7) EXERCISE. Show that *the space* $(\mathcal{B},\|\cdot\|_\infty)$ *of all bounded functions on* [a,b] *with the uniform norm* - see pages 13 and 14 - *is a Banach space.* [Hint: It is similar to, though easier than, the proof in 6) above.]

DEFINITION: Let (X,d) be a metric space and A a given subset of X. We say that $x \in X$ is a *limit point* of A if there exists a sequence of points of A, (a_n), which converges to x.

EXAMPLE. Since in the space $(C[0,1],\|\cdot\|_1)$ we have $f_n(t) = e^{-nt} \overset{d_1}{\to} 0$ [The Example on page 33], we see that the zero function is a limit point of the set A of all strictly positive continuous functions on [0,1]

$$A = \{f \in C[0,1] : f(t) \gneq 0 \text{ for all } t \in [0,1]\}.$$

[Is the same true if $\|\cdot\|_1$ is replaced by $\|\cdot\|_\infty$?]

From this example we see that a limit point of A *need not belong to* A.

Intuitively, limit points of A are those points which can be approached (approximated) arbitrarily well by points of A.

REMARK. *Every point of* A *is a limit point of* A. To see this note that for $a \in A$ the constant sequence a,a,a, ... ,a, ... converges to a.

We now give a useful characterization of limit points, which avoids the use of sequences.

LEMMA 4. *Let* (X,d) *be a metric space.* $x \in X$ *is a limit point of* $A \subseteq X$

if and only if,

for every $\varepsilon > 0$ *there exists* $a \in A$ *with* $d(x,a) < \varepsilon$.

Proof. ($\Rightarrow$) If x is a limit point of A, there exist (a_n) with $a_n \in A$ and $a_n \to x$. So given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n,x) < \varepsilon$ for any $n > N$.

($\Leftarrow$) For $n \in \mathbb{N}$, taking $\varepsilon = \frac{1}{n}$ there exists an element of A, call it a_n, with $d(x,a_n) < \frac{1}{n}$. The sequence (a_n) thus constructed converges to x and so x is a limit point of A.

∎

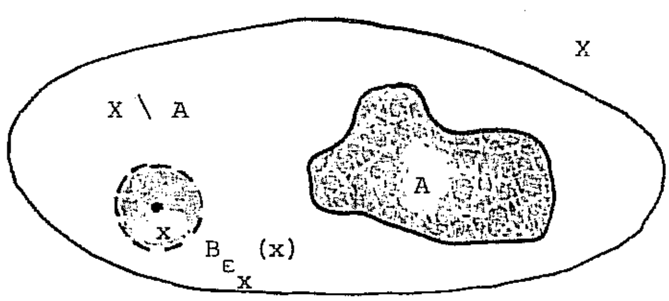
DEFINITION. The set of all limit points of A, denoted by $\bar{A}$ is termed the *closure* of A.

From the preceding remark we see that always $A \subseteq \bar{A}$.

DEFINITION. A subset A of the metric space (X,d) is *closed* if $A = \bar{A}$; that is, if it contains all its limit points.

Thus A is closed if and only if its complement $X \setminus A$ contains no limit points of A. Now by Lemma 4 above, a point x is not a limit point of A if and only if there exists some $\varepsilon_0 > 0$ such that for all $a \in A$ we have $d(a,x) \geq \varepsilon_0$, or equivalently $B_{\varepsilon_0}(x) \cap A = \emptyset$, and so we have the following.

PROPOSITION 5 : *The subset* A *of the metric space* (X,d) *is closed if and only if for each point* x *in the complement* $X \setminus A$ *of* A *there exists* $\varepsilon_x > 0$ *such that* $B_{\varepsilon_x}(x) \subseteq X \setminus A$.



EXERCISES.

1) Let A be the open interval (0,1). In the metric space $\mathbb{R}$ with the usual metric $d(x,y) = |x - y|$ show that : both 0 and 1 are limit points of A; that no other point $x \notin A$ is a limit point of A, and hence conclude that the closure of A is the closed interval [0,1].

2) i) If d is the discrete metric on any set X show that every subset of (X,d) is closed.

   ii) Show that every subset of the metric space in Example 6 on page 5 is closed.

3) Show that in any metric space (X,d) the (closed) ball $B_r[x]$ is in the above sense a closed subset of X.

*4) In any metric space show that any finite subset (a subset containing only a finite number of points) is a closed set.

5) Let (X,d) be a complete metric space and A a closed subset of X. With d restricted to A show that (A,d) is a complete metric space [indeed a subspace of (X,d)].

6) Prove that the intersection of two closed subsets of a metric space is itself closed.

We conclude this section with a useful characterization of the closure of a subset A.

THEOREM 6 *Let* (X,d) *be a metric space and* $A \subseteq X$. *Then* $\bar{A}$ *is the smallest closed subset of* X *containing* A. *That is* $\bar{A}$ *is a closed set and if* B *is any closed set containing* A *then* $\bar{A} \subseteq B$.

Proof. We first show the closure of A, $\bar{A}$, is a closed set. Thus, let x be a limit point of $\bar{A}$, then for $\varepsilon > 0$ there exists $a_1 \in \bar{A}$ with $d(x,a_1) < \frac{\varepsilon}{2}$ and further since $a_1 \in \bar{A}$ i.e. a_1 is a limit point of A there exists $a_2 \in A$ with $d(a_1,a_2) < \frac{\varepsilon}{2}$, whence

$$
\begin{aligned}
d(x,a_2) &\leqslant d(x,a_1) + d(a_1,a_2) \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon, \text{ where } a_2 \in A
\end{aligned}
$$

and so by the lemma $x \in \bar{A}$.

Now let B be a closed set with $A \subseteq B$ and let a be a limit point of A, so there exists a sequence (a_n) with $a_n \in A \subseteq B$ and $a_n \to a$, thus a is a limit point of B, but B is closed and so $a \in B$. Therefore $a \in \bar{A} \Rightarrow a \in B$ or $\bar{A} \subseteq B$.

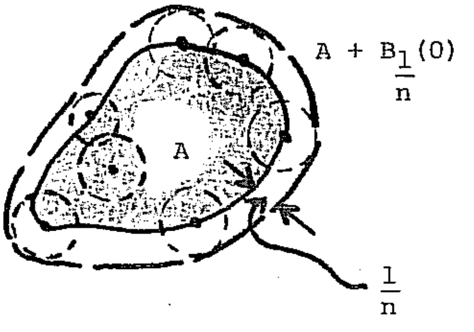
■

EXERCISES *1) For any subset A of the metric space (X,d) show that $\bar{A} = \cap B$ where the intersection is taken over all closed subsets of X which contain A.

*2) For any subset A of the normed linear space (X,d), let

$$A + B_{\frac{1}{n}}(0) = \{a+x : a\in A \text{ and } x\in B_{\frac{1}{n}}(0)\}$$

Thus $A + B_{\frac{1}{n}}(0)$ is the set of all points which are closer to a point of A than $\frac{1}{n}$.



Show that

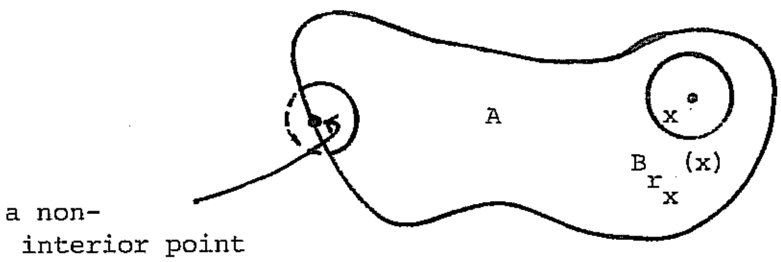
$$\bar{A} = \bigcap_n (A+B_{\frac{1}{n}}(0))$$

§1.4 *The Elements of Topology in Metric spaces - open sets, interiors and boundaries*

In proposition 5 of §1.3 we saw that the complement of a closed set is characterized by the property that each of its points is the centre of an (open) ball which lies entirely in it. We will refer to such points as "interior points".

DEFINITION: For any subset A of the metric space (X,d) the point x is an *interior point* of A if there exists $r_x > 0$ such that

$$B_{r_x}(x) \subseteq A.$$



The set of all interior points of A is known as the *interior of* A, and will be denoted by Int A.

A subset A of (X,d) is said to be *open* if int A = A. That is, if every point of A is an interior point of A, or equivalently, if each point of A is the centre of some (open) ball which is contained in A.

EXERCISES

1) In $\mathcal{R}$ with the usual metric $d(x,y) = |x-y|$, show that the open interval $(a,b) = \{x \in \mathcal{R} : a < x < b\}$ is an open subset of $(\mathcal{R},d)$ in the above sense.

[Hint: For $x \in (a,b)$ consider $r_x = \text{Min}\,\{x-a, b-x\}$.]

2) In any metric space (X,d) show that an open ball, $B_r(x) = \{y \in X : d(x,y) < r\}$, is an open subset of X in the above sense.

3) Let d be the discrete metric on any set X, show directly that every subset of (X,d) is an open set. Prove a similar result for the metric space of Example 6 on page 5. Give alternative proofs for each of these results based on the following theorem.

In terms of the above definitions proposition 5 of section 1.3 may be restated as follows:

THEOREM 1: *In any metric space* (X,d) *a subset* A *is closed if and only if its complement* X \ A *is open.*

In much of analysis open sets appear more fundamental than closed sets. The above theorem provides an important bridge between the two concepts. It is however essential that the two concepts are not seen as mutually exclusive properties for a set A. Indeed any of the following can happen.

(i) A is open but not closed [e.g. (a,b) in $(\mathcal{R},d_1)$ - prove this]

(ii) A is closed but not open [e.g. [a,b] in $(\mathcal{R},d_1)$ - prove this]

(iii) A is neither open or closed [e.g. [a,b) in $(\mathcal{R},d_1)$ - prove this]

(iv) A is both open and closed [e.g. every subset of any set equipped with the discrete metric is both open and closed].

Thus, in general, from a knowledge that A is open (closed) nothing can be inferred as to whether or not A is closed (open).

We now collect together a number of useful characterizations of an open set.

THEOREM 2: *In any metric space* (X,d) *the following are equivalent statements about the subset* A.

i) A *is open*

ii) Int A = A

iii) *every point of* A *is an interior point of* A

iv) *for every* $x \in A$ *there exists* $r_x > 0$ *such that* $B_{r_x}(x) \subseteq A$

v) X \ A *is closed*

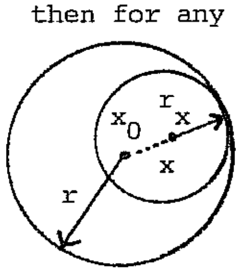
→ vi) A *is a union of open balls.*

Proof. It is immediate from the definitions that

i) ⇔ (ii) ⇔ (iii) ⇔ (iv), while the equivalence of (i) and (v) has been given in Theorem 1. Thus it is sufficient to establish the equivalence of (vi) with each of the others. To do this we show that (vi) ⇒ (iv) ⇒ (vi). To prove vi) ⇒ iv) we first establish the special case when A is itself an open ball, $B_r(x_0)$.

LEMMA. *If* $x \in B_r(x_0)$ - *an open ball in the metric space* (X,d) - , *then* $x \in \text{Int}\, B_r(x_0)$ i.e. *there exists* $r_x > 0$ *such that* $B_{r_x}(x) \subseteq B_r(x_0)$,

Proof (of lemma). Since $x \in B_r(x_0)$, $d(x,x_0) < r$, set $r_x = r - d(x,x_0)$ (> 0), then for any $y \in B_{r_x}(x)$, $d(y,x_0) \leqslant d(y,x) + d(x,x_0)$



$$< r - d(x,x_0) + d(x,x_0)$$

$$= r \quad , \qquad \text{so } y \in B_r(x_0), \text{ all } y \in B_{r_x}(x),$$

whence $B_{r_x}(x) \subseteq B_r(x_0)$ as required

□

Returning to the proof of (vi) ⇒ (iv), since $A = \bigcup_{\lambda \in \Lambda} B_\lambda$ for some family of open balls $\{B_\lambda : \lambda \in \Lambda\}$, if $x \in A$ then $x \in B_\lambda$ for some $\lambda \in \Lambda$. Whence by the lemma, there exists $r_x > 0$ such that $B_{r_x}(x) \subseteq B_\lambda \subseteq A$, as required.

To see that (iv) ⇒ (vi) it suffices to note that, if for each $x \in A$ we let $r_x > 0$ be such that $B_{r_x}(x) \subseteq A$, then we have

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} B_{r_x}(x) \subseteq A$$

and so A is a union of open balls.

■

We now develop what might be called a "set theoretic Algebra" for open sets.

THEOREM 3:

Let (X,d) *be a metric space, then*

(i) *φ and* X *are open sets*

(ii) *A union of open sets is an open set,* and

(iii) *the intersection of a* finite *number of open sets is an open set.*

Proof.

(i) Since ϕ contains no points, it is clear that every point of ϕ is the centre of an open ball contained in ϕ.

While, for any $x \in X$, $B_r(x) \subseteq X$ for every $r > 0$, thus

$X = \bigcup_{x \in X} B_1(x)$ and so X is open.

(ii) Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of open sets of (X,d).

Then, since each G_λ is open we have

$G_\lambda = \bigcup_{\gamma \in \Gamma_\lambda} B_\gamma$ where B_γ is an open ball (all $\gamma \in \Gamma_\lambda$).

Thus, the union of open sets,

$$G = \bigcup_{\lambda \in \Lambda} G_\lambda = \bigcup_{\lambda \in \Lambda} \left(\bigcup_{\gamma \in \Gamma_\lambda} B_\gamma \right)$$

is a union of open balls and consequently is itself open.

(iii) Let $\{G_k : k = 1,2, \ldots, n\}$ be a finite family of open sets,

$x \in G = \bigcap_{k=1,2,\ldots,n} G_k \Rightarrow x \in G_k$ (all $k=1,2, \ldots, n$) then,

since each G_k is open, there exists $r_k > 0$ such that

$B_{r_k}(x) \subseteq G_k$ $(k=1,2, \ldots, n)$.

Let $r = \text{Min}\{r_1, r_2, \ldots, r_n\}$ (which exists and is strictly positive, since there are only a finite number of the r_k),

then clearly

$x \in B_r(x) \subseteq B_{r_k}(x)$ (all k) and so $B_r(x) \subseteq G = \bigcap_k G_k$.

But x was any point of G, so G is open. ■

<u>The finiteness condition in (iii) above cannot be dropped.</u>

EXAMPLE.

$\{B_{\frac{1}{n}}(0) : n \in \mathbb{N}\}$ is an infinite family of open sets in $(\mathbb{R}, d_1)$ such that their intersection

$\bigcap_{n \in \mathbb{N}} B_{\frac{1}{n}}(0) = \{0\}$ (prove)

which is not an open set (prove).

REMARK.

Any family of subsets $\mathcal{T}$ of the set X which have the properties (i), (ii) and (iii) of Theorem 4.2,

(That is (i) $\phi, X \in \mathcal{T}$;

(ii) $\bigcup_{\lambda \in \Lambda} T_\lambda \in \mathcal{T}$ whenever $T_\lambda \in \mathcal{T}$, all $\lambda \in \Lambda$;

(iii) $\bigcap_{\lambda \in F} T_\lambda \in \mathcal{T}$ whenever $T_\lambda \in \mathcal{T}$, all $\lambda \in F$ - a finite index set)

is termed a TOPOLOGY for X, and X equipped with $\mathcal{T}$ is called a *Topological space.*

The notion of a topological space is due to Hausdorff (in 1914) who built on an idea used by Hilbert in 1902 while developing an axiomatic approach to plane Euclidean geometry.

Theorem 3 shows that the family of open subsets of a metric space is a Topology for the space X.

In general, however, there are topologies which do not coincide with the family of open sets generated by any possible metric on X. *Thus topological spaces are more general than metric spaces and their study forms an important branch of modern mathematics. The question of characterizing those topologies which do arise from a metric is known as the metrization problem, which was completely answered only in the late 1940's.*

Using de Morgan's Rules of set theory and Theorem 1 the algebra of open sets (Theorem 3) leads to a corresponding algebra for closed sets.

THEOREM 4 (the 'algebra' of closed sets):

Let (X,d) *be a metric space, then*

(i) ϕ *and* X *are closed;*

(ii) *An intersection of closed sets is a closed set;*

(iii) *The union of a finite number of closed sets is a closed set.*

Proof.

(i) $X\backslash\phi = X$ is open, so ϕ is closed

similarly $X\backslash X = \phi$ is open, so X is closed.

(ii) Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of closed sets, then

$$F = \bigcap_{\alpha\in\Lambda} F_\alpha = X \setminus \bigcup_{\alpha\in\Lambda} (X\setminus F_\alpha) \quad \text{(deMorgan's Theorem)}$$

but, $X \setminus F_\alpha$ is open, and so $\bigcup_{\alpha\in\Lambda} (X \setminus F_\alpha)$ is open, whence F is the complement of an open set and so F is closed.

(iii) Let $\{F_1, F_2, \ldots, F_n\}$ be a finite family of closed sets, then

$$F = \bigcup_{m=1}^{n} F_m = X \setminus \bigcap_{m=1}^{n} (X \setminus F_m) \qquad \text{(de Morgan)}$$

and so, since finite intersections of open sets are open $\bigcap_{m=1}^{n} (X \setminus F_m)$ is open, whence F is closed. ∎

NOTE: Consistency demands that as with finite intersections of open sets, the finiteness condition in (iii) cannot be dropped.

EXAMPLE. In $(\mathcal{R}, d_1)$

$$\bigcup_{n=2}^{\infty} [\tfrac{1}{n}, 1 - \tfrac{1}{n}] = (0,1) \qquad \text{(prove)}$$

which is not closed (prove).

EXERCISES:

1) Prove that in any metric space the complement of any singleton set is open, and hence or otherwise show that the complement of any finite set is open.

2) In a given metric space (X,d) prove that every subset of X is open if and only if every singleton set is open.

*3) (a) Show that the singleton subsets of any non-trivial normed linear space cannot be open sets with respect to the metric induced by the norm.

(b) Show that except for $\{\underset{\sim}{0}\}$ all other singleton sets of $\mathcal{R}^2$ are open ~~with respect to the Post Office Metric.~~

This gives an alternative proof that there is no norm on $\mathcal{R}^2$ which induces the Post Office Metric.

4) DEFINITION. $A \subseteq X$ *is a* *dense* subset in the metric space (X,d) if $\bar{A} = X$. (e.g. Q is dense in R.)

Prove the following statements are equivalent in (X,d).

(i) $A \subseteq X$ is dense;

(ii) the only closed superset of A is X;

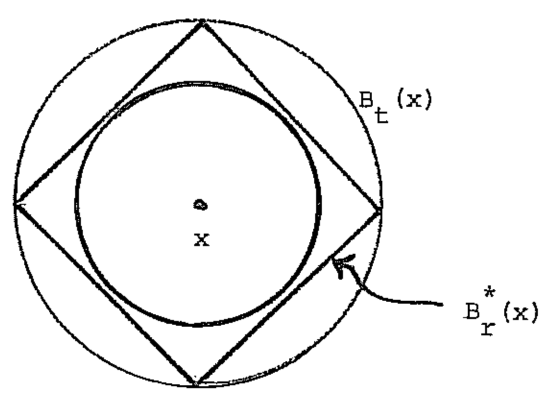
(iii) the only open set disjoint from A is ϕ;

(iv) A has a non-trivial intersection with every non-empty open set of (X,d).

*5) Two metrics d, d' on the set X are said to be equivalent metrics if they give rise to the same family of open sets.

(i) Prove that for any metric space (X,d) d and d^* are equivalent metrics, where $d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$

[Hint: First observe that it is sufficient to show that for any ball $B_r^*(x)$ in (X,d^*) there exists $t > 0$ such that the ball $B_t(x)$ in (X,d) is contained in $B_r^*(x)$, and similarly for any ball $B_t(x)$ in (X,d) there exists a ball $B_r^*(x)$ in (X,d^*) which is contained in $B_t(x)$.]



(ii) Show that two norm functions, $\|.\|$ and $\|.\|^*$, on the same vector space X induce equivalent metrics (are equivalent norms) if and only if there exist positive constants m and M such that for all $x \in X$
$m\|x\| \leqslant \|x\|^* \leqslant M\|x\|$.

**6. For a normed linear space, show that the only sets which are both open and closed, with respect to the metric induced by the norm, are the whole space and the empty set.

Our next theorem provides a characterization for the interior of any set.

THEOREM 5: *For* $A \subseteq X$, (X,d) *a metric space, the interior of* A, $\text{Int } A$ *is the 'largest' open set contained in* A, i.e. if G is any open set contained in A, then $G \subseteq \text{Int } A$.

Proof. We must first show that $\text{Int } A$ is open for any set $A \subseteq X$. Accordingly, $x \in \text{Int } A \Rightarrow$ there exists $r_x > 0$ such that $B_r(x) \subseteq A$. Now $B_{r_x}(x)$ is an open set so for $y \in B_{r_x}(x)$ there exists $r_y > 0$ such that $B_{r_y}(y) \subseteq B_{r_x}(x)$ which is contained in A. Thus each $y \in B_{r_x}(x)$ is an interior point of A or $B_{r_x}(x) \subseteq \text{Int } A$. *But* x was an arbitrary point of $\text{Int } A$ so every point of $\text{Int } A$ is the centre of an open ball $(B_{r_x}(x))$ contained in $\text{Int } A$ which is therefore open. Now, let $G \subseteq A$ be an open subset of A, then $x \in G \Rightarrow$ there exists $r_x > 0$ such that $B_{r_x}(x) \subseteq G \subseteq A \Rightarrow x \in \text{Int } A$ so $G \subseteq \text{Int } A$, establishing the maximality of $\text{Int } A$.

∎

EXERCISES:

1) For any subset A of the metric space (X,d) prove that $X \setminus \text{int } A = \overline{X \setminus A}$

2) <u>The 'algebra' of interiors</u>

   In a metric space (X,d) with $A, B \subseteq X$ prove

   (i) if $A \subseteq B$ then $\text{Int } A \subseteq \text{Int } B$;

   (ii) $\text{Int } (A \cap B) = (\text{Int } A) \cap (\text{Int } B)$;

   (iii) $\text{Int } (A \cup B) \supseteq (\text{Int } A) \cup (\text{Int } B)$;

   (iv) Construct a counter-example to show that the reverse inclusion to that of part (iii) need not hold in general.

Corresponding to this we have an analogous Algebra of closures.

Show that,

(i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$, what can you say about the reverse inclusion?

A derived concept of importance in general analysis is that of the "boundary" of a set.

DEFINITION. For (X,d) a metric space and $A \subseteq X$, the *boundary of* A, denoted by bdry A is given by

$$\text{bdry } A = \bar{A} \cap \overline{(X \setminus A)}.$$

Thus bdry A consists of all those points which are arbitrarily close to both A and its complement $X \setminus A$.

EXAMPLES. In $(\mathbb{R},d_1)$ the boundary of an open interval (a,b) is what we might expect, viz

$$\text{bdry } (a,b) = \{a,b\}$$

$$(= \overline{(a,b)} \cap \overline{(\mathbb{R}\setminus (a,b))}$$

$$= [a,b] \cap \big((-\infty,a] \cup [b,\infty)\big),$$

as $\mathbb{R}\setminus (a,b) = (-\infty,a] \cup [b,\infty)$ is the complement of an open set and so is closed.

However this is not always the case, e.g.

$$\text{bdry } \mathbb{Q} = \bar{\mathbb{Q}} \cap \overline{(\mathbb{R}\setminus \mathbb{Q})}$$

$$= \mathbb{R} \cap \mathbb{R} = \mathbb{R}.$$

while, for $A \subseteq X$, X any set and d the discrete metric

$$\text{bdry } A = \bar{A} \cap \overline{(X\setminus A)} = A \cap (X\setminus A) = \phi.$$

Despite these observations the boundary of a set does behave in an intuitively pleasing way as the following exercise illustrates.

EXERCISE:

Let A be any subset of the metric space (X,d) show that

$$\text{bdry } A = \overline{A} \setminus \text{Int } A.$$

Hence conclude that;

(i) $(\text{Int } A) \cap (\text{bdry } A) = \phi$, and

(ii) $\overline{A} = (\text{Int } A) \cup (\text{bdry } A)$.

We conclude this section by characterizing the open subsets of $(\mathcal{R},d_1)$. The proof of the following characterization need only be studied by honours students.

THEOREM 6:

In $(\mathcal{R},d_1)$ every open set is the union of a countable family of disjoint open intervals.

[Note. Since open intervals correspond to open balls, it is true by definition that every open set is a union of open intervals.]

Proof.

Take G an open subset of $(\mathcal{R},d_1)$ and x any point in G. Let I_x equal the union of all open intervals (open balls) which contain x and are contained in G.

Then

(i) $I_x \neq \phi$, since G is open and so there exists an $r_x > 0$ such that

$x \in B_{r_x}(x) \subseteq G$, i.e. $(x - r_x, x + r_x) \subseteq I_x$.

(ii) Clearly I_x is an open set (why?). In fact I_x is an open interval. To show this it suffices to prove $(a,b) \subseteq I_x$ whenever $a < b$ and $a,b \in I_x$. Now, if $a,b \in I_x$ then, by the definition of I_x, there exist open intervals (c,d) and

(c',d') in I_x such that

$x \in (c,d) \cap (c',d')$ with $a \in (c,d)$ and $b \in (c',d')$ hence

$a,b \in (c,d) \cup (c',d') \subseteq I_x$ but $(c,d) \cup (c',d')$ is an open

interval and so $(a,b) \subseteq I_x$.

(iii) <u>If $y \in I_x$ then $I_y = I_x$</u>, for I_x is an open interval (by (ii)) containing y and contained in G, so by definition of I_y, $I_x \subseteq I_y$ and similarly $I_y \subseteq I_x$.

(iv) <u>For $x,y \in G$, either $I_x = I_y$ or $I_x \cap I_y = \phi$.</u>

Assume $z \in I_x \cap I_y$ then $z \in I_x$ so by (iii) $I_x = I_z$. Similarly $z \in I_y$ so $I_y = I_z$, whence $I_x = I_y$. We have therefore proved that the family of sets $\{I_x : x \in G\}$ is a family of disjoint open intervals and clearly

$G = \bigcup_{x \in G} I_x$ (as $x \in I_x \subseteq G$ for all $x \in G$)

it therefore only remains to prove that there are only a countable number of distinct I_x's.

Let $\mathcal{Q}_G = \mathcal{Q} \cap G$ (the <u>countable</u> set of rational numbers in G), define $f: \mathcal{Q}_G \to \{I_x : x \in G\}$ by $f(q) = I_q$ (which is unique by (iv)), then clearly f is onto, since each I_x, being an interval, contains a rational point q_x and so $I_x = I_{q_x} = f(q_x)$ whence $\{I_x : x \in G\}$ is countable. ∎

EXERCISE:

Give a similar characterization for the closed subsets of $(\mathcal{R},d_1)$.