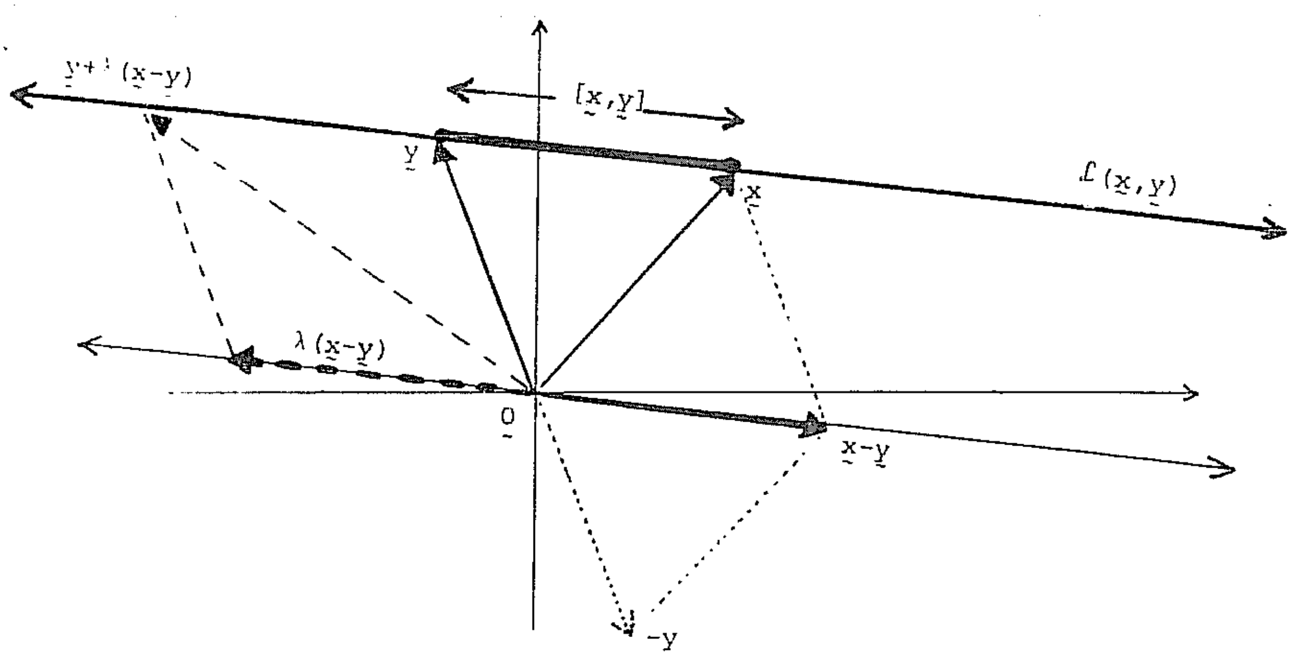
DEFINITION (See your first year work on two and three dimensional vector geometry for motivation):

The *line*, $\mathcal{L}(x,y)$, through the two points x and y of the normed linear space $(X,\|.\|)$ is

$$\mathcal{L}(x,y) = \{z\in X : z = y + \lambda(x-y) \quad \text{for} \quad \lambda \in \mathbb{R}\}$$
$$= \{z\in X : z = \lambda x + (1-\lambda)y \quad \text{for} \quad \lambda \in \mathbb{R}\}$$

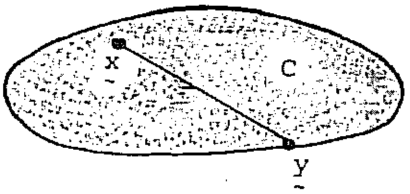


The *line* <u>*segment*</u>, [x,y], between the two points x and y of $(X,\|.\|)$ consists of all those points on $\mathcal{L}(x,y) = \{z\in X : z=\lambda x+(1-\lambda)y,\ \lambda \in \mathbb{R}\}$ which correspond to values of λ between 0 and 1 (see above diagram). Consequently we have

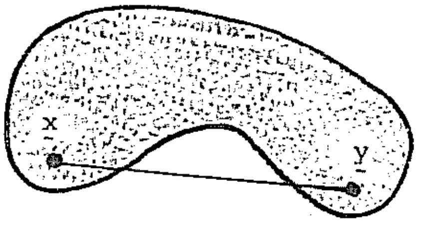
$$[x,y] = \{z\in X : z = \lambda x+(1-\lambda)y \text{ where } 0 \le \lambda \le 1\}$$

An important concept in the study of normed linear spaces is that of a convex set.

DEFINITION: A subset C of the normed linear space $(X,\|.\|)$ is said to be *convex* if whenever x and y are two points of C the line segment between x and y lies entirely in C.



A convex subset of $\mathcal{R}^2$



A non-convex subset of $\mathcal{R}^2$

Thus C is convex if and only if $x,y \in C \Rightarrow [x,y] \subseteq C$, or equivalently

C is convex if and only if

$\lambda x + (1-\lambda)y \in C$ whenever $x,y \in C$ and $\lambda \in [0,1]$.

PROPOSITION: *Any ball in a normed linear space is a convex set.*

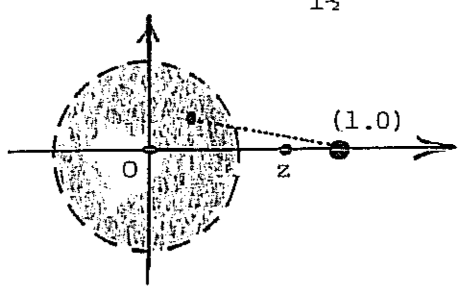
Proof. We will prove the result for <u>open</u> balls. The proof for closed balls is similar and left as an EXERCISE. Let $B_r(x_0)$ be a ball in the normed linear space $(X,\|.\|)$. Then if $z = \lambda x+(1-\lambda)y$ where $x,y \in B_r(x_0)$ and $\lambda \in [0,1]$

$$\begin{aligned}
d(z,x_0) &= \|z-x_0\| \\
&= \|\lambda x+(1-\lambda)y-x_0\| \\
&= \|\lambda(x-x_0) + (1-\lambda)(y-x_0)\| \\
&\leq \|\lambda(x-x_0)\|+\|(1-\lambda)(y-x_0)\| \\
&= |\lambda|\|x-x_0\| + |1-\lambda|\|y-x_0\| \\
&< |\lambda|r + |1-\lambda|r \quad (\text{as } x,y \in B_r(x_0), \text{ so } \|x-x_0\| < r \text{ and } \|y-x_0\| < r) \\
&= \lambda r + (1-\lambda)r \quad (\text{as } 0 \leq \lambda \leq 1 \text{ so } \lambda \text{ and } 1-\lambda \text{ are both positive}) \\
&= r
\end{aligned}$$

and so $z \in B_r(x_0)$, as required to establish convexity. ∎

Note : *The conclusion of the above proposition is* ***not*** *valid in general for metric spaces.* For example; let (X,d) denote R^2 with the "post-office" metric, then the ball

$$B = B_{1\frac{1}{2}}(1,0) \quad \text{is not convex.}$$



To see this, observe that $(0,0)$ and $(1,0) \in B$

$[\ d((0,0),(1,0)) = 1 < 1\frac{1}{2} \quad \text{and} \quad d((1,0),(1,0)) = 0 < 1\frac{1}{2}\ ]$

while $z = \frac{1}{4}(0,0) + \frac{3}{4}(0,1) = (0,\frac{3}{4}) \notin B$

$[\ d((0,\frac{3}{4}),(1,0)) = 1\frac{3}{4} \not< 1\frac{1}{2}\ ]$.

EXERCISES. 1) Sketch the set

$$B = \{(x_1,x_2) \in R^2 : |x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}} = 1\}$$

and show that it is not convex. Hence conclude that

$\|\underset{\sim}{x}\|_{\frac{1}{2}} = (|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}})^2$ does not define a norm on R^2.

[*This shows that the formula of page 10,* $\|\underset{\sim}{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, *does not define a norm for* $p=\frac{1}{2}$. *A similar conclusion applies for* $0 < p < 1$. *However, it can be shown that for this range of* p *values*

$$d_p(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=1}^{n} |x_i - y_i|^p \qquad \textit{is a metric.}]$$

2) i) In any normed linear space $(X,\|.\|)$, if $z \in [x,y]$ show that $\|x-z\| + \|y-z\| = \|x-y\|$.

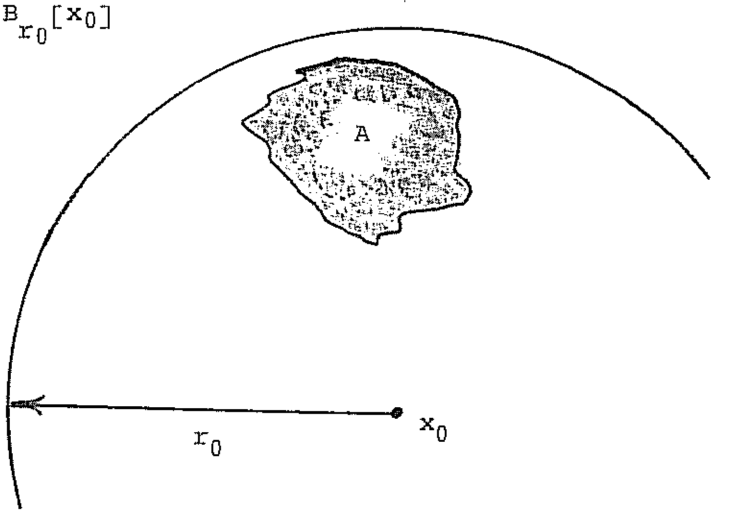
**ii) Show that the converse of i) holds in any inner-product space. That is, if $\|x\| = \sqrt{\langle x,x\rangle}$ for some inner-product $\langle .,.\rangle$ then if z is such that $\|x-z\| + \|y-z\| = \|x-y\|$ we have $z \in [x,y]$. [Hint: You may need the fact (which you should try to prove) that equality holds in Minkowski's inequality $\|x+y\| \le \|x\| + \|y\|$, if and only if x and y are related by $y = \beta x$ or $x = \beta y$ some $\beta \ge 0$.]

iii) The condition in i) and ii) has been used to extend the idea of a line segment into general metric spaces. Thus one takes as the "line-segment" between x,y the set

$$\{z\in X : d(x,z) + d(y,z) = d(x,y)\}.$$

Using this definition sketch the "line-segment" between (1,0) and (0,1) in ℓ_1^2 and ℓ_∞^2.

DEFINITION: A subset A of the metric space (X,d) is *bounded* if A is contained in some ball. That is A is bounded if and only if there exists $x_0 \in X$ and $r_0 > 0$ such that $A \subseteq B_{r_0}[x_0]$



THEOREM. *For a subset* A *of the metric space* (X,d) *the following are equivalent.*

i) A *is bounded.*

ii) *there exists* $M > 0$ *such that* $d(a_1,a_2) \leq M$ *for all* $a_1,a_2 \in A$.

iii) *Sup* $\{d(a_1,a_2):a_1,a_2\in A\} < \infty$, in which case we say

$\text{diam}(A) = \text{Sup}\,\{d(a_1,a_2):a_1,a_2\in A\}$ is the <u>diameter</u> of A.

iv) *For <u>every</u> point* $x\in X$ *there exists an* $r > 0$ *such that* $A \subseteq B_r[x]$.

If in addition $d(x,y) = \|x-y\|$ *where* $\|.\|$ *is a norm on* X,

then the following is equivalent to each of the above conditions.

v) *There exists* $m > 0$ *such that* $\|a\| \leq m$ *for all* $a\in A$.

Proof. We first show that i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) $\Rightarrow$ i).

If (i) holds, then for some $x_0 \in X$ and $r_0 > 0$ we have $A \subseteq B_{r_0}[x_0]$.

Let a_1, a_2 be any two points of A, then we have

$$d(a_1,a_2) \le d(a_1,x_0) + d(x_0,a_2) \qquad \text{(triangle inequality)}$$

$$\le \quad r_0 \quad + \quad r_0 \qquad (\text{as } a_1,a_2 \ A \subseteq B_{r_0}(x_0))$$

So taking $M = 2r_0$ establishes ii).

If ii) holds, then by the definition of supremum (or least upper bound) we have $\text{diam}(A) \le M$, establishing iii)

Now, assume iii) holds, then for all $a_1,a_2 \in A$ we have $d(a_1,a_2) \le \text{diam}(A) < \infty$.

Choose an element a_1 from A, then given any $x \in X$ for $a \in A$ we have

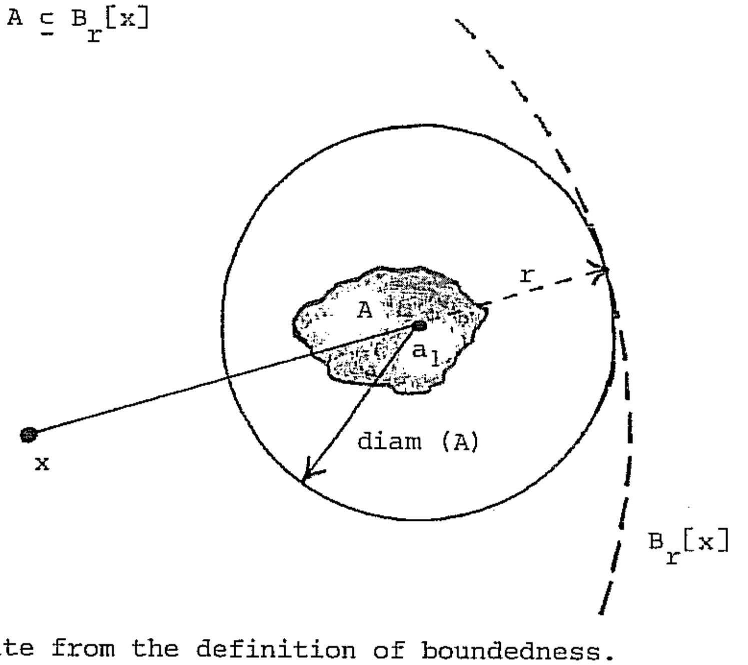
$$d(x,a) \le d(x,a_1) + d(a_1,a)$$

$$\le d(x,a_1) + \text{diam}(A).$$

Since $d(x,a_1)$ is a fixed real number, taking $r = d(x,a_1) + \text{diam}(A)$, we therefore have, for all $a \in A$ that $d(x,a) \le r$ or equivalently that

$$A \subseteq B_r[x]$$

and so iii) $\Rightarrow$ iv).



That iv) $\Rightarrow$ i) is immediate from the definition of boundedness.

To complete the proof we observe that

v) $\Rightarrow$ $A \subseteq B_m(0) \Rightarrow$ i)

$\Rightarrow$ iv) (by above)

$\Rightarrow$ v) (taking $x = 0$ and m equal to the appropriate r.)

EXERCISES. 1) Observe that any ball $B_r[x]$ in (X,d) is bounded and show that $\operatorname{diam} B_r[x] \leq 2r$. (Need equality always hold here?)

2) If A is bounded, show that for any point $a \in A$ we have

$$A \subseteq B_{\operatorname{diam}(A)}[a]$$

Remark: *perhaps surprisingly this cannot in general be improved upon. For example in an arbitrary normed linear space it is not true that for each bounded convex set A there is an $a \in A$ such that*

$$A \subseteq B_{\frac{1}{2}\operatorname{diam}(A)}[A]$$

[Can you give an example?]

§1.3 *Convergent sequences, Cauchy sequences, Completeness and Closed sets.*

By a *sequence* (of elements) of the set X we mean a <u>function</u> $N \to X : n \mapsto x_n$, here as usual we write x_n for the image of $n \in N$ under the function instead of $x(n)$.

NOTATION. The sequence $N \to X : n \mapsto x_n$ is denoted by

$$(x_n)_{n=1}^{\infty} \equiv x_1, x_2, x_3, \ldots, x_n, \ldots.$$

[when the context makes it clear we will sometimes write (x_n) instead of $(x_n)_{n=1}^{\infty}$].

DEFINITION. A sequence (x_n) of points of the metric space (X, d) is *convergent* if there is a point $x \in X$ for which, given any $\varepsilon > 0$ there exists an $N \in N$ such that

$$n \geqslant N \Rightarrow d(x_n, x) < \varepsilon.$$

or equivalently, if

$$x_n \in B_\varepsilon(x) \text{ for all } n \geq N.$$

In this case we say the sequence (x_n) converges to (has limit) x and write $d(x_n, x) \to 0$. Provided the metric space within which we are working is clearly understood we may write $\operatorname{Limit}_{n\to\infty} x_n = x$ or simply $x_n \to x$ to mean the sequence (x_n) converges to x. (Sometimes, to emphasize the metric w.r.t. which convergence is taking place we may write $x_n \overset{d}{\to} x$.)

NOTE. This definition of convergence corresponds to the definition of convergence in R with our general concept of distance replacing the usual one in R, <u>i.e.</u> $|x_n - x|$ becomes $d(x_n, x)$.

EXAMPLE

In the space $C[0,1]$ with norm $\|f\|_1 = \int_0^1 |f|$ the sequence (f_n) where

$f_n(t) = e^{-nt}$ is convergent to the zero function $0(t) = 0$. To see this observe that

$$\|f_n - 0\|_1 = \int_0^1 e^{-nt}dt = \frac{1}{n}[1 - e^{-n}] \to 0 \quad \text{as} \quad n \to \infty.$$

However, the same sequence is not convergent with respect to the norm $\|f\|_\infty = \max_{t\in[0,1]} |f(t)|$, since $\|f_n\|_\infty = \max_{t\in[0,1]} e^{-nt} = 1$ for all n.

This shows that <u>the property of convergence is not inherent in a sequence but depends on the metric used</u>. It also depends on the space X as the following example illustrates.

For $X = \mathbb{R}$ and d_1 the usual metric on $\mathbb{R}$,

$$x_n = \frac{1}{n} \to 0 \in X.$$

However, if we take $X = (0,1]$, with the same metric $x_n = \frac{1}{n}$ does not converge, as the point toward which the sequence is "tending" (0) is not a member of X. In a more general situation it may be difficult to identify the "missing" limit point and so this can represent a real problem.

THEOREM 1: *A convergent sequence* (x_n) *of the metric space* (X,d) *has a unique limit.*

Proof. Assume $x_n \to x$ and $x_n \to y$, then for any $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \Rightarrow d(x_n,x) < \frac{\varepsilon}{2}$$
$$n \geq N_2 \Rightarrow d(x_n,y) < \frac{\varepsilon}{2}$$

(use $\frac{\varepsilon}{2}$ in place of ε in the definition of convergence)

<u>But</u> then, for any $n \geq \text{Max}\{N_1,N_2\}$

$$d(x,y) \leq d(x,x_n) + d(x_n,y), \text{ by the triangle inequality}$$
$$< \varepsilon$$

and, since ε is arbitrary, this implies $d(x,y) = 0$ or $x = y$. ∎

THEOREM 2: *In a metric space the points of a convergent sequence form a bounded set.*

Proof. Let $x_n \to x$ in (X,d), then there exists $N \in \mathbb{N}$ such that $d(x_n,x) < 1$ for all $n > N$. (Note, the use of 1 here is for convenience any other positive number would do.)

So, let $r = \text{Max}\{d(x_1,x),\ d(x_2,x),\ \ldots,\ d(x_N,x), 1\}$ ($<\infty$, why?)

then $d(x_n,x) \le r$ for all $n \in \mathbb{N}$

whence $d(x_n,x_m) \le d(x_n,x) + d(x,x_m)$

$\le 2r$ for all $n, m \in \mathbb{N}$

That is, the points of the sequence form a bounded set of diameter less than or equal to 2r. ∎

EXERCISES:

1. Show that in $(C[0,1],d_\infty)$ the sequence of Taylor polynomials

   $p_n(x) = \sum_{m=0}^{n} x^m/m!$ is convergent to exp. Is the same true in $(C[0,1],d_1)$?

   [Hint: see the exercise on p.16.]

2. (i) In any metric space (X,d) show that a <u>constant sequence</u> $x,x,\ldots,x,\ldots$ is convergent with limit x.

   (ii) Show that the same conclusion holds if it is only assumed that the sequence is "eventually" constant; that is, there exists some $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we have $x_n = x$.

3. Show that in a metric space (X,d), where d is the discrete metric, a sequence (x_n) is convergent if (by 2ii above) and only if it is eventually constant.

4. Using the discrete metric, give a further example to show that convergence depends on the choice of metric.

5. In a normed linear space $(X,\|.\|)$ show that for $\alpha,\beta \in \mathbb{R}$ $\alpha x_n + \beta y_n \to \alpha x + \beta y$ whenever $x_n \to x$ and $y_n \to y$.

6. Let $x_n \to x$ and $y_n \to y$ in the metric space (X,d).

   Show that $d(x_n,y_n) \to d(x,y)$ in $(\mathcal{R},d_1)$.

We now introduce a property for sequences which is independent of X (but not of d).

DEFINITION. A sequence (x_n) of the metric space (X,d) is a *Cauchy Sequence* if given $\varepsilon > 0$ there exists an $N \in \mathcal{N}$ such that

$$m,\ n \geq N \Rightarrow d(x_m,x_n) < \varepsilon,$$

We frequently abreviate this by writting

$$d(x_m,x_n) \to 0 \text{ as } m,n \to \infty.$$

Intuitively, in a Cauchy sequence the points in the "tail" of the sequence become arbitrarily 'close' together.

THEOREM 3: *Every convergent sequence of a metric space is a Cauchy sequence.*

Proof. Let $x_n \to x$ in (X,d), then given $\varepsilon > 0$ there exists $N \in \mathcal{N}$ such that

$$d(x_n,x) < \frac{\varepsilon}{2}\ ,\ \text{all } n \geq N$$

whence, for $m,n \geq N$ we have

$$d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) \quad \text{(triangle inequality)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

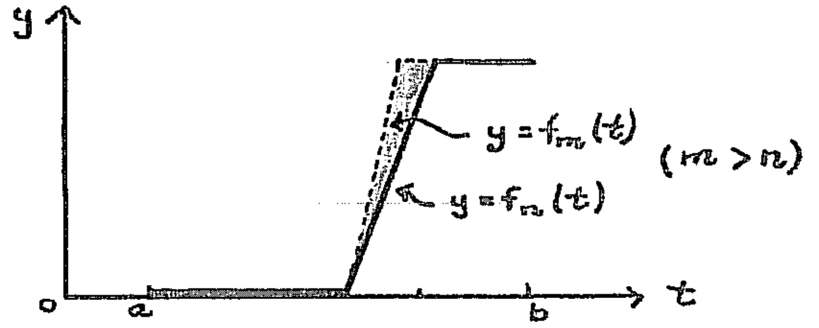
as required. ■

The converse of this need not be true. For example:

1). For $X = (0,1]$ with the usual metric $d_1(x,y) = |x-y|$ the sequence $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n},\ \ldots$ is a Cauchy sequence (although from above it is not a convergent sequence). To see that $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence, note that $|\frac{1}{n} - \frac{1}{m}| < \varepsilon$ whenever $m,\ n > \frac{1}{\varepsilon}$.

2). The sequence defined by

$$f_n(t) = \begin{cases} 0 \text{ for } a \le t \le \frac{b+a}{2} \\ n(t - \frac{a+b}{2}) \text{ for } \frac{b+a}{2} < t < \frac{b+a}{2} + \frac{1}{n} \\ 1 \text{ for } \frac{b+a}{2} + \frac{1}{n} \le t \le b \end{cases}$$



is a Cauchy sequence, in C[a,b] with d_1 as metric, since for $m > n$

$$d_1(f_n, f_m) = \| f_n - f_m \|_1$$

$$= \int_{\frac{a+b}{2}}^{\frac{a+b}{2} + \frac{1}{n}} m(t - \frac{b+a}{2}) - n(t - \frac{b+a}{2}) \, dt$$

$$\le \frac{1}{2n} \to 0 \text{ as } m, n \to \infty$$

To see that *it does not converge in* C[a,b] assume the contrary. That is, assume there exists a continuous function f such that

$$\| f_n - f \|_1 = \int_a^b |f_n - f| \to 0 \quad (\text{as } n \to \infty)$$

Then we have

$$\int_a^{\frac{a+b}{2}} |f(t)| dt + \int_{\frac{a+b}{2}}^{\frac{a+b}{2} + \frac{1}{n}} |n(t - \frac{a+b}{2}) - f(t)| dt + \int_{\frac{a+b}{2} + \frac{1}{n}}^{b} |1 - f(t)| dt \to 0$$

Since each of the three terms is positive this requires that each of them tend to zero separately. Thus we must have:

(i) $\int_a^{\frac{a+b}{2}} |f(t)|dt \to 0$ (as $n \to \infty$), which clearly requires that

$f(t) = 0$ for all t with $a < t < \frac{a+b}{2}$

[(ii) For the middle term we have, by the continuity of each f_n and of f and the fact that each f_n is between 0 and 1, that

$$\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} \left|n(t - \tfrac{a+b}{2}) - f(t)\right|dt \le \left|1 + \max_{\frac{a+b}{2}<t<\frac{a+b}{2}+\frac{1}{n}} |f(t)|\right| . \frac{1}{n}$$

$$\to 0 \text{ as } n \to \infty.$$

So this term automatically tends to zero and imposes no constraint on f.]

iii) By the continuity of f we see that

$$\int_{\frac{a+b}{2}+\frac{1}{n}}^{b} |1-f(t)|dt \to \int_{\frac{a+b}{2}}^{b} |1-f(t)|dt, \text{ as } n \to \infty.$$

For this limit to be zero, we clearly require that $1-f(t) = 0$ (or $f(t)=1$) for all t with $\frac{a+b}{2} < t < b$.

Thus we have that f must be such that

$$f(t) = \begin{cases} 0 & \text{for } a < t < \frac{a+b}{2} \\ 1 & \text{for } \frac{a+b}{2} < t < b \end{cases}$$

which is impossible for a continuous function f and so we conclude that no such f can exist.

Spaces for which the converse of Theorem 3 does hold are particularly important in analysis and its applications.

DEFINITION: A metric space (X,d) in which every Cauchy sequence is convergent is said to be *complete*. Thus (X,d) is a complete metric space if, whenever the sequence (x_n) is such that $d(x_n,x_m) \to 0$ as n and $m \to \infty$, then there esists an $x \in X$ with $x_n \xrightarrow{d} x$.

A normed linear space, $(X,\|.\|)$ which is complete with respect to the induced metric is termed a <u>Banach space</u>. $[(X,\|.\|)$ is a Banach space if $\|x_n - x_m\| \to 0$ implies there exists $x \in X$ with $\|x_n - x\| \to 0.]$

An inner-product space X which is complete with respect to the metric induced by the inner-product generated norm $(\|x\| = \sqrt{<x,x>})$ is known as a <u>Hilbert Space</u>.

<u>EXAMPLES</u>

1) We take it as an assumed property (axiom) of the real number system that R *together with the usual metric* $d(x,y) = |x-y|$ *is a complete space. For any metric space* (X,d) *it is possible to find a minimal complete superspace* $(\tilde{X},\tilde{d})$ *known as the completion of* (X,d). (*<u>i.e.</u>* $X \subseteq \tilde{X}$ *and* $\tilde{d}(x,y) = d(x,y)$ *for all* $x,y \in X$.)

One construction of $(\tilde{X},\tilde{d})$ *from* (X,d), *due to Cauchy, allows, as a special case, the real numbers* R *to be axiomatically derived from the rational numbers as their completion* $\tilde{Q}$.

EXERCISE. Show that (Q,d) the metric space of rational numbers equipped with the usual metric $d(p,q) = |p-q|$ is <u>not</u> complete.

2) The previous two examples show that neither of the spaces $((0,1], d_{usual})$ or $(C[a,b], d_1)$ is complete.

Indeed, none of the spaces $(C[a,b],d_p)$ *for* $1 \le p < \infty$ *are complete. In particular* $(C[a,b],d_2)$ *is <u>not</u> a Hilbert space. The problem of adjoining additional functions (and extending the definition of the metric to cover these new functions) so as to "complete" these spaces is a major motivation for Lebesgue's theory of integration and measure.*

3) EXERCISE. For any set X equipped with the discrete metric d and for the space of Example 6 on page 5. show that a sequence is a Cauchy sequence if and only if it is eventually constant. Hence conclude that in both these cases the space is complete.

4) From the completeness of $\mathbb{R}$ follows the completeness of the finite dimensional spaces ℓ_p^n where $p = 1,2$ or ∞. There results follow from a more general result; that all finite dimensional normed linear spaces are complete, which will be established later in the section on compactness. Consequently we will omit direct proofs for the results at this stage.

5) *The sequence spaces* ℓ_p $(1 \le p \le \infty)$ *are Banach spaces.* These results may be established by a careful "passage to the limit" from the finite dimensional cases, however we will omit details from this course. (Honours students should attempt to prove that ℓ_2 is indeed a Hilbert space.)

6) We now establish the completeness (and hence the importance) of the space $(C[a,b],d_\infty)$.

The space $(C[a,b],\|.\|_\infty)$ *is a Banach space.*

To see this, it suffices to show that a uniform limit of continuous functions is continuous.

Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(C[a,b], d_\infty)$. <u>i.e.</u> Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_\infty(f_n, f_m) = \|f_n - f_m\|_\infty = \max_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon$$

whenever $n, m \geqslant N$.

Consequently, for each $x \in [a, b]$

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for } n, m \geqslant N,$$

<u>i.e.</u>

$(f_n(x))_{n=1}^\infty$ is a Cauchy sequence in the complete metric space $(\mathbb{R},d_1)$ and so is convergent to some unique limit which we choose to denote by $f(x)$.

Define a function f on $[a, b]$ by $x \mapsto f(x)$. Then, for every $x \in [a, b]$ $f_n(x) \xrightarrow{d_1} f(x)$ and we say f is the <u>point wise limit</u> of the sequence $\{f_n\}$. (This type of convergence - point wise convergence - is important in real analysis but peripheral to metric analysis, since it does not represent convergence with respect to any of the metrics on $C[a,b]$.)

[In general point wise convergence is 'weaker' than uniform convergence (convergence with respect to the uniform metric d_∞) i.e.

$$\text{(uniform convergence)} \Rightarrow \text{(point wise convergence)}$$

however, it may happen that $f_n \to f$ point wise but $f_n \not\to$ uniformly. Give an example illustrating this.]

Because of the particular construction of f above,

$\left(f(x) = \underset{m\to\infty}{\text{Limit}}\ f_m(x) \text{ where } \{f_m\} \text{ is a Cauchy sequence in } (C[a, b], d_\infty)\right)$

we have, for $n, m \geqslant N$

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for all } x \in [a, b]$$

and so

$$\underset{m\to\infty}{\text{Limit}}|f_n(x) - f_m(x)| = |f_n(x) - \underset{m\to\infty}{\text{Limit}}\ f_m(x)| = |f_n(x) - f(x)| < \varepsilon$$

for all $x \in [a, b]$ and $n \geqslant N$.

Whence $\underset{x\in[a,b]}{\text{Max}} |f_n(x) - f(x)| < \varepsilon$ whenever $n \geqslant N$, or f_n converges uniformly to f, i.e. $d_\infty(f_n - f) \to 0$.

We now show $f \in C[a, b]$ and so establish the completeness of $(C[a, b], d_\infty)$. To do this we must show f is continuous, i.e. given $\varepsilon > 0$ and any $x_0 \in [a, b]$ we must find $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \quad (x \in [a, b]).$$

Now

$$\begin{aligned}|f(x) - f(x_0) &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|\end{aligned}$$

and since $f_n \xrightarrow{d_\infty} f$ there exists $N \in \mathbb{N}$

$$n \geqslant N \Rightarrow |f(x) - f_n(x)|,\ |f_n(x_0) - f(x_0)| \leqslant d_\infty(f_n, f) < \frac{\varepsilon}{3}.$$

So for any fixed $n > N$

$$|f(x) - f(x_0)| \leqslant \frac{\varepsilon}{3} + |f_n(x) - f_n(x_0)| + \frac{\varepsilon}{3}$$

but $f_n \in C[a,b]$ so there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

and so for this δ we have $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, as required, to show f is continuous.