^{\infty} (x-\mu) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx - \mu \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \mu - \mu \times 1 = 0$$

especially important is

$$\sigma^2 = \mu_2^\mu = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$ — the Variance of X, $Var(X)$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu \int_{-\infty}^{\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \mu_2^o - 2\mu^2 + \mu^2$$

$$= \mu_2^o - \mu^2$$

$\sigma = \sqrt{Var(X)}$ is the standard devn of X

Var(X)

ie $E\left((X-E(X))^2\right) = E(X^2) - E(X)^2$

<u>Eg</u> For X = no. showing a dice

$E(X) = 3\frac{1}{2}$

$E(X^2) = \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)$

$= \frac{1}{6}(1+4+9+16+25+36)$

$= \frac{91}{6}$

so $Var(X) = \sigma^2 = \frac{91}{6} - \frac{49}{4}$

$=$

and $\sigma =$

<u>Eg</u> when $p(x) = pe^{-px}$

$\mu = E(X) = \frac{1}{p}$

$E(X^2) = p\int_{-\infty}^{\infty} x^2 e^{-px} = \frac{2}{p^2}$ (Verify using S'gn by parts 2x)

So $Var(X) = \frac{2}{p^2} - \left(\frac{1}{p}\right)^2$

$= 1/p^2$

& $\sigma = 1/p$

Moment generating function

of RV X with density fn f is

$$M_x(t) = E\left(e^{tX}\right)$$

$$= \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(1 + xt + \frac{x^2t^2}{2!} + \dots + \frac{x^nt^n}{n!} + \dots\right) f(x)\,dx$$

$$= \mu_0^0 + \mu_1^0 t + \mu_2^0 \frac{t^2}{2!} + \dots \mu_n^0 \frac{t^n}{n!} + \dots$$

Eg for $f(x) = pe^{-px}$

$$M_x(t) = \int_0^{\infty} e^{xt} pe^{-px}\,dx$$

$$= p\int_0^{\infty} e^{x(t-p)}\,dx$$

$$= \frac{p}{p-t} \quad (t < p)$$

$$= \frac{1}{1-t/p} = 1 + \frac{t}{p} + \frac{t^2}{p^2} + \dots$$

$\frac{2}{p^2} \times \frac{t^2}{2!} + \dots$

So $\mu_0^0 = 1$, $\mu_1^0 = \frac{1}{p}$, $\mu_2^0 = \frac{2}{p^2}$ (as we saw),

$\dots, \mu_n^0 = \frac{n!}{p^n}$

Transformed ~~Ra~~ V's

Normalized RV

$$Z = \frac{X - \mu_x}{\sigma_x}$$

useful:

$$E(a + bX) = a + bE(X)$$

$$Var(a + bX) = b^2 Var(X)$$

$\Rightarrow$

$$E(Z) = 0$$

$$Var(Z) = \sigma_z = 1$$

Some Commonly occuring Distributions

Binomial distribution (or Bernoulli dist)

For $n \in \mathbb{N}$

$X_n \equiv$ $n^{\underline{o}}$ of times a given event E occurred during n independent trials of an experiment (sometimes known as "Bernoulli trials")

Eg • $n^{\underline{o}}$ people expressing support for a given option in an opinion pole of n people.

• $n^{\underline{o}}$ successful outcomes from n similar surgeries

• $n^{\underline{o}}$ of defectives in a sample of n similarly produced items. (Sampling with replacemt)

X_n can take the values $x = 0, 1, 2, \dots, n$.

In any of the identical trials ~~~~ let

$p = P(E)$,

Then the probability function for X_n is

$$P_n(x) = P(X_n = x)$$

$$= \binom{n}{x} p^x q^{n-x}$$

where $q = P(\bar{E}) = 1 - p$ & $\binom{n}{x} = {}^nC_x$

$=$ $n^{\underline{o}}$ ordered selection of x items from n without replacement

$$= \frac{n!}{x!(n-x)!} = \frac{n \cdots (n-x+1)}{x(x-1)\cdots 1}$$

— Binomial coefficient

2

Note: $\displaystyle\sum_{x=0}^{n} P_n(x) = (p+q)^n$ — by Binomial Thm
$= 1$

$E((X-\mu)^2)$
$= E(X^2 - 2\mu X + \mu^2)$
$= E(X^2) - \mu^2$

Moment generating function:

$$M_n(t) = \sum_{x=0}^{n} e^{xt}\binom{n}{x}p^x q^{n-x}$$

$$= \sum_{n=0}^{n}\binom{n}{x}(pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

So mean:

$$\mu_n = M_n'(0) = pn \quad , \text{ as } M_n'(t) = n(pe^t+q)^{n-1}pe^t$$

$$M''(t) = n(n-1)p^2(pe^t+q)^{n-2}e^{2t} + n(pe^t+q)^{n-1}pe^t$$

Variance

$$\sigma_n^2 = E(X_n^2) - \mu_n^2$$

$$= M''(0) - \mu_n^2$$

3 $\frac{1}{8}$

$$= n(n-1)p^2 + np - p^2n^2$$

$$= \cancel{n^2p^2} - np^2 - \cancel{p^2n^2} + np$$

$$= n(p-p^2) = np(1-p) = npq$$

$n = 3$

$p = 0{\cdot}9$ $\quad$ $p = 0{\cdot}5$ $\quad$ $p = 0{\cdot}1$

Ex

Prob. 5 H's in 7 tosses of a fair coin:

$$\binom{7}{5}\frac{1}{2}^5\frac{1}{2}^2 = \frac{7\times 6}{2}\times\frac{1}{2^7}$$

$$= \frac{21}{128}$$

2^2 4
2^4 16
2^5 32
2^6 64
2^7 128

Prob of at least 5 H's:

$$P_7(5)+P_7(6)+P_7(7)$$

$$= \frac{1}{128}\left(\overset{\binom{7}{5}}{21}+\overset{\binom{7}{6}}{7}+\overset{\substack{7\\7}}{1}\right)$$

$$= \frac{29}{128}$$

316
16
96
16
256
128

Poisson Distribution

$$P(x) = \frac{\mu^x}{x!}e^{-\mu}$$

$$E(aX+b) = aE(X)+b$$

1

$$Var(aX+b) = a^2 Var(X)$$

$Z = \dfrac{X-\mu}{\sigma}$ is standardized RV:

$$E(Z)=0, \quad Var(Z)=1$$

<u>Eg's</u> <u>Binomial</u> X = no times event E occurs in n independent trials

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

<u>Eg</u> prob. of producing a defective screw is 0.01
What is prob. that a sample of 100 screws selected at random will contain (a) 1 defective
(b) at most 2 defectives

(a) Inspecting each of the 100 screws constitutes an independent Bernoulli trial so

$$\text{Prob}(1 \text{ defective}) = \binom{100}{1}(0.01)^1(0.99)^{99}$$

$$= 0.370 \quad (0.36972964)$$

(b) prob (≤ 2 defectives) = prob (0 defective) + prob (1 ") + " (2 ") mutually exclusive events

$$= \binom{100}{0}(0.01)^0(0.99)^{100} + \binom{100}{1}(0.01)^1(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98}$$

$$= 0.92$$

Sampling with replacement:

Suppose out of N screws D are defective.

Trial: draw screw at random from n

$p = $ Prob (screw defective) $= \dfrac{D}{N}$

then replace it $\Rightarrow$ trials are indept.

Prob (x defects in n draws)

$$= \binom{n}{x}\left(\frac{D}{N}\right)^{x}\left(1-\frac{D}{N}\right)^{n-x}$$

without replacement

Can draw $\dbinom{N}{n}$ different samples of n from N

Can draw x deffectives from the D in $\dbinom{D}{x}$ ways

$\times \dbinom{N-D}{n-x}$ ways of drawing $n-x$ non-defectives from the $N-D$

So no. distinct samples of size n containing x defects is

$$\binom{D}{x}\binom{N-D}{n-x}$$

out of $\dbinom{N}{n}$ possible samples so

Prob of drawing x defects out of n draws without replacement is $\dbinom{D}{x}\dbinom{N-D}{n-x}\Big/\dbinom{N}{n}$

Show that mean nD/N

variance $nD(N-D)(N-n)/N^2(N-1)$

The hypergeometric

POISSON ③

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$p = \frac{\mu}{n}$$

$$q = \left(1 - \frac{\mu}{n}\right)$$

Let $n \to \infty$ keeping mean μ fixed

i.e. $$\binom{n}{x}\left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$$= \underbrace{\frac{n(n-1)\cdots(n-x+1)}{n \; n \cdots n}}_{\to 1} \frac{1}{x!} \mu^x e^{-\mu}$$

Prob | event occurs between t & $t+dt$ $= \alpha\, dt$

Prob more than 1 .... is negligible $(=0)$

$: o(dt^2)$

(4)

$$P_n(t+dt) = P_n(t) \cancel{\times} (1-\alpha\, dt)$$

$$+ \quad P_{n-1}(t) \times \alpha\, dt$$

$$\frac{P_n(t+dt) - P_n(t)}{dt} = \alpha\left(P_{n-1}(t) - P_n(t)\right)$$

$$\frac{dP_n}{dt} = \alpha(P_{n-1} - P_n)$$

$$\frac{dP_n}{dt} + \alpha P_n = \alpha P_{n-1}$$

$$P_0(t) = e^{-\alpha t}$$

$$\frac{d P_1(t)}{dt} + \alpha P_1 = \alpha e^{-\alpha t}$$

$$\mu p' + \mu' p$$

$$= \mu p' + \mu' p$$

$$\frac{\mu'}{\mu} = \alpha$$

$$\mu = e^{\alpha t}$$

$$P_1(t) = e^{-\alpha t}\int \alpha\, dt$$

$$= \alpha t e^{-\alpha t}$$

$q \sim t$

$\mu = \alpha t \qquad n = x$ (5)

$$P(X=x) = P_{\cancel{x}}(t) = \frac{1}{x!}\mu^x e^{-\mu}$$

$$\sum_{x=0}^{\infty} e^{tx} \frac{1}{x!}\mu^x e^{-\mu}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

$$M(t) = e^{-\mu} e^{\mu e^t}$$

$$m'(t) = e^{-\mu} e^{\mu e^{t}} \times \mu e^t$$

$$m'(0) = e^{-\mu} e^{\mu \times 1} \times \mu \times 1$$

$$= \mu$$

$$m''(0) = e^{-\mu} \left(e^{\mu e^t} \left[(\mu e^t)^2 + \mu e^t \right] \right.$$

$$m''(0) = e^{-\mu} e^{\mu} \left[\mu^2 + \mu \right]$$

$$= \mu^2 + \mu$$

$$\alpha^2 = \mu^2 + \mu - \mu^2 = \mu.$$

approximating Binomial
Eg from before want

$$\binom{100}{1}(0.01)^1(0.99)^{99} \qquad x = 1$$

Binomial $\mu = np = 100 \times 0.1 = 1$

so

$$\sim \frac{\mu^1}{1!}e^{-\mu} = \frac{1}{e}$$

$$= 0.\underline{36}78794$$

Queing on average 2 customers ~~per ... customer~~ arrive per minute

Prob that during a given minute ~~...~~ 10 customers will arrive

$\mu = 2 \qquad x = 10$

$$\frac{2^{10}}{10!}e^{-2} = \frac{1024}{3628800\,e^2}$$

$$\doteq 0.000038$$

2 customers arrive,

$$\frac{2^2}{2!} \times \frac{1}{e^2} \doteq 0.27$$

Normal

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_{\odot_R} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^R e^{-r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^R$$

$$= \pi\left[1 - e^{-R^2}\right]$$

$$\to \pi \text{ as } R \to \infty$$

$$= \int e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \left(\int e^{-x^2}\,dx\right)^2$$

$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$

is a distribution with

mean $\underbrace{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x e^{-x^2}\,dx}_{\text{odd fn}} = 0$

variance

$$\sigma^2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \underset{u}{x}\,\underbrace{x e^{-x^2}}_{v'}\,dx$$

$$= \frac{1}{\sqrt{\pi}}\left(\left[-\frac{1}{2}x e^{-x^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{1}{2}e^{-x^2}\,dx\right)$$

$$= \frac{1}{2}$$

$\tilde{X} \sim 0, \frac{1}{2}$

$X = \sqrt{2}\,\tilde{X}$ has mn 0, Var 1

$\frac{1}{\sqrt{2}}X$ has va $\frac{1}{2}$

X has Var 1

$x = \sqrt{2}\,\tilde{x}$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$$

$\Phi(-z)$ $\quad 1-\Phi(z)$

$$\Phi(z) - (1-\Phi(z))$$

$$= 2\Phi(z) - 1$$

~~Z mean μ variance σ^2 has~~

Z mean 0 variance $\sigma^2 = 1$ has
distribution $\Phi(z)$

$$\left(= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \right)$$

$X = \sigma Z + \mu$ has distribution with
mean μ variance σ

density fn of X

$= \text{Prob}(X \leq x)$

$= \text{Prob}(\sigma Z + \mu) \leq x)$

$= \text{Prob}\left(Z \leq \frac{1}{\sigma}(x - \mu)\right)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sigma}(x-\mu)} e^{-\frac{1}{2}t^2}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du$$

$t = \frac{1}{\sigma}(u - \mu)$

~~$x = \sigma t + \mu$~~

$u = \sigma t + \mu$

So $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$E(\sigma Z + \mu)$

$= \sigma E(Z) + \mu$

$= \sigma\mu$

Joint Distributions

Our theory of distribution extends to random variables in $\mathbb{R}^n$:

$$(X_1, X_2, \cdots) : S_1 \times S_2 \times \cdots \longrightarrow \mathbb{R}$$
$$: (x_1, x_2, \cdots) \longmapsto f(x_1, x_2, \cdots)$$

To illustrate we take $n = 2$ & use (X, Y)

Distribution function

$$F(x, y) := P(X \leq x, Y \leq y)$$
$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\, dx\, dy$$

where $f(x, y)$ is density fn

[In discrete case $F(x, y)$

$$= \sum_{\substack{x_i \leq x \\ y_j \leq y}} p(x_i, x_j)$$

$p_{ij} = P(X = x_i, Y = y_j)$]

Eg 1) X = n° on fair dice $\quad Y$ = face of fair coin $H = 1$, $T = -1$

	1	2	3	4	5	6
1						
−1						

$\forall\, p_{ij} = \frac{1}{12}$

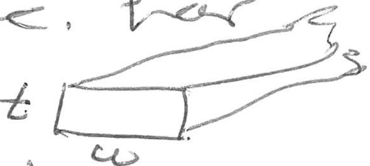
2) 2 balls from urn with 2 R 3 B with replacement

X = n° R $\quad Y$ = n° B

(B) Y \ X (R)	0	1	2
	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$
0	0	0	$\frac{4}{25}$
1	0	$\frac{12}{25}$	0
2	$\frac{9}{25}$	0	0

~~R_1B_1~~
~~R_1B_2~~
~~R_1B_3~~
~~R_2B_1~~
~~R_2B_2~~
~~R_2B_3~~
~~R_1R_1~~
~~R_1R_2~~
~~R_2R_1~~
~~R_2R_2~~

3) X = width
Y = thickness
of a rec. bar



$f(x, y)$

Marginal distributions

density fn

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \qquad F_1$$

$$\left[P_1(x_i) = \sum_{j=1}^{n} p_{ij} \right]$$

Similarly $f_2(y)$ $\qquad F_2$

X, Y independent if

$$F_1(x)\, F_2(y) = F(x, y)$$

$$\Longleftrightarrow \quad f_1(x)\, f_2(x) = f(x, y)$$

$\Longleftarrow \quad \Longrightarrow$

For any function $g : R^2 \to \mathbb{R}$
we can define

$$Z = g(X, Y)$$

$$F(z) = \text{Prob}(Z \le z)$$

$$= \text{Prob}\{(x,y) : g(x,y) \le z\}$$

$$= \sum_{\substack{x,y \\ g(x,y) \le z}} p(x,y)$$

etc.

~~Eg. $g(x,y) = x + y$~~

~~$E(X+Y) =$~~

Expectations

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

$$E(a g(X,Y) + b h(X,Y))$$

$$= a E(g(X,Y)) + b E(h(X,Y))$$

$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x f_1(x)\, dx$

implications

So 1) $E(X+Y) = E(X) + E(Y)$

2) $E(XY) = E(X)E(Y)$ if X & Y are independent

extend to $X_1 + X_2 + \cdots + X_n$

$Z = X + Y$

$$\sigma^2_{X+Y} = \mathrm{Var}(X+Y)$$

$$= E\left((X+Y)^2\right) - E(X+Y)^2$$

$$= E\left(X^2 + 2XY + Y^2\right) - \left(E(X) + E(Y)\right)^2$$

$$= E(X^2) + 2E(XY) + E(Y^2)$$
$$- E(X)^2 + 2E(X)E(Y) + E(Y)^2$$

$$= E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2$$
$$+ 2\left(\underbrace{E(XY) - E(X)E(Y)}_{\sigma^2_{XY}}\right)$$

σ^2_{XY} ← the covariance of X & Y

$$\sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}$$

$= 0$ if X, Y indep

$= \sigma_X^2 + \sigma_Y^2$ if X, Y indept

— addn of variances for indept random variables

Statistics

Experiment

set of all possible outcomes is "population"

prob dist on ↺ is "popul^n distribution"

parameters of ↺ are "population parameters"

Eg "population mean" μ

"population variance" σ^2

μ_k^0 — "k'th moment of population"

Sample of size n is the data (outcomes) from n independent trials of the exp.

Aim of statistics: • To infer info about population distribution from the sample (samples)

Eg give estimates of popul^n parameters

$\bar{x}$ of mean μ

$\bar{\sigma}$ of s.d. σ

- Gauge the reliability likely accuracy of such estimates
- Use such info to answer questions such as we have been considering in prob. theory.

Fisher's maximum likelihood method

X random variable on popul$\underline{n}$ $\theta_1, \theta_2, \ldots \theta_k$ params of pop

Sample: values $x_1, x_2, \ldots, x_n$ observed for X in n indep^t trials.

prob of this in discrete case is

$$l(\theta_1, \ldots, \theta_k, x_1, \ldots, x_n)$$

$$= p(\theta_1, \ldots, \theta_k, x_1)\, p(\ldots, x_2) \cdots p(\ldots, x_n)$$

In cont. case for small Δx

$$\text{Prob}(x_i \le X_i \le x_i + \Delta x \quad i = 1, 2, 3 \ldots n)$$

$$= l(\theta_1, \ldots \theta_k, x_1, \ldots, x_n)\, \Delta x^n$$

choose $\theta_1, \ldots \theta_k$ to maximise l - the likelihood $f\underline{n}$.

usually means solving

$$\frac{\partial l}{\partial \theta_i} = 0 \qquad i = 1, \ldots, k.$$

Since $l > 0$ & $\ln$ is $\uparrow$ $f\underline{n}$

$$\Leftrightarrow \quad \frac{\partial \ln l(\theta_1, \ldots \theta_k, x_1, \ldots, x_n)}{\partial \theta_i} = 0 \qquad i = 1, 2, \ldots$$

[$\ln l$ is often simpler than l.]

Eg

Populn normal distributed mean μ s.d σ

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

so

$$l = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\ln l = -n(\ln 2\pi + \ln\sigma) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$\frac{\partial \ln l}{\partial \mu} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\text{i.e. } \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\overline{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (= \text{avg/mean of the } x_i)$$

also

$$\frac{1}{2}\sigma^{-2}$$

$$\frac{\partial \ln l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 = 0$$

$$\text{so } \bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \quad (*)$$

$\left(= \begin{cases} \text{second moment of data about mean} \\ \text{variance of data} \end{cases} \right)$

Suggests using sample parameters $\Leftrightarrow$ this estimates for those of pop

Note: Considerations of goodness of estimates & bias suggests replacing n by $n-1$ in (*) is better for small ns

(1) $Z \sim N(0,1)$

$$M_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} e^{t^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} dw \qquad \begin{array}{l} w = x - t \\ dw = dx \end{array}$$

$$= e^{t^2/2}.$$

So, $X \sim N(\mu, \sigma^2)$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$w = \frac{x-\mu}{\sigma}$$

$$dx = \sigma \, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma w + \mu)} e^{-\frac{1}{2}w^2} dw$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\sigma t) w} e^{-\frac{1}{2}w^2} dw$$

$$= e^{\mu t} M_Z(\sigma t)$$

$$= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

(2) X & Y independent R.V.'s

$$\Rightarrow \quad M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$M_{X+Y}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x+y)}\,\underbrace{f(x,y)}_{f_1(x)f_2(y)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{tx} f_1(x)\,dx \int_{-\infty}^{\infty} e^{ty} f_2(y)\,dt$$

$$= M_X(t)\,M_Y(t)$$

$$X \sim N(\mu_1, \sigma_1^2), \quad Y \sim N(\mu_2, \sigma_2^2)$$

$$M_X(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \qquad M_Y(t) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$M_{X+Y}(t) = e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

$$\Longleftrightarrow \quad X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

(as $M_X(t) = \cancel{\text{determines}} \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$

determines f — cf. Laplace transform determines fn)

X_i indep. identically dist. R.V.s, $i=1,2,\dots,n$

$\sim N(\mu,\sigma^2)$ — (*)

$$\Longrightarrow \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

even if (*) not true — as $n \to \infty$ | central limit Thm.

So $$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

RV = sample means

So $$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

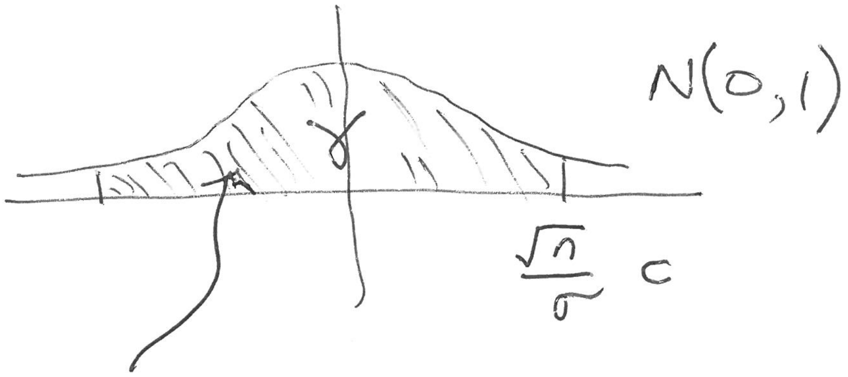
Assume σ known,
estimate μ by $\overline{x}$ (sample mean)

$\gamma\%$ confidence interval

Find c s.t.

$$P(|\mu-\overline{x}| < c) = \gamma\%$$

i.e. $P\left(-\frac{\sqrt{n}}{\sigma}c \le Z \le \frac{\sqrt{n}}{\sigma}c\right) = \gamma$



$$\Phi\left(\frac{\sqrt{n}}{\sigma}c\right) - \left(1 - \Phi\left(\frac{\sqrt{n}}{\sigma}c\right)\right)$$

$$= 2\Phi\left(\frac{\sqrt{n}}{\sigma}c\right) - 1 = \gamma$$

so $$\frac{\sqrt{n}}{\sigma}c = \Phi^{-1}\left(\frac{\gamma+1}{2}\right)$$

$$c = \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(\frac{\gamma+1}{2}\right)$$

Eg Know $X \sim N(\mu, \underbrace{9}_{=\sigma^2})$

Sample of size $\underbrace{100}_{=n}$ is drawn & found to have mean $\bar{x} = 5$

Find 95% confidence interval for μ

$\Downarrow$

$\gamma = 0.95$

$$c = \frac{3}{10}\Phi^{-1}\left(\frac{1.95}{2}\right) = \frac{3}{10}\Phi^{-1}(0.975)$$

$$\Phi^{-1}(0.975) = z(\overset{0.975}{\Phi})$$

$$= 1.960 \quad \text{(from table)}$$

so $c = \frac{3}{10} \times 1.645$

$$= \cancel{0.588}\ 0.4935$$

So with 95% confidence

$$|\mu - 5| \le 0.4935$$

$$\begin{array}{r} 1.00000 \\ .4935 \\ \hline .5065 \end{array}$$

ie $4.5065 \le \mu \le 5.4935$

More realistic is case when neither μ or σ is known.

Then use estimates

$$\hat{\mu} = \bar{x} \quad \& \quad \hat{\sigma}^2 = s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

Need to know how

$$T = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

is distributed.

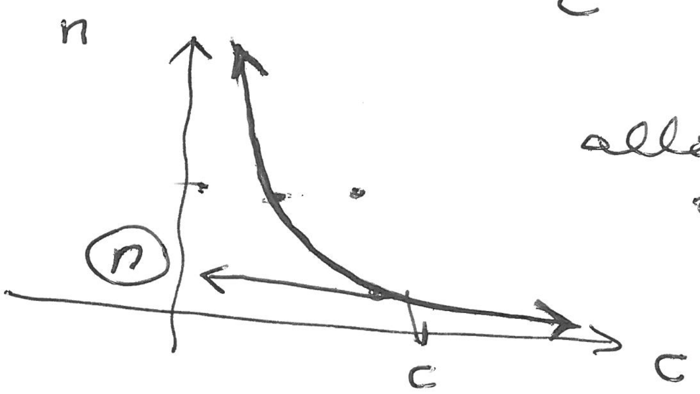
ans. t-distribution (students)

cont at (**)

(*) For given γ level:

$$c = \frac{\sigma}{\sqrt{n}} \Phi^{-1}\left(\frac{\gamma+1}{2}\right) \quad \text{const}, k \text{ say}$$

ie $\quad n = \frac{k^2}{c^2}$



allows sample size to be selected to achieve a given error (c) at given confidence level.

(**) t-dist c/ tailed

distribution fn is

$$P(T \le t) = F(t, m)$$

m = deg. of freedom = n − 1 ← sample size

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\,\Gamma\left(\frac{1}{2}m\right)} \int_{-\infty}^{t} \left(1 + \frac{u^2}{m}\right)^{-(m+1)/2} du$$

$\boxed{\text{Values tabulated}}$

$t \quad F(t, m)$

(2)

To find γ% confidence interval for the mean, μ, read table "backwards" to find the value of t, t_*, such that $F(t) = \frac{\gamma+1}{2}$

then
$$c = \frac{s}{\sqrt{n}} t_*$$

Eg Sample $n=9$, $\bar{x} = 5$, $s^2 = 9$

95% confidence interval

t_* s.t. $F(t_*, 8) = \frac{\gamma+1}{2} = 0.975$

from table is 1.86

So $c = \frac{3}{3} \times 1.86$

c.f. would have been 1.645 if σ had been known to be 9.

sample $x_1, \dots, x_n$ for $N(\mu, \sigma^2)$

$$\hat{\sigma} = \bar{s} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})$$

$(n-1)\dfrac{\bar{s}^2}{\sigma^2}$ is "<u>chi-squared</u>" distributed with $m := (n-1)$ degrees of freedom

χ^2-distribution:

$$F(x) = \frac{1}{2^{m/2}\,\Gamma\left(\frac{m}{2}\right)} \int_0^x e^{-u/2}\, u^{(m-2)/2}\, du$$

γ% confidence for σ^2

$$\gamma = \text{Prob}\left(\frac{(n-1)\bar{s}^2}{F^{-1}\left(\frac{1}{2}(1+\gamma)\right)} \le \sigma^2 \le \frac{(n-1)\bar{s}^2}{F^{-1}\left(\frac{1}{2}(1-\gamma)\right)}\right)$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2$$

<u>Ex</u>

By the induction hypothesis, the right side equals $b[(b - a)A_k + ka^{k-1}] + a^k$. Direct calculation shows that this is equal to

$$(b - a)\{bA_k + ka^{k-1}\} + aka^{k-1} + a^k.$$

From (7b) with $n = k$ we see that the expression in the braces $\{\cdots\}$ equals

$$b^{k-1} + 2ab^{k-2} + \cdots + (k - 1)ba^{k-2} + ka^{k-1} = A_{k+1}.$$

Hence our result is

$$\frac{b^{k+1} - a^{k+1}}{b - a} = (b - a)A_{k+1} + (k + 1)a^k.$$

Taking the last term to the left, we obtain (7) with $n = k + 1$. This proves (7) for any integer $n \geqq 2$ and completes the proof. ■

Section 18.2, page 754

ANOTHER PROOF OF THEOREM 1 *without the use of a harmonic conjugate*

We show that if $w = u + iv = f(z)$ is analytic and maps a domain D conformally onto a domain D^* and $\Phi^*(u, v)$ is harmonic in D^*, then

$$\Phi(x, y) = \Phi^*(u(x, y), v(x, y)) \tag{1}$$

is harmonic in D, that is, $\nabla^2\Phi = 0$ in D. We make no use of a harmonic conjugate of Φ^*, but use straightforward differentiation. By the chain rule,

$$\Phi_x = \Phi_u{}^* u_x + \Phi_v{}^* v_x.$$

We apply the chain rule again, underscoring the terms that will drop out when we form $\nabla^2\Phi$:

$$\begin{aligned}\Phi_{xx} &= \underline{\Phi_u{}^* u_{xx}} + (\Phi^*_{uu}u_x + \underline{\Phi^*_{uv}v_x})u_x \\ &\quad + \underline{\Phi_v{}^* v_{xx}} + (\underline{\Phi^*_{vu}u_x} + \Phi^*_{vv}v_x)v_x.\end{aligned}$$

Φ_{yy} is the same with each x replaced by y. We form the sum $\nabla^2\Phi$. In it, $\Phi^*_{vu} = \Phi^*_{uv}$ is multiplied by

$$u_x v_x + u_y v_y$$

which is 0 by the Cauchy–Riemann equations. Also $\nabla^2 u = 0$ and $\nabla^2 v = 0$. There remains

$$\nabla^2\Phi = \Phi^*_{uu}(u_x{}^2 + u_y{}^2) + \Phi^*_{vv}(v_x{}^2 + v_y{}^2).$$

By the Cauchy–Riemann equations this becomes

$$\nabla^2\Phi = (\Phi^*_{uu} + \Phi^*_{vv})(u_x{}^2 + v_x{}^2)$$

and is 0 since Φ^* is harmonic. ■

APPENDIX 5

Tables

For Tables of Laplace transforms see Secs. 6.8 and 6.9.
For Tables of Fourier transforms see Sec. 11.10.

If you have a Computer Algebra System (CAS), you may not need the present tables, but you may still find them convenient from time to time.

Table A1 Bessel Functions

For more extensive tables see Ref. [GR1] in App. 1.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3391	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

$J_0(x) = 0$ for $x = 2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711, 21.2116, 24.3525, 27.4935, 30.6346$
$J_1(x) = 0$ for $x = 3.83171, 7.01559, 10.1735, 13.3237, 16.4706, 19.6159, 22.7601, 25.9037, 29.0468, 32.1897$

Table A1 *(continued)*

x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

Table A2 Gamma Function [see (24) in App. A3.1]

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000 000	1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384
1.02	0.988 844	1.22	0.913 106	1.42	0.886 356	1.62	0.895 924	1.82	0.936 845
1.04	0.978 438	1.24	0.908 521	1.44	0.885 805	1.64	0.898 642	1.84	0.942 612
1.06	0.968 744	1.26	0.904 397	1.46	0.885 604	1.66	0.901 668	1.86	0.948 687
1.08	0.959 725	1.28	0.900 718	1.48	0.885 747	1.68	0.905 001	1.88	0.955 071
1.10	0.951 351	1.30	0.897 471	1.50	0.886 227	1.70	0.908 639	1.90	0.961 766
1.12	0.943 590	1.32	0.894 640	1.52	0.887 039	1.72	0.912 581	1.92	0.968 774
1.14	0.936 416	1.34	0.892 216	1.54	0.888 178	1.74	0.916 826	1.94	0.976 099
1.16	0.929 803	1.36	0.890 185	1.56	0.889 639	1.76	0.921 375	1.96	0.983 743
1.18	0.923 728	1.38	0.888 537	1.58	0.891 420	1.78	0.926 227	1.98	0.991 708
1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384	2.00	1.000 000

Table A3 Factorial Function and Its Logarithm with Base 10

n	$n!$	$\log(n!)$	n	$n!$	$\log(n!)$	n	$n!$	$\log(n!)$
1	1	0.000 000	6	720	2.857 332	11	39 916 800	7.601 156
2	2	0.301 030	7	5 040	3.702 431	12	479 001 600	8.680 337
3	6	0.778 151	8	40 320	4.605 521	13	6 227 020 800	9.794 280
4	24	1.380 211	9	362 880	5.559 763	14	87 178 291 200	10.940 408
5	120	2.079 181	10	3 628 800	6.559 763	15	1 307 674 368 000	12.116 500

Table A4 Error Function, Sine and Cosine Integrals [see (35), (40), (42) in App. A3.1]

x	erf x	Si(x)	ci(x)	x	erf x	Si(x)	ci(x)
0.0	0.0000	0.0000	∞	2.0	0.9953	1.6054	−0.4230
0.2	0.2227	0.1996	1.0422	2.2	0.9981	1.6876	−0.3751
0.4	0.4284	0.3965	0.3788	2.4	0.9993	1.7525	−0.3173
0.6	0.6039	0.5881	0.0223	2.6	0.9998	1.8004	−0.2533
0.8	0.7421	0.7721	−0.1983	2.8	0.9999	1.8321	−0.1865
1.0	0.8427	0.9461	−0.3374	3.0	1.0000	1.8487	−0.1196
1.2	0.9103	1.1080	−0.4205	3.2	1.0000	1.8514	−0.0553
1.4	0.9523	1.2562	−0.4620	3.4	1.0000	1.8419	0.0045
1.6	0.9763	1.3892	−0.4717	3.6	1.0000	1.8219	0.0580
1.8	0.9891	1.5058	−0.4568	3.8	1.0000	1.7934	0.1038
2.0	0.9953	1.6054	−0.4230	4.0	1.0000	1.7582	0.1410

Table A5 Binomial Distribution

Probability function $f(x)$ [see (2), Sec. 24.7] and distribution function $F(x)$

n	x	$p = 0.1$		$p = 0.2$		$p = 0.3$		$p = 0.4$		$p = 0.5$	
		$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
		0.		0.		0.		0.		0.	
1	0	9000	0.9000	8000	0.8000	7000	0.7000	6000	0.6000	5000	0.5000
	1	1000	1.0000	2000	1.0000	3000	1.0000	4000	1.0000	5000	1.0000
	0	8100	0.8100	6400	0.6400	4900	0.4900	3600	0.3600	2500	0.2500
2	1	1800	0.9900	3200	0.9600	4200	0.9100	4800	0.8400	5000	0.7500
	2	0100	1.0000	0400	1.0000	0900	1.0000	1600	1.0000	2500	1.0000
	0	7290	0.7290	5120	0.5120	3430	0.3430	2160	0.2160	1250	0.1250
3	1	2430	0.9720	3840	0.8960	4410	0.7840	4320	0.6480	3750	0.5000
	2	0270	0.9990	0960	0.9920	1890	0.9730	2880	0.9360	3750	0.8750
	3	0010	1.0000	0080	1.0000	0270	1.0000	0640	1.0000	1250	1.0000
	0	6561	0.6561	4096	0.4096	2401	0.2401	1296	0.1296	0625	0.0625
	1	2916	0.9477	4096	0.8192	4116	0.6517	3456	0.4752	2500	0.3125
4	2	0486	0.9963	1536	0.9728	2646	0.9163	3456	0.8208	3750	0.6875
	3	0036	0.9999	0256	0.9984	0756	0.9919	1536	0.9744	2500	0.9375
	4	0001	1.0000	0016	1.0000	0081	1.0000	0256	1.0000	0625	1.0000
	0	5905	0.5905	3277	0.3277	1681	0.1681	0778	0.0778	0313	0.0313
	1	3281	0.9185	4096	0.7373	3602	0.5282	2592	0.3370	1563	0.1875
5	2	0729	0.9914	2048	0.9421	3087	0.8369	3456	0.6826	3125	0.5000
	3	0081	0.9995	0512	0.9933	1323	0.9692	2304	0.9130	3125	0.8125
	4	0005	1.0000	0064	0.9997	0284	0.9976	0768	0.9898	1563	0.9688
	5	0000	1.0000	0003	1.0000	0024	1.0000	0102	1.0000	0313	1.0000
	0	5314	0.5314	2621	0.2621	1176	0.1176	0467	0.0467	0156	0.0156
	1	3543	0.8857	3932	0.6554	3025	0.4202	1866	0.2333	0938	0.1094
	2	0984	0.9841	2458	0.9011	3241	0.7443	3110	0.5443	2344	0.3438
6	3	0146	0.9987	0819	0.9830	1852	0.9295	2765	0.8208	3125	0.6563
	4	0012	0.9999	0154	0.9984	0595	0.9891	1382	0.9590	2344	0.8906
	5	0001	1.0000	0015	0.9999	0102	0.9993	0369	0.9959	0938	0.9844
	6	0000	1.0000	0001	1.0000	0007	1.0000	0041	1.0000	0156	1.0000
	0	4783	0.4783	2097	0.2097	0824	0.0824	0280	0.0280	0078	0.0078
	1	3720	0.8503	3670	0.5767	2471	0.3294	1306	0.1586	0547	0.0625
	2	1240	0.9743	2753	0.8520	3177	0.6471	2613	0.4199	1641	0.2266
7	3	0230	0.9973	1147	0.9667	2269	0.8740	2903	0.7102	2734	0.5000
	4	0026	0.9998	0287	0.9953	0972	0.9712	1935	0.9037	2734	0.7734
	5	0002	1.0000	0043	0.9996	0250	0.9962	0774	0.9812	1641	0.9375
	6	0000	1.0000	0004	1.0000	0036	0.9998	0172	0.9984	0547	0.9922
	7	0000	1.0000	0000	1.0000	0002	1.0000	0016	1.0000	0078	1.0000
	0	4305	0.4305	1678	0.1678	0576	0.0576	0168	0.0168	0039	0.0039
	1	3826	0.8131	3355	0.5033	1977	0.2553	0896	0.1064	0313	0.0352
	2	1488	0.9619	2936	0.7969	2965	0.5518	2090	0.3154	1094	0.1445
	3	0331	0.9950	1468	0.9437	2541	0.8059	2787	0.5941	2188	0.3633
8	4	0046	0.9996	0459	0.9896	1361	0.9420	2322	0.8263	2734	0.6367
	5	0004	1.0000	0092	0.9988	0467	0.9887	1239	0.9502	2188	0.8555
	6	0000	1.0000	0011	0.9999	0100	0.9987	0413	0.9915	1094	0.9648
	7	0000	1.0000	0001	1.0000	0012	0.9999	0079	0.9993	0313	0.9961
	8	0000	1.0000	0000	1.0000	0001	1.0000	0007	1.0000	0039	1.0000

Table A6 Poisson Distribution

Probability function $f(x)$ [see (5), Sec. 24.7] and distribution function $F(x)$

x	$\mu = 0.1$		$\mu = 0.2$		$\mu = 0.3$		$\mu = 0.4$		$\mu = 0.5$	
	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
	0.		**0.**		**0.**		**0.**		**0.**	
0	9048	0.9048	8187	0.8187	7408	0.7408	6703	0.6703	6065	0.6065
1	0905	0.9953	1637	0.9825	2222	0.9631	2681	0.9384	3033	0.9098
2	0045	0.9998	0164	0.9989	0333	0.9964	0536	0.9921	0758	0.9856
3	0002	1.0000	0011	0.9999	0033	0.9997	0072	0.9992	0126	0.9982
4	0000	1.0000	0001	1.0000	0003	1.0000	0007	0.9999	0016	0.9998
5							0001	1.0000	0002	1.0000

x	$\mu = 0.6$		$\mu = 0.7$		$\mu = 0.8$		$\mu = 0.9$		$\mu = 1$	
	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
	0.		**0.**		**0.**		**0.**		**0.**	
0	5488	0.5488	4966	0.4966	4493	0.4493	4066	0.4066	3679	0.3679
1	3293	0.8781	3476	0.8442	3595	0.8088	3659	0.7725	3679	0.7358
2	0988	0.9769	1217	0.9659	1438	0.9526	1647	0.9371	1839	0.9197
3	0198	0.9966	0284	0.9942	0383	0.9909	0494	0.9865	0613	0.9810
4	0030	0.9996	0050	0.9992	0077	0.9986	0111	0.9977	0153	0.9963
5	0004	1.0000	0007	0.9999	0012	0.9998	0020	0.9997	0031	0.9994
6			0001	1.0000	0002	1.0000	0003	1.0000	0005	0.9999
7									0001	1.0000

x	$\mu = 1.5$		$\mu = 2$		$\mu = 3$		$\mu = 4$		$\mu = 5$	
	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
	0.		**0.**		**0.**		**0.**		**0.**	
0	2231	0.2231	1353	0.1353	0498	0.0498	0183	0.0183	0067	0.0067
1	3347	0.5578	2707	0.4060	1494	0.1991	0733	0.0916	0337	0.0404
2	2510	0.8088	2707	0.6767	2240	0.4232	1465	0.2381	0842	0.1247
3	1255	0.9344	1804	0.8571	2240	0.6472	1954	0.4335	1404	0.2650
4	0471	0.9814	0902	0.9473	1680	0.8153	1954	0.6288	1755	0.4405
5	0141	0.9955	0361	0.9834	1008	0.9161	1563	0.7851	1755	0.6160
6	0035	0.9991	0120	0.9955	0504	0.9665	1042	0.8893	1462	0.7622
7	0008	0.9998	0034	0.9989	0216	0.9881	0595	0.9489	1044	0.8666
8	0001	1.0000	0009	0.9998	0081	0.9962	0298	0.9786	0653	0.9319
9			0002	1.0000	0027	0.9989	0132	0.9919	0363	0.9682
10					0008	0.9997	0053	0.9972	0181	0.9863
11					0002	0.9999	0019	0.9991	0082	0.9945
12					0001	1.0000	0006	0.9997	0034	0.9980
13							0002	0.9999	0013	0.9993
14							0001	1.0000	0005	0.9998
15									0002	0.9999
16									0000	1.0000

Table A7 Normal Distribution

Values of the distribution function $\Phi(z)$ [see (3), Sec. 24.8]. $\Phi(-z) = 1 - \Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
	0.		0.		0.		0.		0.		0.
0.01	5040	0.51	6950	1.01	8438	1.51	9345	2.01	9778	2.51	9940
0.02	5080	0.52	6985	1.02	8461	1.52	9357	2.02	9783	2.52	9941
0.03	5120	0.53	7019	1.03	8485	1.53	9370	2.03	9788	2.53	9943
0.04	5160	0.54	7054	1.04	8508	1.54	9382	2.04	9793	2.54	9945
0.05	5199	0.55	7088	1.05	8531	1.55	9394	2.05	9798	2.55	9946
0.06	5239	0.56	7123	1.06	8554	1.56	9406	2.06	9803	2.56	9948
0.07	5279	0.57	7157	1.07	8577	1.57	9418	2.07	9808	2.57	9949
0.08	5319	0.58	7190	1.08	8599	1.58	9429	2.08	9812	2.58	9951
0.09	5359	0.59	7224	1.09	8621	1.59	9441	2.09	9817	2.59	9952
0.10	5398	0.60	7257	1.10	8643	1.60	9452	2.10	9821	2.60	9953
0.11	5438	0.61	7291	1.11	8665	1.61	9463	2.11	9826	2.61	9955
0.12	5478	0.62	7324	1.12	8686	1.62	9474	2.12	9830	2.62	9956
0.13	5517	0.63	7357	1.13	8708	1.63	9484	2.13	9834	2.63	9957
0.14	5557	0.64	7389	1.14	8729	1.64	9495	2.14	9838	2.64	9959
0.15	5596	0.65	7422	1.15	8749	1.65	9505	2.15	9842	2.65	9960
0.16	5636	0.66	7454	1.16	8770	1.66	9515	2.16	9846	2.66	9961
0.17	5675	0.67	7486	1.17	8790	1.67	9525	2.17	9850	2.67	9962
0.18	5714	0.68	7517	1.18	8810	1.68	9535	2.18	9854	2.68	9963
0.19	5753	0.69	7549	1.19	8830	1.69	9545	2.19	9857	2.69	9964
0.20	5793	0.70	7580	1.20	8849	1.70	9554	2.20	9861	2.70	9965
0.21	5832	0.71	7611	1.21	8869	1.71	9564	2.21	9864	2.71	9966
0.22	5871	0.72	7642	1.22	8888	1.72	9573	2.22	9868	2.72	9967
0.23	5910	0.73	7673	1.23	8907	1.73	9582	2.23	9871	2.73	9968
0.24	5948	0.74	7704	1.24	8925	1.74	9591	2.24	9875	2.74	9969
0.25	5987	0.75	7734	1.25	8944	1.75	9599	2.25	9878	2.75	9970
0.26	6026	0.76	7764	1.26	8962	1.76	9608	2.26	9881	2.76	9971
0.27	6064	0.77	7794	1.27	8980	1.77	9616	2.27	9884	2.77	9972
0.28	6103	0.78	7823	1.28	8997	1.78	9625	2.28	9887	2.78	9973
0.29	6141	0.79	7852	1.29	9015	1.79	9633	2.29	9890	2.79	9974
0.30	6179	0.80	7881	1.30	9032	1.80	9641	2.30	9893	2.80	9974
0.31	6217	0.81	7910	1.31	9049	1.81	9649	2.31	9896	2.81	9975
0.32	6255	0.82	7939	1.32	9066	1.82	9656	2.32	9898	2.82	9976
0.33	6293	0.83	7967	1.33	9082	1.83	9664	2.33	9901	2.83	9977
0.34	6331	0.84	7995	1.34	9099	1.84	9671	2.34	9904	2.84	9977
0.35	6368	0.85	8023	1.35	9115	1.85	9678	2.35	9906	2.85	9978
0.36	6406	0.86	8051	1.36	9131	1.86	9686	2.36	9909	2.86	9979
0.37	6443	0.87	8078	1.37	9147	1.87	9693	2.37	9911	2.87	9979
0.38	6480	0.88	8106	1.38	9162	1.88	9699	2.38	9913	2.88	9980
0.39	6517	0.89	8133	1.39	9177	1.89	9706	2.39	9916	2.89	9981
0.40	6554	0.90	8159	1.40	9192	1.90	9713	2.40	9918	2.90	9981
0.41	6591	0.91	8186	1.41	9207	1.91	9719	2.41	9920	2.91	9982
0.42	6628	0.92	8212	1.42	9222	1.92	9726	2.42	9922	2.92	9982
0.43	6664	0.93	8238	1.43	9236	1.93	9732	2.43	9925	2.93	9983
0.44	6700	0.94	8264	1.44	9251	1.94	9738	2.44	9927	2.94	9984
0.45	6736	0.95	8289	1.45	9265	1.95	9744	2.45	9929	2.95	9984
0.46	6772	0.96	8315	1.46	9279	1.96	9750	2.46	9931	2.96	9985
0.47	6808	0.97	8340	1.47	9292	1.97	9756	2.47	9932	2.97	9985
0.48	6844	0.98	8365	1.48	9306	1.98	9761	2.48	9934	2.98	9986
0.49	6879	0.99	8389	1.49	9319	1.99	9767	2.49	9936	2.99	9986
0.50	6915	1.00	8413	1.50	9332	2.00	9772	2.50	9938	3.00	9987

Table A8 Normal Distribution

Values of z for given values of $\Phi(z)$ [see (3), Sec. 24.8] and $D(z) = \Phi(z) - \Phi(-z)$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$
1	−2.326	0.013	41	−0.228	0.539	81	0.878	1.311
2	−2.054	0.025	42	−0.202	0.553	82	0.915	1.341
3	−1.881	0.038	43	−0.176	0.568	83	0.954	1.372
4	−1.751	0.050	44	−0.151	0.583	84	0.994	1.405
5	−1.645	0.063	45	−0.126	0.598	85	1.036	1.440
6	−1.555	0.075	46	−0.100	0.613	86	1.080	1.476
7	−1.476	0.088	47	−0.075	0.628	87	1.126	1.514
8	−1.405	0.100	48	−0.050	0.643	88	1.175	1.555
9	−1.341	0.113	49	−0.025	0.659	89	1.227	1.598
10	−1.282	0.126	50	0.000	0.674	90	1.282	1.645
11	−1.227	0.138	51	0.025	0.690	91	1.341	1.695
12	−1.175	0.151	52	0.050	0.706	92	1.405	1.751
13	−1.126	0.164	53	0.075	0.722	93	1.476	1.812
14	−1.080	0.176	54	0.100	0.739	94	1.555	1.881
15	−1.036	0.189	55	0.126	0.755	95	1.645	1.960
16	−0.994	0.202	56	0.151	0.772	96	1.751	2.054
17	−0.954	0.215	57	0.176	0.789	97	1.881	2.170
18	−0.915	0.228	58	0.202	0.806	97.5	1.960	2.241
19	−0.878	0.240	59	0.228	0.824	98	2.054	2.326
20	−0.842	0.253	60	0.253	0.842	99	2.326	2.576
21	−0.806	0.266	61	0.279	0.860	99.1	2.366	2.612
22	−0.772	0.279	62	0.305	0.878	99.2	2.409	2.652
23	−0.739	0.292	63	0.332	0.896	99.3	2.457	2.697
24	−0.706	0.305	64	0.358	0.915	99.4	2.512	2.748
25	−0.674	0.319	65	0.385	0.935	99.5	2.576	2.807
26	−0.643	0.332	66	0.412	0.954	99.6	2.652	2.878
27	−0.613	0.345	67	0.440	0.974	99.7	2.748	2.968
28	−0.583	0.358	68	0.468	0.994	99.8	2.878	3.090
29	−0.553	0.372	69	0.496	1.015	99.9	3.090	3.291
30	−0.524	0.385	70	0.524	1.036			
31	−0.496	0.399	71	0.553	1.058	99.91	3.121	3.320
32	−0.468	0.412	72	0.583	1.080	99.92	3.156	3.353
33	−0.440	0.426	73	0.613	1.103	99.93	3.195	3.390
34	−0.412	0.440	74	0.643	1.126	99.94	3.239	3.432
35	−0.385	0.454	75	0.674	1.150	99.95	3.291	3.481
36	−0.358	0.468	76	0.706	1.175	99.96	3.353	3.540
37	−0.332	0.482	77	0.739	1.200	99.97	3.432	3.615
38	−0.305	0.496	78	0.772	1.227	99.98	3.540	3.719
39	−0.279	0.510	79	0.806	1.254	99.99	3.719	3.891
40	−0.253	0.524	80	0.842	1.282			

Table A9 t-Distribution

Values of z for given values of the distribution function $F(z)$ (see (8) in Sec. 25.3). Example: For 9 degrees of freedom, $z = 1.83$ when $F(z) = 0.95$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.90	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.40	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81
0.975	12.7	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.8	6.96	4.54	3.75	3.36	3.14	3.00	2.90	2.82	2.76
0.995	63.7	9.92	5.84	4.60	4.03	3.71	3.50	3.36	3.25	3.17
0.999	318.3	22.3	10.2	7.17	5.89	5.21	4.79	4.50	4.30	4.14

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.54	0.54	0.54	0.54	0.54	0.54	0.53	0.53	0.53	0.53
0.8	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
0.9	1.36	1.36	1.35	1.35	1.34	1.34	1.33	1.33	1.33	1.33
0.95	1.80	1.78	1.77	1.76	1.75	1.75	1.74	1.73	1.73	1.72
0.975	2.20	2.18	2.16	2.14	2.13	2.12	2.11	2.10	2.09	2.09
0.99	2.72	2.68	2.65	2.62	2.60	2.58	2.57	2.55	2.54	2.53
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.85
0.999	4.02	3.93	3.85	3.79	3.73	3.69	3.65	3.61	3.58	3.55

$F(z)$	Number of Degrees of Freedom									
	22	24	26	28	30	40	50	100	200	∞
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.25	0.25	0.25	0.25
0.7	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.52
0.8	0.86	0.86	0.86	0.85	0.85	0.85	0.85	0.85	0.84	0.84
0.9	1.32	1.32	1.31	1.31	1.31	1.30	1.30	1.29	1.29	1.28
0.95	1.72	1.71	1.71	1.70	1.70	1.68	1.68	1.66	1.65	1.65
0.975	2.07	2.06	2.06	2.05	2.04	2.02	2.01	1.98	1.97	1.96
0.99	2.51	2.49	2.48	2.47	2.46	2.42	2.40	2.36	2.35	2.33
0.995	2.82	2.80	2.78	2.76	2.75	2.70	2.68	2.63	2.60	2.58
0.999	3.50	3.47	3.43	3.41	3.39	3.31	3.26	3.17	3.13	3.09

Table A10 Chi-square Distribution

Values of x for given values of the distribution function $F(z)$ (see Sec. 25.3 before (17)). Example: For 3 degrees of freedom, $z = 11.34$ when $F(z) = 0.99$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.005	0.00	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16
0.01	0.00	0.02	0.11	0.30	0.55	0.87	1.24	1.65	2.09	2.56
0.025	0.00	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.70	3.25
0.05	0.00	0.10	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94
0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48
0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21
0.995	7.88	10.60	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.005	2.60	3.07	3.57	4.07	4.60	5.14	5.70	6.26	6.84	7.43
0.01	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26
0.025	3.82	4.40	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59
0.05	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85
0.95	19.68	21.03	22.36	23.68	25.00	26.30	27.59	28.87	30.14	31.41
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.99	24.72	26.22	27.69	29.14	30.58	32.00	33.41	34.81	36.19	37.57
0.995	26.76	28.30	29.82	31.32	32.80	34.27	35.72	37.16	38.58	40.00

$F(z)$	Number of Degrees of Freedom									
	21	22	23	24	25	26	27	28	29	30
0.005	8.0	8.6	9.3	9.9	10.5	11.2	11.8	12.5	13.1	13.8
0.01	8.9	9.5	10.2	10.9	11.5	12.2	12.9	13.6	14.3	15.0
0.025	10.3	11.0	11.7	12.4	13.1	13.8	14.6	15.3	16.0	16.8
0.05	11.6	12.3	13.1	13.8	14.6	15.4	16.2	16.9	17.7	18.5
0.95	32.7	33.9	35.2	36.4	37.7	38.9	40.1	41.3	42.6	43.8
0.975	35.5	36.8	38.1	39.4	40.6	41.9	43.2	44.5	45.7	47.0
0.99	38.9	40.3	41.6	43.0	44.3	45.6	47.0	48.3	49.6	50.9
0.995	41.4	42.8	44.2	45.6	46.9	48.3	49.6	51.0	52.3	53.7

$F(z)$	Number of Degrees of Freedom							
	40	50	60	70	80	90	100	> 100 (Approximation)
0.005	20.7	28.0	35.5	43.3	51.2	59.2	67.3	$\frac{1}{2}(h - 2.58)^2$
0.01	22.2	29.7	37.5	45.4	53.5	61.8	70.1	$\frac{1}{2}(h - 2.33)^2$
0.025	24.4	32.4	40.5	48.8	57.2	65.6	74.2	$\frac{1}{2}(h - 1.96)^2$
0.05	26.5	34.8	43.2	51.7	60.4	69.1	77.9	$\frac{1}{2}(h - 1.64)^2$
0.95	55.8	67.5	79.1	90.5	101.9	113.1	124.3	$\frac{1}{2}(h + 1.64)^2$
0.975	59.3	71.4	83.3	95.0	106.6	118.1	129.6	$\frac{1}{2}(h + 1.96)^2$
0.99	63.7	76.2	88.4	100.4	112.3	124.1	135.8	$\frac{1}{2}(h + 2.33)^2$
0.995	66.8	79.5	92.0	104.2	116.3	128.3	140.2	$\frac{1}{2}(h + 2.58)^2$

In the last column, $h = \sqrt{2m - 1}$, where m is the number of degrees of freedom.

Table A11 F-Distribution with (m, n) Degrees of Freedom

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.95**

Example: For (7, 4) d.f., $z = 6.09$ if $F(z) = 0.95$.

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$
1	161	200	216	225	230	234	237	239	241
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88