3) *Function spaces.* (These are perhaps the most important examples and the ones to which our ensuing theory has the most important and immediate applications - some of which will be taken up in later chapters.)

Recall that the set of all real valued functions with common domain $D \subseteq \mathbb{R}$, usually a closed interval $[a,b]$, forms a vector space with respect to the point-wise defined operations:

$$"f+g":D \to \mathbb{R} : x \mapsto f(x)+g(x)$$

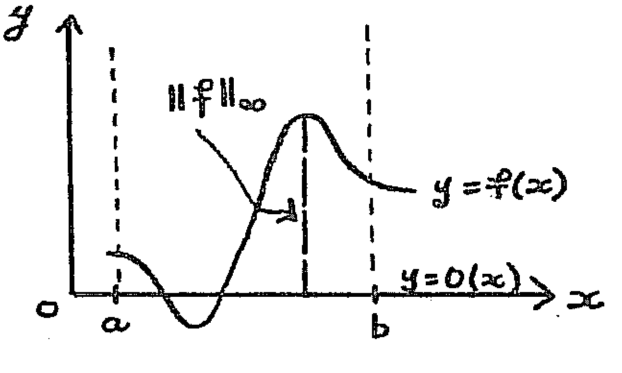
$$"\lambda f": D \to \mathbb{R} : x \mapsto \lambda f(x) .$$

Let $\mathcal{B}$ denote the vector subspace of all bounded functions on D; that is, functions f for which there exists a constant $M_f$ such that

$$|f(x)| \le M_f, \text{ for all } x \in D.$$

Defining a norm on $\mathcal{B}$ amounts to providing a measure of the proximity of a function f to the zero function $0[0(x) = 0$, all $x \in D]$. Now at any $x \in D$ the function f differs in value from the zero function by $|f(x)|$. In many applications it seems reasonable to take the "largest" such difference in value to be the norm of f. Accordingly we define the <u>uniform</u> (or Tchebyscheff) <u>norm</u> on $\mathcal{B}$ by

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$$



This is the analogue of the $\|.\|_\infty$ in the last two examples. To see that it is indeed a norm function we proceed as follows.

Clearly: $\|f\|_\infty \geq 0$; $\|f\|_\infty = 0 \Leftrightarrow |f(x)| = 0$ all x

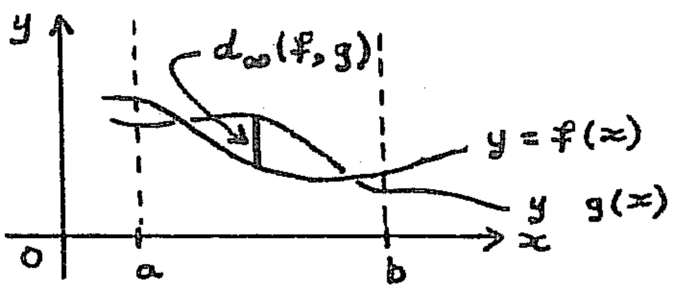
$$\Leftrightarrow f = 0;$$

$$\|\lambda f\|_\infty = \sup_{x\in D} |\lambda f(x)| = \sup_{x\in D} |\lambda|\,|f(x)| = |\lambda|\,\|f\|_\infty;$$

$$\begin{aligned}\|f+g\|_\infty &= \sup_{x\in D} |f(x)+g(x)| \\ &\leq \sup_{x\in D} (|f(x)| + |g(x)|) \\ &\leq \sup_{x\in D} |f(x)| + \sup_{x\in D} |g(x)| \\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

and so $\|.\|_\infty$ is indeed a norm on $\mathcal{B}$, inducing the *uniform metric*.

$$d_\infty(f,g) = \|f-g\|_\infty = \sup_{x\in D} |f(x) - g(x)|.$$




We will later show that any continuous function defined on a closed interval [a,b] is bounded. Thus an important subspace of $(\mathcal{B}, \|.\|_\infty)$ is $(C[a,b], \|.\|_\infty)$ where $C[a,b]$ denotes the space of all continuous real valued functions with domain [a,b]. By replacing summation with integration it is possible to define other norms on $C[a,b]$ in analogy with $\|.\|_p$ (p=1, or 2) of the last two examples. Thus for $C[a,b]$ we have the uniform norm defined above,

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

and in addition:

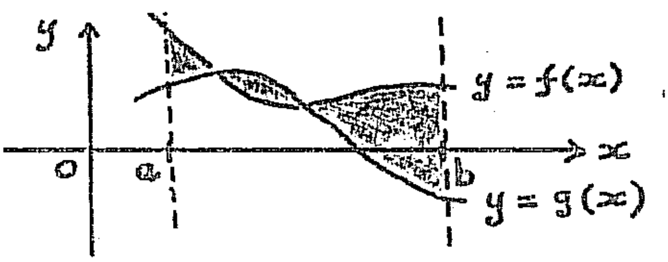
$$\|f\|_1 = \int_a^b |f(x)|dx \ ;$$

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2 \, dx}$$

A proof that $\|.\|_2$ is a norm will be given in the next section on inner-products. That $\|.\|_1$ satisfies (N1) to (N4) is left as an easy EXERCISE.
NOTE: $\|.\|_1$ takes the "absolute area" between f and the zero function 0 (the x-axis) as a measure of the distance between these two functions.
The metric induced by this norm,

$$d_1(f,g) = \int_a^b |f - g|$$

is represented by the area of the shaded region in the following sketch




EXAMPLE: For $f(x) = x^3+x+1$ and $g(x) = x^3 + x^2 + \frac{1}{2}x + 1$ in $C[0,1]$
we have

$$d_\infty(f,g) = \max_{0\le x\le 1} |\tfrac{1}{2}x - x^2|$$

$$= \tfrac{1}{2} \qquad \text{(check this)}$$

while,

$$d_1(f,g) = \int_0^1 \left|\tfrac{1}{2}x-x^2\right| dx = \int_0^{\frac{1}{2}} (\tfrac{1}{2}x-x^2)\,dx + \int_{\frac{1}{2}}^1 (x^2-\tfrac{1}{2}x)\,dx$$

$$= \frac{1}{8}\,.$$

EXERCISE: (a) In $C[0,1]$ determine the values of $d_\infty(f,g)$ and $d_1(f,g)$ when $f(x) = \sin x$ and $g(x) = x$, and also when $g(x) = x - x^3/6$.

(b) Using Taylor's Theorem with remainder, obtain estimates for

$$d_\infty(f,p_n) \text{ and } d_1(f,p_n) \text{ in } C[0,1]$$

when

$$f = \exp \text{ and } p_n(x) = \sum_{m=0}^{n} x^m/m!$$

INNER PRODUCT SPACES.

Just as metrics are induced by the richer structure of a norm, a norm itself sometimes results because of other structure carried by the space. In particular this is so when the space has an inner-product defined on it.

DEFINITION. An inner-product for the vector space X is a mapping from ordered pairs of elements of X into the real field;

$X \times X \to \mathcal{R}$: $(x,y) \mapsto \langle x,y\rangle$, which satisfies:

(IP1) $\langle x,x\rangle > 0$ for all $x \in X$ and $x \neq 0$. (positivity)

(IP2) $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y \in X$. (symmetry)

(IP3) $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ for all $x,y \in X$ and $\lambda \in \mathcal{R}$ (homogeneity)

(IP4) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ for all $x,y,z \in X$. (additivity)

EXAMPLES

(1) On $X = \mathcal{R}^n$ the usual "dot" or scalar product of two vectors $(\underset{\sim}{x}.\underset{\sim}{y})$ is an inner-product: $\langle \underset{\sim}{x},\underset{\sim}{y}\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$

where $\underset{\sim}{x} = (x_1,x_2,\dots,x_n)$ and $\underset{\sim}{y} = (y_1,y_2,\dots,y_n)$

[You should verify this as an EXERCISE.]

This is not the only inner-product which can be defined on $R^n$, indeed for any set of n strictly positive numbers $w_1, w_2, \ldots, w_n$ an inner-product is defined by

$$<\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{n} w_i x_i y_i \ .$$

*Such "weighted" inner-products are of considerable importance in some areas of statistics.*

(2) An inner-product on $\ell_2$ the space of all square summable sequences (see Example 2 of normed linear spaces) may be defined by

$$<\underset{\sim}{x}, \underset{\sim}{y}> = \sum_{n=1}^{\infty} x_n y_n$$

[That this expression is finite, for all sequences $\underset{\sim}{x} = x_1, x_2, \ldots, x_n, \ldots$ and $\underset{\sim}{y} = y_1, y_2, \ldots, y_n, \ldots$ for which $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} y_n^2$ are finite, follows from the Cauchy - Schwarz - Bunyakowski inequality to be established below. That it satisfies the four axioms of an inner-product is readily checked and so is left as an (optional) EXERCISE.]

(3) For $X = C[a,b]$ we can define

$$<f,g> = \int_a^b f(x)g(x)dx \text{ for all } f,g \in C[a,b].$$

With the exception of (IP1) the axioms of an inner-product are easily verified:

(IP1) If $f \in C[a,b]$ is not the zero function, then there exists some $x_0 \in [a,b]$ for which $f(x_0) \neq 0$. By the continuity of f there exists $\delta > 0$ such that for $x \in [a,b]$

$$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < |f(x_0)|/_2.$$

Consequently, for $x_0-\delta < x < x_0 + \delta$ we have

$$f(x)^2 > \frac{1}{4} |f(x_0)|^2 > 0 \ ,$$

and so

$$\langle f,f\rangle = \int_a^b f(x)^2 dx$$

$$= \int_a^{x_0-\delta} f(x)^2 dx + \int_{x_0-\delta}^{x_0+\delta} f(x)^2 dx + \int_{x_0+\delta}^{b} f(x)^2 dx$$

$$\geqslant \int_{x_0-\delta}^{x_0+\delta} f(x)^2 dx \qquad \text{(as } f(x)^2 \geqslant 0 \text{ for all } x\text{)}$$

$$> \frac{1}{4}|f(x_0)|^2 \times 2\delta$$

$$> 0 .$$

(IP2) For $f,g \in C[a,b]$ we have

$$\langle g,f\rangle = \int_a^b g(x)f(x)dx$$

$$= \int_a^b f(x)g(x)dx$$

$$= \langle f,g\rangle .$$

(IP3) For $f,g \in C[a,b]$ and $\lambda \in \mathbb{R}$

$$\langle \lambda f,g\rangle = \int_a^b \lambda f(x)g(x)dx$$

$$= \lambda\int_a^b f(x)g(x)dx$$

$$= \lambda\langle f,g\rangle$$

(IP4) For $f,g,h \in C[a,b]$

$$\langle f+g,h\rangle = \int_a^b [f+g](x)h(x)dx$$

$$= \int_a^b [f(x)+g(x)]h(x)dx$$

$$= \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx$$

$$= \langle f,h\rangle + \langle g,h\rangle .$$

The following useful properties of an inner-product are immediate consequences of the above axioms, which you should prove for yourself as EXERCISES.

(1) $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

(2) $\langle x, \lambda y\rangle = \lambda\langle x,y\rangle$

(3) $\langle x,x\rangle = 0$ if and only if $x = 0$

(4) $y = 0$ if and only if $\langle x,y\rangle = 0$ for all $x \in X$.

A vector space X together with an inner-product $\langle .\,,.\rangle$ will be referred to as an <u>inner-product space</u>.

*Inner-product spaces were implicitly studied by many mathematicians [For example; the two German mathematicians, David Hilbert (1862-1943) and Erhard Schmidt (1876-1959) and the Hungarian Friederich Riesz (1880-1956)] during the first three decades of the twentieth century, however, the axioms were not made explicit until 1929 when they were expounded by John von Neumann (1903-1957) as a basis for his axiomatic development of quantum mechanics.*

The importance of an inner-product space for our purposes is that *the formula*

$$\|x\| = \sqrt{\langle x,x\rangle}$$

*defines a norm on* X. The axioms (N1), (N2) and (N3) are easily established:

(N1) $\langle x,x\rangle$ is greater than 0 if $x \neq 0$ and equals 0 if $x = 0$ consequently, for all x, $\langle x,x\rangle \geqslant 0$ and so $\|x\| \geqslant 0$ .

(N2) $\|x\| = 0 \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0$

(N3) $$\|\lambda x\| = \sqrt{\langle\lambda x, \lambda x\rangle} = \sqrt{\lambda^2\langle x,x\rangle}$$
$$= \sqrt{\lambda^2}\,\sqrt{\langle x,x\rangle}$$
$$= |\lambda|\,\|x\| \;.$$

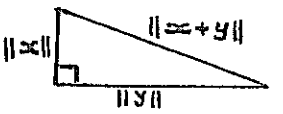
Thus, it only remains to establish (N4), the triangle inequality. This is done as part (ii) of the following theorem.

<u>Theorem 1</u>. *In any inner-product space the following are true:*

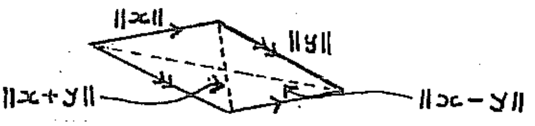
(i) $|\langle x,y\rangle| \leqslant \|x\|\,\|y\|$ *(Cauchy-Schwarz-Bunyakowski inequality)*

(ii) $\|x + y\| \leqslant \|x\| + \|y\|$ *(triangle inequality)*

(iii) If $\langle x,y\rangle = 0$ $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ *(Pythagorean identity)*




(iv) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *(Parallelogram law)*




Proof.

*(i)* If $y = 0$ both sides are zero and so the result is immediate.

If $y \neq 0$ we proceed as follows. For any scalar $\alpha$ we have

$0 \le \|x+\alpha y\|^2 = \langle x+\alpha y, x+\alpha y\rangle$

$$= \langle x,x\rangle + \langle \alpha y,x\rangle + \langle x,\alpha y\rangle + \langle \alpha y,\alpha y\rangle$$

$$= \|x\|^2 + \alpha\langle x,y\rangle + \alpha[\langle x,y\rangle + \alpha\|y\|^2]$$

So, choosing $\alpha = -\langle x,y\rangle/\|y\|^2$ (possible as $\|y\| \neq 0$) we see that the term in square brackets is zero and $0 \le \|x\|^2 + \alpha\langle x,y\rangle = \|x\|^2 - \frac{\langle x,y\rangle}{\|y\|^2}\langle x,y\rangle$.

Rearranging we therefore have

$$\langle x,y\rangle^2 \le \|x\|^2\|y\|^2$$

Taking square roots we therefore have

$$|\langle x,y\rangle| \le \|x\|\|y\| \quad \text{as required.}$$

(ii) $\|x + y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$

$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

$$\leqslant \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \text{(by i)}$$

$$= (\|x\| + \|y\|)^2$$

so taking the square root of both sides we obtain the triangle inequality, which is otherwise known as <u>Minkowski's inequality</u> in this context.

(iii) Follows immediately from the first two lines of the proof in (ii) with $\langle x,y \rangle = 0$.

(iv) From the first two lines of the proof in (ii)

$$\|x + y\|^2 = \|x\|^2 + 2\langle x,y \rangle + \|y\|^2 \quad .$$

Similarly by expanding $\langle x-y,x-y \rangle$ we obtain

$$\|x - y\|^2 = \|x\|^2 - 2\langle x,y \rangle + \|y\|^2 \quad .$$

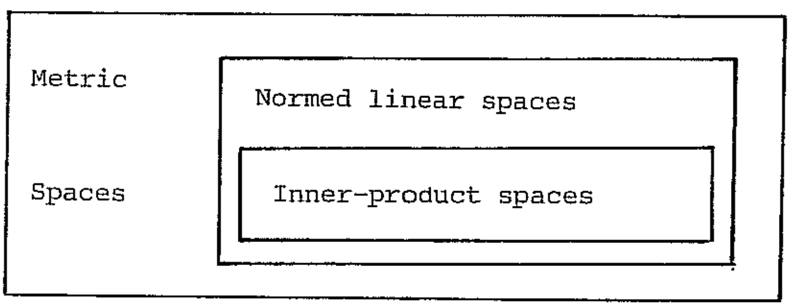
Adding these two identities yields the parallelogram law. ∎

The Euclidean norm in $R^n$ and the norms denoted by $\|.\|_2$ in $\ell_2$ and in $C[a,b]$ arise from the above formula for the appropriately defined inner-products. Thus, we have finally arrived at a proof of the fact that these are indeed norm functions. All of these norms have special properties; for example, they satisfy the parallelogram law, which are not satisfied by all norm functions;

For example in $\ell_1^2$ the norm $\|\underset{\sim}{x}\|_1 = |x_1| + |x_2|$ does not satisfy the parallelogram rule. To see this observe that for $\underset{\sim}{x} = (1,0)$ and $\underset{\sim}{y} = (0,1)$ we have

$$2(\|\underset{\sim}{x}\|^2 + \|\underset{\sim}{y}\|^2) = 4 \quad \text{while} \quad \|\underset{\sim}{x} + \underset{\sim}{y}\|^2 + \|\underset{\sim}{x} - \underset{\sim}{y}\|^2 = 8 \ .$$

This shows that $\|.\|_1$ does <u>not</u> arise from any inner-product according to the formula $\sqrt{\langle x,x \rangle}$. Diagrammatically we have the following situation.




*Indeed, any norm satisfying the parallelogram law may be shown to arise from a suitably defined inner-product. Thus the parallelogram law characterizes inner-product spaces (the Jordan-von Neumann characterization). Honours students may like to attempt proving this as an exercise. (See exercise 2 on the next page for a hint.)*

EXERCISES

1) Motivated by (iii) of the previous theorem and the ordinary idea of perpendicularity in $R^n$, viz. $\underset{\sim}{x}.\underset{\sim}{y} = 0$, we define two vectors $x,y$ in the inner-product space X to be *orthogonal* if $\langle x,y \rangle = 0$.

i) Show that x and y are orthogonal if [and only if - by iii) of the theorem] they satisfy the Pythagorean identity

$\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

*ii) Show that x is orthogonal to y if and only if $\|x\| \leqslant \|x+\lambda y\|$ for all scalars $\lambda \in \mathbb{R}$. (Can you give a geometric interpretation to this result.)

*The condition $\|x\| \leqslant \|x+\lambda y\|$ all $\lambda$ is often used as a generalized definition of x being orthogonal to y in any normed linear space (R.C. James, 1947).*

iii) If $x \neq 0$ and y are elements of X verify that x and

$y - \frac{\langle x,y\rangle}{\langle x,x\rangle} x$ are orthogonal.

2) For any inner-product space, verify the "polarization identity":

$$\langle x,y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

3) In the space $C[0,1]$, show that

$$d_2(x,\sin x) = \sqrt{\langle x - \sin x, x - \sin x\rangle} = \sqrt{\int_0^1 (x-\sin x)^2 dx} \doteq 0.061$$

while

$$d_2(x-\frac{x^3}{6}, \sin x) \doteq 0.002.$$

4) In terms of the Euclidean norm $\|.\|_2$ on $\mathbb{R}^2$ note that the "post-office" metric is given by

$$d(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 0 & \text{if } \underset{\sim}{x} = \underset{\sim}{y} \\ \|\underset{\sim}{x}\|_2 + \|\underset{\sim}{y}\|_2 & \text{otherwise.} \end{cases}$$

Use this to verify that d is indeed a metric.

[Hint. When proving (M4) consider the cases: $\underset{\sim}{x} = \underset{\sim}{z}$; $\underset{\sim}{x} = \underset{\sim}{y}$ or $\underset{\sim}{y} = \underset{\sim}{z}$; $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are all distinct.]

§1.2 <u>*Geometry in Metric spaces.*</u> *Balls, convexity and boundedness.*

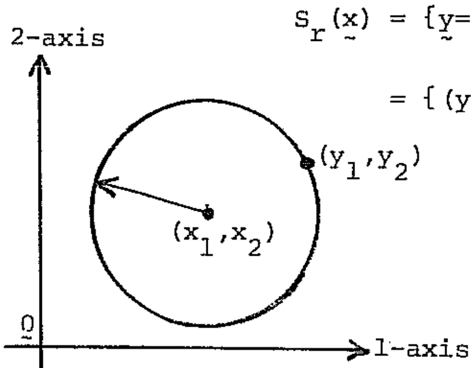
In ordinary (Euclidean) geometry a circle (or sphere) is defined to be the set of points equidistant from a given point - the centre. It is possible to generalise this notion into any metric space.

DEFINITION: Let $(X,d)$ be a metric space. The <u>*sphere* of radius $r > 0$ and centre $x$</u> is the set $\{y \in X : d(x,y) = r\}$ which we denote by $S_r(x)$.

NOTES 1) Under this definition we continue to use the term sphere regardless of the dimension of the space. Thus for example, in $\ell_2^2$ ($R^2$ with the Euclidean metric) the "sphere" centre $\underset{\sim}{x} = (x_1,x_2)$ and radius r is what would conventionally be referred to as the circle with centre x and radius r:

$$S_r(\underset{\sim}{x}) = \{\underset{\sim}{y}=(y_1,y_2)\in R^2 : d_2(\underset{\sim}{x},\underset{\sim}{y}) = r\}$$

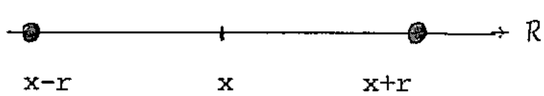
$$= \{(y_1,y_2) : (y_1-x_1)^2+(y_2-x_2)^2=r^2\}$$




While, in $R$ with the usual metric, $d_1(x,y) = |x-y|$, we see that the "sphere" $S_r(x)$ consists of two points

$$S_r(x) = \{y \in R : |x-y| = r\}$$

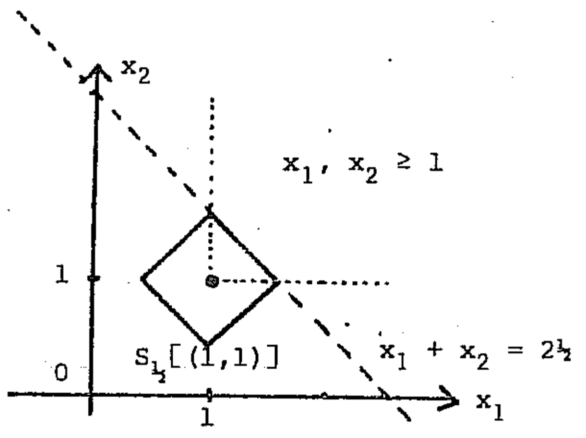
$$= \{x-r, x+r\}$$




2) The "shape" of a sphere varies with the particular norm used. For example, in $\ell_1^2$ the sphere (circle) with centre (1,1) and radius ½ is

$$S_{\frac{1}{2}}(1,1) = \{\underset{\sim}{x} = (x_1,x_2) \in \mathbb{R}^2 : d_1(\underset{\sim}{x},(1,1)) = \|\underset{\sim}{x} - (1,1)\|_1 = \tfrac{1}{2}\}$$
$$= \{(x_1,x_2) : |x_1 - 1| + |x_2 - 1| = \tfrac{1}{2}\}$$

which is the diamond illustrated below




[To see this; consider the case $x_1,x_2 \geqslant 1$ then $|x_1 - 1| = x_1 - 1$ and $|x_2 - 1| = x_2 - 1$ and so we need $x_1 - 1 + x_2 - 1 = \frac{1}{2}$ or $x_1 + x_2 = 2\frac{1}{2}$. Similarly consider the other three cases: $x_1 \geqslant 1$, $x_2 < 1$; $x_1 < 1$, $x_2 \geqslant 1$; $x_1,x_2 < 1$.]

EXERCISES: 1) For any set X with the discrete metric d, show that, for any $x \in X$ we have

$$S_r(x) = \begin{cases} \emptyset \text{ (the empty set)} & \text{if } r \neq 1 \\ X\setminus\{x\} \text{ if } & r = 1 \end{cases}$$

2) In $\ell^2_\infty$ sketch the sphere $S_{\frac{1}{2}}(1,1)$.

3) In $\mathbb{R}^2$ with the "post-office" metric sketch each of the following spheres:

$S_1(0,0)$, $S_3(2,0)$, $S_{\frac{1}{2}}(1,0)$, $S_1(1,0)$

Of particular importance when $(X,\|.\|)$ is a <u>normed linear space</u> is the *unit sphere of X*, $S_1(0)$ which we sometimes denote by $S(X)$. Thus

$$S(X) = \{x \in X : \|x\| = 1\} .$$

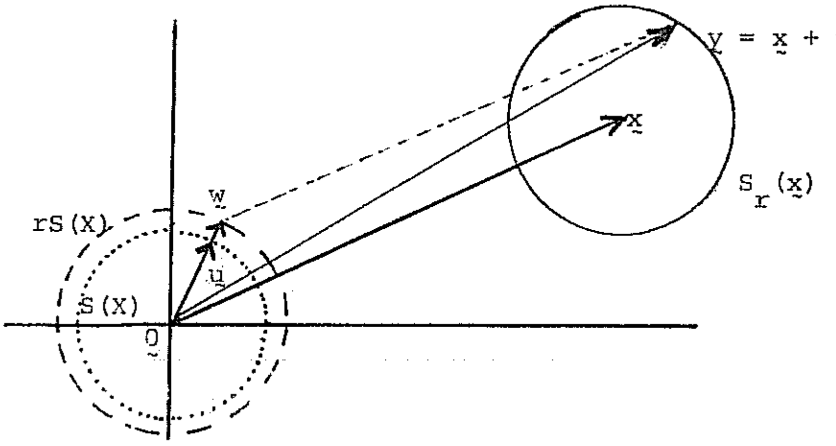
Once the unit sphere $S(X)$ is known all other spheres in the normed linear space $(X,\|.\|)$ are essentially determined. Indeed all other spheres are translates of dilates of the unit sphere.

In $(X,\|.\|)$

$$y \in S_r(x) \Leftrightarrow \|y - x\| = r$$

$$\Leftrightarrow y = x + w \quad \text{where} \quad w \ (= y - x) \in S_r(0), \ \underline{ie}. \ \|w\| = r$$

$$\Leftrightarrow y = x + ru \quad \text{where} \quad u \ (= w/r) \ \text{has} \ \|u\| = 1 \ \underline{ie}. \quad u \in S_1(X)$$




Thus if we write $rS(X)$ for the dilate $\{ru : u \in S(X)\}$ and $x + rS(X)$ for the translate $\{x + w : w \in rS(X)\}$ we have

$$S_r(x) = rS(X) + x$$

EXERCISE: Sketch the unit sphere in $\ell_1^2$ . Hence deduce that the sketch of $S_{\frac{1}{2}}(1,1)$ obtained above is essentially correct.

More important than the concept of a sphere for the study of metric spaces is that of a "ball".

DEFINITION: Let $(X,d)$ be a metric space. The *open ball of radius r and centre x* is

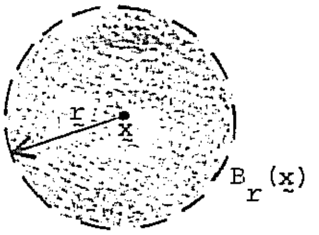
$$B_r(x) = \{y \in X : d(y,x) < r\} .$$

The *closed ball of radius r and centre x* is

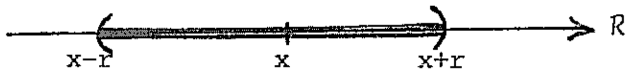
$$B_r[x] = \{y \in X : d(y,x) \leqslant r\} .$$

Thus in $R^2$ with the Euclidean metric $B_r(x)$ is the "disk"

$$\{(y_1,y_2) : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\}$$



In $\mathcal{R}$ with the usual metric $B_r(x)$ is the open interval $(x - r, x + r)$ - verify this.




[Note, $B_r[x] = B_r(x) \cup S_r(x)$ or equivalently

$B_r(x) = B_r[x] \backslash S_r(x)$.]

The (closed) <u>*unit ball* of a normed linear space</u> $(X,\|.\|)$ is

$$B[X] = \{x \in X : \|x\| \leqslant 1\} \quad (= B_1[0]).$$

As with spheres, in the case of a <u>normed linear space</u> we have

$$B_r[x] = rB[X] + x$$

$$= \{y \in X : y = ru + x \text{ for } u \in B[X]\} .$$

Similarly, $B_r(x) = rB_1(0) + x$.

EXERCISES:

1) Verify the claim that in the normed linear space $(X,\|.\|)$

$$B_r[x] = x+rB[X] .$$

2) (a) Sketch the (closed) unit ball for $\ell_1^2$ and $\ell_\infty^2$

(b) Sketch the (closed) unit ball in the normed linear space resulting from $\mathcal{R}^2$ equipped with the inner-product

$$<\underset{\sim}{x},\underset{\sim}{y}> = x_1y_1 + 2x_2y_2$$

where

$$\underset{\sim}{x} = (x_1,x_2) \text{ and } \underset{\sim}{y} = (y_1,y_2).$$

(c) Show that the (closed) unit ball in $(\mathcal{C}[a,b],\|.\|_\infty)$ consists of all continuous functions on $[a,b]$ whose graphs lie entirely between the two lines $y = 1$ and $y = -1$.

3) Sketch the ball $B_2(1,1)$ in each of the following spaces.

$\ell_1^2, \ell_2^2, \ell_\infty^2$, $\mathcal{R}^2$ with the "post-office" metric.

Because of the vector space structure present when $(X,\|.\|)$ is a <u>normed</u> linear space, we can define the notion of a line as well as those of spheres and balls.