Summary of achievements

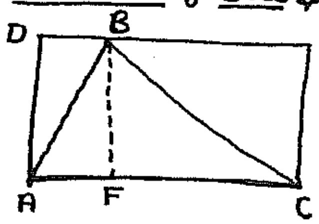
Egyptian "facts" on areas of Δ's & quad^{ls}
& vol^{ms} of some pyrimidal solids
(obtained by "ind^{n}" from simple cases)
Expressed "algebraically: Eg "To find the content of a Δ multiply the tp-r (side chosen as base) (measured in cubits) by the mryt (⊥ ht. ?) and take half."
(Rhind Papyrus. 1,650 B.C.)

N.B. results expressed in terms of standard units (implicit) some volum calc's. even involved conversion factors between units.

Thales → ~624 – 547 B.C.

Greece introduction of "deductive" proofs.
(gave cond^{ns} for 2 Δ's to be "equal in all respects")
Area taken as an implicit property of a surface not requiring definition. He, as did all the latter Hellenic mathematicians, considered the ratio of two surfaces (areas) not the area of one figure.*

Eg



(area) Δ ABC : (area) ADEC = 1:2
the Δ AFB is half the rect. ADBF
and the Δ FBC is half the rect FBEC.
But Δ's AFB and FBC together make Δ ABC and similarly rects. ADBF and FBEC together make ADEC.

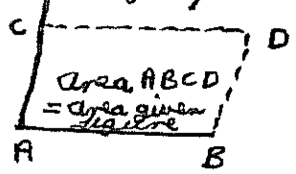
(* This could be due to a Babylonian influence on Greek mathematics pre-thales. However it seems the Babylonian's, like the Egyptians determined areas numerically using standard units of length measurement – Eg Tablet YBC 4612) ‡

‡ although in other contexts they "may" have had some notion of proportion & ratio (in trig.)

Pythagoras (& his "school")

(~572 ~ 495 B.C.)

1) refined the theory of proportions used to express the ratio of commensurable lines and surfaces.
2) discovered the existence of incommensurables (side to the diagonal of a square.), blocking the use of arithmetical reasoning in geometry untill a satisfactory extension to the concept of number had been made (such was not fully the case till [illegible] although, by admitting "symbols" such as √5, π etc. which could be "formally" handled like numbers, the difficulty had ceased to be an obstical much earlier.)
3) Developed the theory of "application of areas"
This "theory" is really the definition of a class of problems and the solution of some particular ones.
viz:
i) (The case of immediate interest to us)
To apply (construct) on a given st. line AB as base a parallelogram containg a given angle (usually a rt. ∠, and hence a rectangle) and equal (in area) to a given figure.



Note If ABCD and ABC'D' result from the application of areas for two figures, Then the area of one figure to that of the other is as AC is to AC' and so the problem of comparing areas is reduced to that of comparing the "lengths" of st. lines.

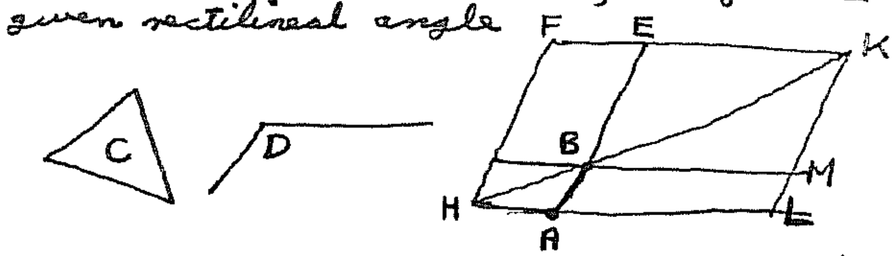
This and the related problem of constructing from a given figure an equal square provided the essence of the Post Pythagorus treatment of areas.

To illustrate application of areas consider

EUCLID: Proposition 44 of Book I. (Probably of a Pythagorean origin)

"To a given st. line to apply, in a given rectilineal angle, a parallelogramme equal to a given triangle."

Let AB be the given st. line, C the given Δ and D the given rectilineal angle



Let the ∥^{l}ogram BEFG be constuct equal to Δ C in the angle EBG which is equal to D(*); let it be so placed that BE is in a st. line with AB.
Let FG be drawn through H and let AH be drawn through A ∥^{l} BG. Let HB be joined.
Since HF falls upon the ∥^{ls} AH, EF
∠ AHF and ∠ HFE equal two rt. ∠'s.
∴ ∠ BHG & ∠ GFE together are less than two rt. angles; and st. lines produced indefinately from angles less than 2 rt ∠'s meet;
∴ HB and FE when produced will meet at K.
Through K draw KL ∥^{l} EA and let HA and GB be produced to the points L and M on KL. Then HLKF is a paralelogram as are AHGB and BEKM. While the ∥^{l}ograms ABML and GFEB are so called complements about HK and therefore equal(*). But GFEB equals C, ∴ ABML equals C, while ∠ ABM = D as does ∠ ALM.
∴ the ∥^{l}ogram ABML equal to C and with angle D has been applied to AB
QED

(*) denote constructions or results previously established.

more generally the method of application of areas was

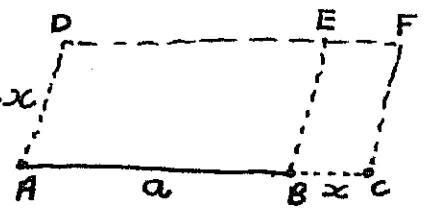
ii) To apply to a given st. line a $\parallel^{l}$ogram equal to a given figure (area A) and either

① exceeding (ὑπερβάλλειν)

or ② falling-short (ἐλλείπειν)

by a $\parallel^{l}$ogram similar to a given $\parallel^{l}$ogram (usually a square)

Eg Given AB and $\parallel^{l}$ogram P to find ABFC equal to A but exceeding the $\parallel^{l}$ogram ADEB on AB itself by the parallelogram BEFC similar to D.



N.B. ① corresponds to solving

$A = (a+x)x = x^2 + ax$ (the equation of an hyperbola in the A – x plane)

and ② to

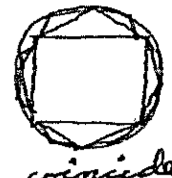
$A = (a-x)x = ax - x^2$ (the equation of an ellipse in the A – x plane)

in the case of P being a square. We see here the origins of our names for the conic sections, ___ (being the equivalent of) the greek for exceeds, ___ applies, ___ , falls short

This theory provided a "geometric algebra" capable of solving problems which today would be formulated as quadratic equa

Antiphon (contemporary of Socrates) (469 – 399 B.C.)

attempted the quadrature of the ⊙ by inscribing successive regular polygons in it



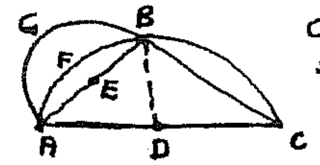
"believing" that in this way the area of the ⊙ would be used up, and we should sometime have a polygon the sides of which would, owing to their smallness, coincide with the circumference of the ⊙. (a similar procedure using inscribed & excribed figures was used by BRYSON a student of Socrates / Euclid).

DEMOCRITUS ($\sim$465 $\sim$ 375) known for his "atomic theory" of "matter" enunciated the propositions "that the volume of a pyramid on any polygonal base is $\frac{1}{3}$ the volume of the prism with the same base and height, and similarly for the volume of a cone to the corresponding circular cylinder," though he was unable to give "scientific" proofs. In part we must show that pyramids (or cones or cylinders) on the same base and having equal heights have the same volume. Here Democritus may have reasoned: "If two pyramids be cut respectively by planes $\parallel^{l}$ to the base and dividing the heights in the same ratio, then the corresponding sections are equal. ∴ the 2 pyramids contain the same "infinite" number of equal plane sections (or infinitely thin laminae) and are therefore equal in content. (Heath M. of G.M.).

Note He did not mathematical magnitudes (Eg the length of a line, or volume of a cylinder) as made up of "indivisibles" – "mathematical atoms".

Indeed he argued against such atomism; holding lines to be divisible ad infinitum, in opposition to the view taken by Xenocrates (396 – 314 B.C.) and some others. Indeed the idea of "mathematical atoms" was severely routed by the major Greek philosophers and so reasoning like the above, or that of Antiphon, had to be expunged from geometry. Indeed the criticism by George Berkeley of the similar intuitive and unfounded ideas employed by Newton, is pale compared to the objections of Greek philosophers. Thus of Antiphon's argument, Aristotle concluded "it was an error which was even beneath the notice of geometers."

Hippocrates ($\sim$440 / $\sim$470 $\sim$) in attempting to squre the ⊙ (?) showed how to square certain lunes. Eg



area of lune AGBF = area Δ ADB.

he also enunciated: "2 ⊙'s are to one another as the sqs. on their diameters" though it seems he failed to give scientific proof.

Plato (

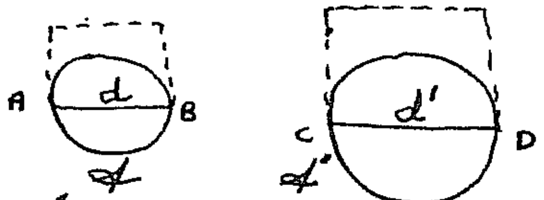
EUDOXUS ($\sim$408 $\sim$ 355 B.C.) developed the theory of proportion as given in Euclid Book V defining equal ratios in a way analogous to that used by Dedekind and Weierstrass in their axiomatic development of real numbers. Perhaps more importantly however he resolved the above dilema of mathematical atomism providing a scientific approach to the proof the results such as those of Democritus by

The "method of Exhaustion", based upon the Lemma (deducible from the Axiom of Archimedes): Euclid Book X Proposition I

"Two unequal magnitudes being set out, if from the greater there being subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out," (N.B. this once and for all dispells any idea of mathematical atomism. It is also "equivalent" to the modern statement of $ar^n \to 0$ as $n \to \infty$ $(r < \frac{1}{2})$ – and as we shall see was often used by the Greeks to circumvent points in an argument where we would appeal to a limitting idea). The method by means of a "reductio ad absurdum" allowed the comparison of areas for non-polygonal figures. It is best illustrated by means of an example: EUCLID Book XII Proposition 2. (although the proof is almost certainly attributable to Eudoxus)

"Circles are to one another as the squares on the diameters." As the proof of this requires considerable space, I will here give only a summary of the argument – However the student urged to at least inspect the full argument.

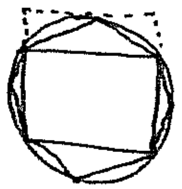
Let AB and CD be ~~re~~spective diameters of the 2 ⊙'s.



If the square on AB is not to the square on CD as the circle on AB is to that on CD, then as the square on AB is to the square on CD so will the circle on AB be either to some less area than the circle on CD, or to a greater.

First let it be in that ratio to a less area S.
(i.e. $d^2 : d'^2 = \mathcal{A} : S < \mathcal{A}'$, the existence of such a lesser area S is an assumption which seems to require more careful attention than it here seems to be given.)

Euclid now proves that an inscribed square in the ⊙ on CD has more than half the area of $\mathcal{A}'$ and that the regular inscribed octagon formed be erecting isosceles Δ's on each side of the square when subtracted takes away more than half what was left by the square.



He then infers that the same thing will happen whenever the number of sides of the inscribed polygon is doubled.

Thus by the above lemma we may continue the construction of such polygons until we arrive at one, P' leaving over segments together less than the excess of $\mathcal{A}'$ over S.
i.e. $\mathcal{A}' >$ area of polygon $P' > S$

Let a similar polygon P to P' be inscribed in the circle on AB
Then by Euclid's previous proposition (XII 1)

$$\text{area } P : \text{area } P' = d^2 : d'^2 = \mathcal{A} : S$$

But area $P < \mathcal{A}$

$\therefore$ area $P' < S$

which is impossible, Hence S cannot be less than $\mathcal{A}'$.

Now suppose the other case where the square on AB to that on CD is in the ratio of the circle on AB to an area S greater than the circle on CD.

Since $d^2 : d'^2 = \mathcal{A} : S$
we have $d'^2 : d^2 = S : \mathcal{A}$
supposing $S : \mathcal{A} = \mathcal{A}' : T$
we have since $S > \mathcal{A}'$ that $\mathcal{A} > T$
and so
$d'^2 : d^2 = \mathcal{A}' : T$ where $T < \mathcal{A}$

the impossibility of which follows by exactly the same argument as that used above, for it corresponds to the same problem with the role of the two circles interchanged.

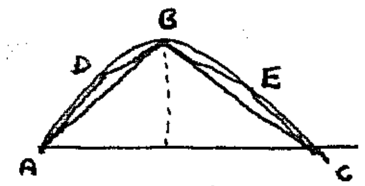
latter archimedes replaced this by an independent argument based on circumscribed polygons

While it will be appreciated that this method is rigorous and was to be used by all subsequent greek geom. when "formally" establishing comparisons between areas of non-polygonal figures, we can appreciate that it is "difficult of application" and fails to provide a means for deriving the answer only allowing the truth to be demonstrated (once guessed).

ARISTOTLE (384 – 322 B.C.)
EUCLID (~ 300 B.C.) compiled into a systematic exposition the work of his predecessors.
~365 ~ ?
~330 ~ 275 B.C.

ARCHIMEDES (287 – 212 B.C.) gave Greek maths some of its crowning achievements. On problems of content he obtained quadratures of parabolic sections, results on the volume of paraboloids of revolution and on the surface area of the sphere. In the scientific proof of these results he employed the method of exhaustion using arguments which, in that context, anticipated many [redacted] results on infinite series latter used for the founding of the calculus.

Eg In formally setting out the quadrature of the parabola he argued thus:



He shows
(area) Δ ADB + (area) Δ BEC = $\frac{1}{4}$ (area) Δ ABC
and similarly each addition of a similar kind to the inscribed figure adds $\frac{1}{4}$ that of the last.

Next he shows that given any number of areas a, b, c, ···, z of which a is greatest and each is 4 times the next in order (i.e. a = 4b, b = 4c, ···) then

$$a + b + c + \cdots + z + \tfrac{1}{3}z = \tfrac{4}{3}a$$

(i.e. $a\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \frac{1}{3}\frac{1}{4^n}\right) = \frac{4}{3}a$ or equivalently

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} = \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}} = \frac{4}{3} - \frac{1}{3}\left(\frac{1}{4^n}\right).\ )$$

and so the "error" term $\frac{1}{3}z$ is more than halved at each addition and hence by a reductio ad absurdum he is able to show that the inscribed figure exhausts the segment and so segment : area Δ ABC = 4 : 3.

In the determination of the volume of revolution of a parabolic segment, and the area bounded by the polar axis and one turn of an (archimedean) spiral respectively he similarly treated the "series"

$$\frac{n^2}{1 + 2 + \cdots + n} \quad \text{and} \quad \frac{1 + 4 + \cdots + n^2}{n^3} \quad \text{as } n \to \infty$$

and so in effect (though not in spirit) determined $\int_0^1 x\,dx$ & $\int_0^1 x^2\,dx$, he even may have found the corresponding result for sums of cubes as so $\int_0^1 x^3\,dx$.

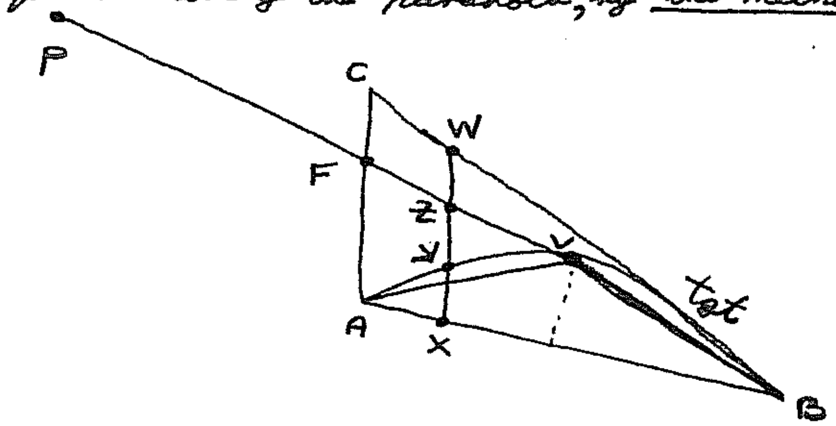
Thus it is proved that as the ratio of the square on AB is to that on CD neither is the ratio of the circle on AB to a less or a greater area than that of the circle on CD
$\therefore$ as the square on AB is to the square on CD so is the ⊙ on AB to the ⊙ on CD. Q.E.D. (From Heath's commentary on the Elements)

... used by all Greek ... when "formally" establishing comparisons between areas,

Of overriding importance in accessing Archimedes nearness to the modern idea of integral is a treatise addressed to Eratosthenes which remained largely unknown (and so had little influence on the latter course of mathematics) until "discovered" in 1906 in which he outlines the heuristic procedure whereby he was first led to many of his results. This was

<u>the</u> (mechanical) <u>method of Archimedes</u> which is most clearly shown by an example.

quadrature of the parabola, by <u>the method</u>:



In the opposite construction (where PF = FB) for any pt X we have, from the properties of the parabola, that

$$\frac{XW}{XY} = \frac{AB}{AX} = \frac{BF}{FZ} = \frac{FP}{FZ}$$

But Z is the c of g of XW, so from the law of the lever with F as fulcrum, we have that XW in its present position will balance XY moved to P.

Thus inasmuch as ΔABC consists of lines XW and the segment similarly of XY we conclude that ΔABC in its present position will balance the segment moved so as to have c of g at P (or equivalently moved so as to be all "condensed" at P). Now the c of g of ΔABC is on BF and $\frac{1}{3}$ of the distance from F to B so

$$\frac{\text{segment}}{\Delta ABC} = \frac{\frac{1}{3}FB}{PF}$$ again by the law of the lever

or segment $= \frac{1}{3}\Delta ABC = \frac{4}{3}\Delta AVB$ (as before)