§1.5 *Mappings between Metric spaces and Continuity*

DEFINITION. A *mapping (function)* from the metric space (X,d) into the metric space (Y,d') associates with each point $x \in X$ a unique point $y \in Y$ which is often denoted by $f(x)$.

It will be convenient to use the following suggestive

NOTATIONS.

(1) $f: X \to Y$, $f: x \mapsto f(x)$ or even $f: X \to Y : x \mapsto f(x)$, indicates the mapping f from X into Y 'carrying' x to $f(x)$.

(2) For $A \subseteq X$

$f(A) = \{f(a) : a \in A\}$

$= \{y \in Y : \text{there exists an } x \in X \text{ with } f(x) = y\}$.

$f(A)$ is termed the *image of* A *under* f.

(3) For $B \subseteq Y$

$f^{-1}(B) = \{x \in X : f(x) \in B\}$, this is not to be confused with the inverse function of f which may or may not exist.

$f^{-1}(B)$ is known as the *pre-image* (or inverse image) *of* B *under* f.

DEFINITION. For metric spaces (X,d) and (Y,d') $f : X \to Y$ is *continuous* at $x_0 \in X$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$d(x,x_0) < \delta \Rightarrow d'(f(x),f(x_0)) < \varepsilon$,

or equivalently,

$x \in B_\delta(x_0) \Rightarrow f(x) \in B'_\varepsilon(f(x_0))$

or

$f(B_\delta(x_0)) \subseteq B'_\varepsilon(f(x_0))$.

NOTE. This definition extends the familiar definition of local continuity for a $f:\mathcal{R} \to \mathcal{R}$ which is the special case $(X,d) = (\mathcal{R},d_1)$.

Local continuity in metric spaces can be characterized in terms of sequences, as the next theorem shows.

THEOREM 1 (SEQUENTIAL CONTINUITY):

Let (X,d), (Y,d') *be metric spaces, then* $f : X \to Y$ *is continuous at* $x_0 \in X$ *if and only if for every sequence* $\{x_n\}$ *with* $x_n \to x_0$, $f(x_n) \to f(x_0)$.

Proof.

($\Rightarrow$) Since f is continuous at x_0, for any $\varepsilon > 0$ there is a $\delta > 0$ with $f(B_\delta(x_0)) \subseteq B'_\varepsilon(f(x_0))$

Now if $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that

$d(x_n,x_0) < \delta$ for all $n \geq N$ and so

$f(x_n) \in B'_\varepsilon(f(x_0))$ for all $n \geq N$

or $f(x_n) \to f(x_0)$.

($\Leftarrow$) Assume $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$, but f is not continuous at x_0 <u>i.e.</u> there exists an $\varepsilon > 0$ such that $f(B_r(x_0)) \not\subseteq B'_\varepsilon(f(x_0))$ for all $r > 0$. Thus in particular for each $n \in \mathbb{N}$, there exists an $x_n \in B_{\frac{1}{n}}(x_0)$ such that

$f(x_n) \notin B'_\varepsilon(f(x_0))$.

The sequence $\{x_n\}$ so constructed is such that $d(x_n,x_0) < \frac{1}{n}$

so $x_n \to x_0$, but $d(f(x_n),f(x_0)) > \varepsilon$ all n, so $f(x_n) \not\to f(x_0)$

a contradiction to our assumption. ∎

DEFINITION: For metric spaces (X,d), (Y,d'), $f:X \to Y$ is <u>*continuous*</u> if f is continuous at each $x \in X$

(This is sometimes referred to as f being <u>globally continuous.</u>)

COROLLARY (to Theorem 1): *If* (X,d) and (Y,d') *are metric spaces, then* <u>$f : X \to Y$ *is continuous if and only if f preserves convergent sequences*</u>

<u>i.e.</u> for any sequence $\{x_n\}$ convergent to x we have $f(x_n) \to f(x)$.

NOTE. (1) It is <u>not</u> true, that for continuous $f : X \to Y$ if $f(x_n) \to f(x)$ then $x_n \to x$. [E.G. in $(\mathbb{R},d_1)$ for $f : x \mapsto x^2$ and $x_n = (-1)^n$, $f(x_n) = 1 \to f(1)$ but $-1, 1, -1, 1, -1, \ldots, \not\to 1$.]

(2) This corollary often provides the simplest way of proving a mapping is discontinuous at x; viz. by selecting a sequence $x_n \to x$ for which $f(x_n) \not\to f(x)$.

EXERCISE. The Corollary to theorem 1 asserts that convergent sequences are preserved under continuous mappings. Show that this is not necessarily true for Cauchy sequences, i.e. it may happen that $\{x_n\}$ is a Cauchy sequence in (X,d), $f : X \to Y$ is continuous and $\{f(x_n)\}$ is not a Cauchy sequence in (Y,d').
(Hint: Consider $f : (0,\infty) \to (0,\infty) : x \mapsto 1/x$.)

EXAMPLES:

1) *Lipschitz Mappings*

Let $f : X \to Y$, where (X,d) and (Y,d') are metric spaces, be such that $d'(f(x),f(y)) \leqslant Md(x,y)$ for all $x, y \in X$ and some $M > 0$, (Such an f is said to satisfy a *Lipschitz condition* with Lipschitz constant M), then f is continuous.

[Clearly given $\varepsilon > 0, f(B_{\frac{\varepsilon}{M}}(x)) \subseteq B_\varepsilon(f(x))$ for all $x \in X$.]

[*Remark: The mean value theorem for derivatives asserts that every differentiable functions on* (a,b) *satisfies a Lipschitz condition with* $(X,d) = ((a,b), d_1)$ *and* $(Y,d') = (R,d_1)$, *can you prove this?*]

Of special importance later, will be the case when $M < 1$, $X = Y$ and $d = d'$, in which case f is called a *strict contraction* on (X,d) - the effect of f is to every where decrease the distance between points.

Another particular case of special interest, occurs when $d'(f(x), f(y)) = d(x,y)$ for all $x, y \in X$. Such an f is called an *isometry* from (X,d) into (Y,d').

Note: An isometry is necessarily 1 to 1, i.e. $f(x) = f(y) \Rightarrow x = y$ or $f^{-1}(\{x\})$ is singleton [prove this].

If f is an isometry of (X,d) into (Y,d') we can readily see that restricting d' to $f(X)$, the metric space $(f(X),d')$ is essentially the same as (X,d) and we speak of Y containing a (isometric) copy of X, viz $f(X)$.

Note further, that if f is also onto, i.e. $f(X) = Y$, then the inverse map $f^{-1}: Y \to X$ exists and

$$d(f^{-1}(x), f^{-1}(y)) = d'(f(f^{-1}(x)), f(f^{-1}(y))) = d'(x,y)$$

so in this case f^{-1} is an isometry (and so continuous mapping) from Y onto X.

2) *Linear Mappings between Normed Linear Spaces*

Let X be a normed linear space, with norm $\|.\|$, and

Y be a normed linear space with norm $\|.\|'$.

RECALL. $T : X \to Y$ is *linear* if

$$T(x + \lambda y) = T(x) + \lambda T(y) \text{ for all } x, y \in X \text{ and scalars } \lambda.$$

(In this context the term mapping (or function) is sometimes replaced by transformation or operator.)

DEFINITION. A linear mapping $T : X \to Y$ between normed linear spaces is *bounded* if for all $x \in X$ $\|T(x)\|' \leq M\|x\|$ for some $M > 0$.

NOTE: This is not quite the same as the "usual" meaning of "bounded": for real valued functions f is bounded if there exists some constant M ~~such that $|f(x)| \leq M$ for all x; indeed the R.H.S. of the inequality~~ in the above definition of boundedness depends on x and may become arbitratily large [to see this, for any $x \neq 0$ consider $M\|nx\|$ where

$n \in \mathbb{N}$]. *There is however a connection between the two notions of bounded. It is an easy* EXERCISE *to show that the linear mapping* T *is bounded if and only if* $\|Tx\| \leq M$ *for all* $x \in B[X]$. *Thus,* T *is bounded in the above sense if and only if the mapping* $x \mapsto \|Tx\|$ *is a bounded function (in the usual sense) on the unit ball of* $X, B[X]$.

THEOREM 2: *A linear mapping between normed linear spaces is continuous if and only if it is bounded.*

(Here 'continuous' means continuous w.r.t. the metrics induced by the respective norms.)

Proof. ($\Leftarrow$) Given $\varepsilon > 0$ and any $x \in X$, if $y \in B_{\frac{\varepsilon}{M}}(x)$ i.e. $\|x-y\| < \frac{\varepsilon}{M}$, then

$$\|T(x) - T(y)\|' = \|T(x-y)\|' \qquad \text{(by linearity)}$$
$$\leq M\|x-y\| \qquad \text{(by boundedness)}$$

and so $\quad T(y) \in B_{\varepsilon}(T(x))$, whence T is continuous.

($\Rightarrow$) Since T is continuous, it is in particular continuous at 0 i.e. given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x,0) < \delta \Rightarrow d'(T(x), T(0)) < \varepsilon$$

or

$$\|x\| < \delta \Rightarrow \|T(x)\|' < \varepsilon \qquad \text{(as } T(0) = 0\text{, by linearity).}$$

Now for any $x \in X$,

$$\left\|\frac{\delta}{2\|x\|}x\right\| = \frac{\delta}{2\|x\|}\|x\| < \delta \qquad \text{(by (N3).)}$$

so

$$\left\|T\left(\frac{\delta}{2\|x\|}x\right)\right\|' = \frac{\delta}{2\|x\|}\|T(x)\|' \qquad \text{(by the linearity of } T \text{ and (N3).)}$$
$$< \varepsilon$$

or

$$\|T(x)\|' < \frac{2\varepsilon}{\delta}\|x\|.$$

whence T is bounded with $M = \frac{2\varepsilon}{\delta}$

EXERCISES:

1) i) Let (X,d) be a metric space and x_0 a fixed element of X. Show that the mapping

$$f : X \to R : x \mapsto d(x,x_0)$$

is continuous.

ii) Let $(X,\|.\|)$ be a normed linear space, show the mapping $x \mapsto \|x\|$ is continuous from (X,d) to (R,d_1) where d is the metric induced by the norm $\|.\|$.)

2) If T is a linear transformation from R^n to R^m then

$$T(x_1,x_2,\ldots,x_n) = \left(\sum_{i=1}^{n} t_{1i}x_i, \sum_{i=1}^{n} t_{2i}x_i, \ldots, \sum_{i=1}^{n} t_{mi}x_i\right)$$

for some $m \times n$ matrix $[t_{ji}]$ (refer linear algebra).
Show that T defines a bounded (hence continuous) linear mapping from ℓ_1^n to ℓ_1^m.

3) i) Show that the "evaluation functional"

$$F : C[a,b] \to R : f \mapsto f(x_0) \quad (x_0 \text{ a fixed point of } [a,b]),$$

is a continuous mapping from $(C[a,b],d_\infty)$ into (R,d_1).
Is this still true if $C[a,b]$ is considered with the metric d_1.

**ii) Prove that $T : C[0,1] \to C[0,1]$ defined by $T(f)(x) = \int_0^x F(t,f(t))dt$ is continuous with respect to the metric d_∞ if $F : R^2 \to R$ is continuous.

4) Show that, *a linear mapping between normed linear spaces is continuous if and only if it is continuous at* 0.

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5) Prove that $Ker(T) = T^{-1}(\{0\})$ is a closed subset if T is a continuous linear mapping between normed linear spaces.

6) Let (X,d), (Y,d'), $(Z,d")$ be metric spaces and $f : X \to Y$, $g : Y \to Z$ be continuous mappings.

Show the composite $g o f : X \to Z$ is continous.

Remark: The algebraic structure of any vector space X corresponds to a number of 'natural' mappings. From the additive structure we derive the translations

$$t_y : X \to X : x \mapsto x + y \ (= y + x), \text{ for each } y \in X.$$

Similarly, scalar multiplication produces

$$d_\lambda : X \to X : x \mapsto \lambda x, \text{ for each scalar } \lambda,$$

and $f_x : R \to X : \lambda \mapsto \lambda x$, *for each* $x \in X$

These last two mappings are readily seen to be linear.
A metric (or topology) on X is said to be compatible with the algebraic structure of X if the above mappings are continuous with respect to it when R has the usual metric.
For a normed linear space $(X, \|.\|)$, *the continuity of* t_y *follows easily while that of* d_λ *and* f_x *is equivalent to (N3), which asserts boundedness for these mappings.*
Thus in a normed linear space the algebraic structure and the metric induced by the norm are compatible.

The next theorem provides a very general, and powerful characterization of continuous mappings which is often used as a definition of continuity, particularly in the setting of topological spaces (see comment on page 50).

THEOREM 3: *Let* (X,d) *and* (Y,d') *be metric spaces, then* $f: X \to Y$ *is continuous if and only if for any open set* $G \subseteq Y$, $f^{-1}(G) = \{x \in X: f(x) \in G\}$ *is an open set of* X.

i.e. the inverse image of open sets under f are open.

PROOF. ($\Rightarrow$) Let G be an open subset in Y, then for any $x \in f^{-1}(G), f(x) \in G$ which is open and so there exists $r_x > 0$ with $B'_{r_x}(f(x)) \subset G$.

Now there exists a $\delta_x > 0$ with $f(B_{\delta_x}(x)) \subseteq B'_{r_x}(f(x)) \subseteq G$ (taking $\varepsilon = r_x$ in the definition of continuity), so $B_{\delta_x}(x) \subseteq f^{-1}(G)$ and $f^{-1}(G)$ is open.

($\Leftarrow$) Since, $f^{-1}(G)$ is open whenever G is, for any $x \in X$ and $\varepsilon > 0$, we have $f^{-1}(B'_\varepsilon(f(x))$ is an open set containing x and so there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B'_\varepsilon(f(x))$ or $f(B_\delta(x)) \subseteq B'_\varepsilon(f(x))$ and f is continuous. ∎

NOTE: This theorem does not assert that the image of an open set under f is open. This is demonstrated in the following exercise.

EXERCISE:

A mapping $f: X \to Y$ between the two metric spaces (X,d), (Y,d') is *open* if $f(A)$ is an open subset of Y whenever A is an open subset of X (i.e. f carries open sets to open sets).

Show that not every continuous mapping need be open.

(Hint: Consider $f: \mathbb{R} \to \mathbb{R}: x \mapsto c$, a constant.)

If $f: X \to Y$ is a 1-1 and onto mapping (so $f^{-1}: Y \to X$ exists) between the metric spaces (X,d) and (Y,d') such that both f and f^{-1} are continuous, then f is termed a *homeomorphism* of X onto Y and we say X and Y are *homeomorphic* or *topologically equivalent*. [A particular case of this occurs when f is an isometry of X onto Y.] The term "topologically equivalent" is an appropriate one: From the above theorem and the continuity of f and f^{-1} it is an easy EXERCISE to see that a subset of X is open if and only if its image under f is also open.

A property P is a *topological invariant* for metric spaces if, whenever X, Y are homeomorphic under f and $A \subseteq X$ has P then $f(A)$ also has P.

EXAMPLES. It is easily seen that the property of "being open" is a topological invariant (prove), as also is "being closed", however "boundedness" is not [give an example to show this. Hint: consider $(\mathbb{R},d_1)$ and $(\mathbb{R},d^*)$, where d^* is the metric derived from d_1 as in Exercise 2 of page 5.].

EXERCISES:

1) i) Prove that the metric spaces $\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), d_1\right)$ and (R, d_1) are homeomorphic. [Hint: consider $f \equiv \tan$]

ii) Using i) conclude that "completeness" is <u>not</u> a topological invariant.

2) If f is a homeomorphism of (X,d) onto (Y,d') show that

$$d''(x,y) = d'(f(x),f(y))$$

defines an equivalent metric to d on X (see exercise 5 on page 51).

*3) Let X and Y be normed linear spaces with norms $\|.\|$ and $\|.\|'$ respectively. Show that the linear mapping $T: X \to Y$ is a homeomorphism if and only if ther exists $m, M > 0$ such that

$$m\|x\| \leqslant \|T(x)\|' \leqslant M\|x\| \quad \text{for all } x \in X.$$

*4) Let M denote the family of <u>all</u> metric spaces. Show that "(X,d) is homeomorphic to (Y,d')" defines an equivalence relationship on M (and so metric spaces may be partitioned into classes of homeomorphic spaces).

REMARK:

Let X *be any vector space of finite dimension n over the field* R, $\|.\|$ *a norm on* X *and* $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{b}_n\}$ *a basis for* X.
It may be shown that the 'natural' isomorphism

$$\phi: X \to R^n: (\lambda_1\underset{\sim}{b}_1 + \lambda_2\underset{\sim}{b}_2 + \ldots + \lambda_n\underset{\sim}{b}_n) \mapsto (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

is a homeomorphism between X *and* ℓ_2^n.

Thus X *and* ℓ_2^n *are topologically equivalent. In particular then the family of spaces* ℓ_p^n $(1 \leqslant p \leqslant \infty$, n fixed) *are topologically equivalent.*
~~*Further, since in this case* ϕ *can be taken to be the identity*~~ *map, each of the metrics* d_p $(1 \leqslant p \leqslant \infty)$ *give rise to the same open sets (although of course the open balls differ from metric to metric).*

Consequently there is only one norm topology possible for R^n.
(Although there are many different norms they all induce equivalent metrics.)
In fact, it can be shown that there is only one topology for R^n compatible with the algebraic structure.
With respect to this unique topology any linear transformation (mapping) $T : R^n \to R^m$ is continuous. By the previous remark, this implies the continuity of any linear transformation between finite dimensional vector spaces. This explains the relative unimportance of metric, or continuity, arguments in finite dimensional linear algebra.
(Precisely the same remarks apply to finite vector spaces over the complex field C.)

§1.6 *Compactness in Metric Spaces*

Before proceeding to the main work of this section we must first clarify the notion of a "subsequence".

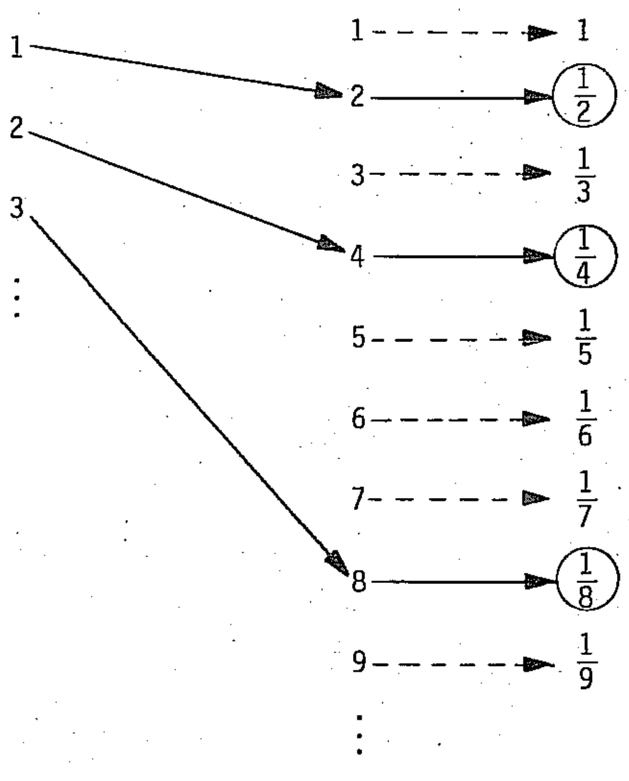
DEFINITION. A subsequence of the sequence (a_n); that is, of the function α: $n \mapsto a_n$, is the composite $\alpha \circ \eta$ where η is a strictly increasing sequence of natural number; η: $k \mapsto n_k \in N$ with $n_k > n_\ell$ whenever $k > \ell$. [Note: this implies that $n_k \geq k$ for all k. An observation which is often helpful in the course of proofs.]

We will denote such a subsequence by

$$\left(a_{n_k}\right)_{k=1}^{\infty}, \quad \left(a_{n_k}\right) \text{ or } a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$$

EXAMPLE: Let $a_n = \frac{1}{n}$ and let η be the (strictly increasing) sequence of natural numbers $k \mapsto 2^k$, then the composit $\alpha \circ \eta$ is the function

$$k \mapsto \alpha(\eta(k)) = \alpha(2^k) = a_{2^k} = {}^1/_{2^k}$$



Thus the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^k}, \ldots$ is a subsequence of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$.

Intuitively a subsequence of (a_n) is obtained by deleting points from $a_1, a_2, \ldots, a_n, \ldots$. while preserving the order, provided that the deletions are such that an infinite number of the original terms are always retained.

EXERCISES: 1) In any metric space (X , d) show that the following are equivalent.

i) the sequence (a_n) converges to a.

ii) every subsequence of (a_n) converges to a

iii) (a_n) is a Cauchy sequence and there is at least one subsequence (a_{n_k}), of (a_n), which converges to a.

2) Show that a subsequence of a subsequence of (a_n) is itself a subsequence of (a_n).

DEFINITION. In a metric space (X,d) the subset $K \subseteq X$ is *compact* (strictly speaking, *sequentially compact*) if every sequence of points in K has a subsequence which converges to a point of K.

As we will subsequently see compact sets have many very nice properties. *The fundamental role compactness plays in mathematics is excellently set forth in Courant and Robbins, "What is Mathematics?". A book which you should certainly try to inspect.*

At this stage it is difficult to give examples of compact sets. We remark however that the closed interval [0,1] in $(\mathbb{R},d_1)$, indeed any closed bounded subset of a finite dimensional normed linear space, is compact. [The proof of these assertions will occupy much of this section.]

It is however easy to give examples of non-compact sets. Thus in $(\mathbb{R},d_1)$ the whole space $\mathbb{R}$ and the set of positive real numbers $\mathbb{R}^+$ are not compact. [To see this, note that either set contains the sequence of natural numbers 1,2,3, ..., n, ... and that no subsequence of these can converge.]

EXERCISES. 1) Show that the intersection of two compact sets is compact.

2) Show that the union of two compact sets is compact. [Hint. For any sequence in $A \cup B$ note that either an infinite number of the terms must lie in A or in B. Hence there is a subsequence contained entirely in A or B.]

3) Show that any finite subset (that is, a subset with only a finite number of elements) of the metric space (X,d) is compact.

4) For any set X with the discrete metric d show that the only compact subsets of (X,d) are the finite subsets.

Before turning to the problem of characterizing compact sets we prove one extremely useful property of such sets (others will be given in the next chapter).

Theorem 1. *Let (X,d) be a metric space, K a compact subset of X and $f: K \to \mathbb{R}$ a continuous real valued mapping on K, then f assumes a maximum and a minimum value on K.*

Remark: That this is not true for general K is shown by the function $f: (0,\infty) \to \mathbb{R}: x \mapsto \frac{1}{x}$ which is continuous (indeed differentiable) but assumes neither a maximum or minimum value. Incidentally this also serves to show that $(0,\infty)$ is not a compact subset of $\mathbb{R}$ with absolute value as norm.

Proof. Noting that a minimum of f is a maximum of $-f$ it suffices to show f assumes a maximum. Let $M = \sup\{f(x): x \in K\}$, i.e. M is the supremum or l.u.b. of the range of f; we must show there exists $x_M \in K$ such that $f(x_M) = M$. Now, by the definition of supremum there exists a sequence of points (x_k) of K such that $f(x_k) \to M$ as $k \to \infty$, and by the compactness of K there is a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ convergent to some point x_M of K. Whence, by the continuity of f on K we have $f(x_M) = \lim_{n\to\infty} f(x_{k_m}) = M$ as required. ∎

We begin our search for compact sets with a series of lemmas. The first two give necessary conditions for a set to be compact while the next two provide means whereby new compact sets may be derived from known ones.

LEMMA 1: *A compact subset of a metric space is closed.*

Proof. Let K be a compact subset of the metric space (X,d) and (x_n) a sequence of points of K convergent to $x \in X$ we must show $x \in K$. By the compactness of K there is a subsequence of points (x_{n_k}) convergent to a point y of K, but then by exercise 1 on page 68 $x = y$ so $x \in K$ as required. ∎

LEMMA 2: *A compact subset of a metric space is bounded.*

Proof. Let K be a compact subset of the metric space (X,d) and choose any point $x_o \in X$.

$$\text{Let } R = \text{Sup } \{d(x_o,x) : x \in K\}.$$

Then, by definition of supremum $d(x_o,x) \le R$ for all $x \in K$ and there exists a sequence (x_n) of points of K with $d(x_o,x_n) \to R$ as $n \to \infty$. By the compactness of K there exists a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots$ convergent to some $x \in K$. Thus

$$\begin{aligned} R = \lim_{k\to\infty} d(x_o, x_{n_k}) &\le \lim_{k\to\infty} (d(x_o,x) + d(x, x_{n_k})) \\ &= d(x_o,x) + \lim_{k\to\infty} d(x, x_{n_k}) \\ &= d(x_o,x) + 0, \end{aligned}$$

and so we conclude that $R < \infty$, proving the result. ∎

EXERCISE: By first proving that the mapping

$Q:(X,d) \to (R,d_1) : x \mapsto d(x_o,x)$ [x_o a fixed point of X] is continuous, give an alternative proof of lemma 2 based on Theorem 1 above.

LEMMA 3: *Let (X,d) and (Y,d') be metric spaces, K a compact subset of X and f: K → Y a continuous function on K into Y. Then*

$f(K) = \{f(x) : x \in K\}$ *is a compact subset of* Y, *i.e. the continuous image of a compact subset is compact or compactness is preserved under continuous mappings.*

Proof. Let (y_n) be a sequence of points in $f(K)$, then for each $n \in N$ $y_n = f(x_n)$ for some point $x_n \in K$ (definition of $f(K)$) and so we arrive at a sequence of points (x_n) in K which, by the compactness of K, contains a subsequence $\left(x_{n_k}\right)_{k=1}^{\infty}$ convergent to some point $x \in K$.

Now, the sequence $\left(f(x_{n_k})\right)_{k=1}^{\infty}$ is clearly a subsequence of (y_n) and by the continuity of f, $f(x_{n_k}) \to f(x)$, i.e. (y_n) contains a convergent subsequence and so f(K) is compact. ∎

LEMMA 4. *A closed subset of a compact set is compact.*

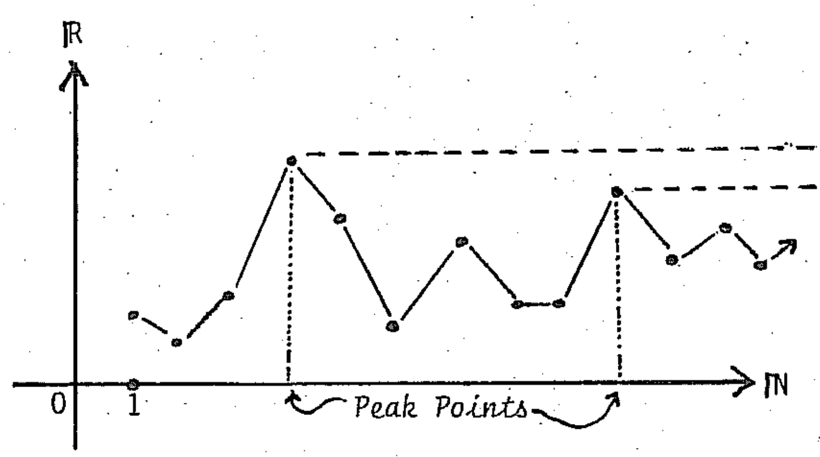
Proof. Let K be a compact subset of X, $A \subseteq K$ a closed subset and (a_n) a sequence of points of A. Then (a_n) is also a sequence of points of K and so by its compactness there exists a subsequence $\left(a_{n_k}\right)_{k=1}^{\infty}$ convergent to $a \in K$, thus, since $a_{n_k} \in A$ each k, a is a limit point of A and so $a \in A$ (A is closed). Whence (a_n) has a subsequence convergent to $a \in A$ and so A is compact. ∎

We have so far not given any examples of a compact set, our next results remedy this situation at least in the case of finite dimensional normed linear spaces.

We begin with the classical Theorem of Bolzano-Weierstrass.

Theorem 5. For $a < b \in R$ *the closed interval* $[a,b]$ *is a compact subset of* R *with absolute value as norm.*

Proof. Let (x_n) be any sequence of points of $[a,b]$, it suffices to show: there is a subsequence $\left(x_{n_k}\right)_{k=1}^{\infty}$ which is either decreasing or increasing, for then, (x_{n_k}) is bounded both above and below (by b and a) and so, by the first year result "an increasing (decreasing) sequence bounded above (below) is convergent", $x_{n_k} \to x$ for some $x \in \mathbb{R}$ which, since $[a,b]$ is closed, is an element of $[a,b]$. I.e. (x_n) contains a subsequence $x_{n_k} \to x \in [a,b]$ whence $[a,b]$ is compact.



all subsequent terms lie beneath the dotted lines

Call $n \in N$ a "peak point" of (x_n) if $x_n > x_m$ for all $m > n$, then we have two possibilities:

(1) (x_n) has an infinite number of peak points at $n_1 < n_2 < n_3 < \ldots..$, in which case $x_{n_1} > x_{n_2} > x_{n_3} > \ldots$ and so $\left(x_{n_k}\right)_{k=1}^{\infty}$ is the required decreasing subsequence.

(2) (x_n) has only finitely many peak points at $n_1 < n_2 < \ldots < n_m$. ~~In this case let $N_1 > n_m$, then N_1 is not a peak point so~~ there exists $N_2 > N_1$ with $x_{N_1} < x_{N_2}$, further N_2 is not a

peak point so there exists $N_3 > N_2$ with $x_{N_2} < x_{N_3}$, continuing inductively in this way we arrive at the required increasing subsequence $x_{N_1} < x_{N_2} < x_{N_3} < \ldots$. ∎

Our next results generalise this result into higher dimensions.

LEMMA 6. *Let* $X = \ell_\infty^n$, *then the* *<u>unit Ball</u>* *of* X,

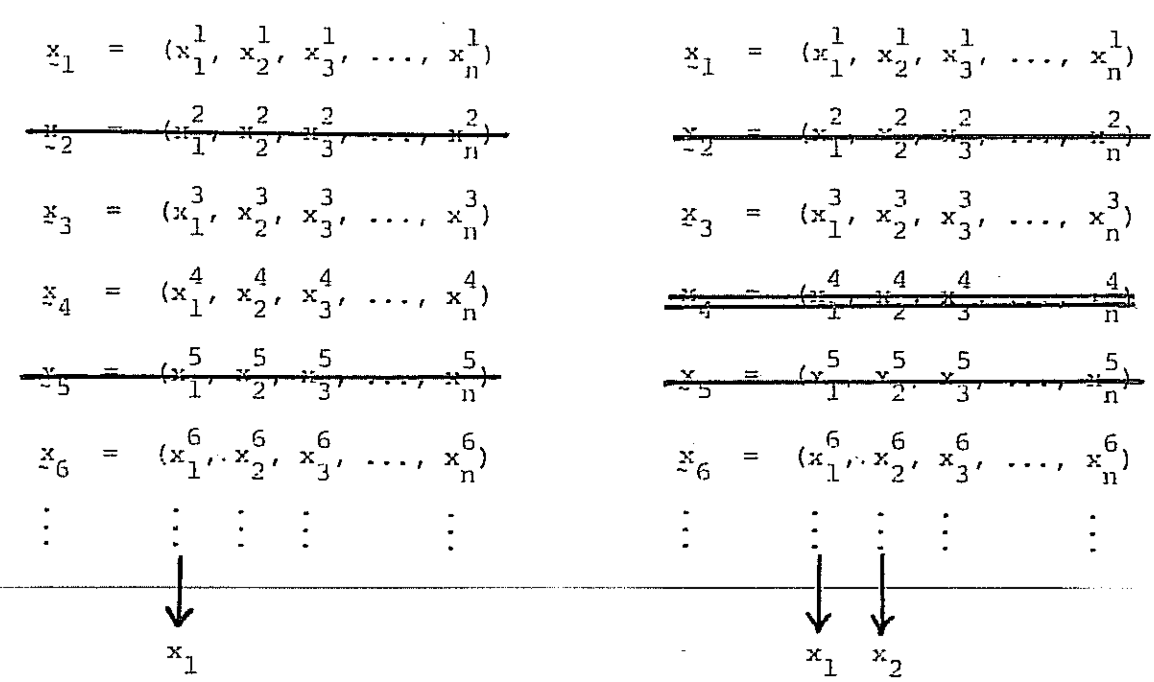
$B = \{\underset{\sim}{x} \in X: \ \|\underset{\sim}{x}\|_\infty \leq 1\}$, *is compact.*

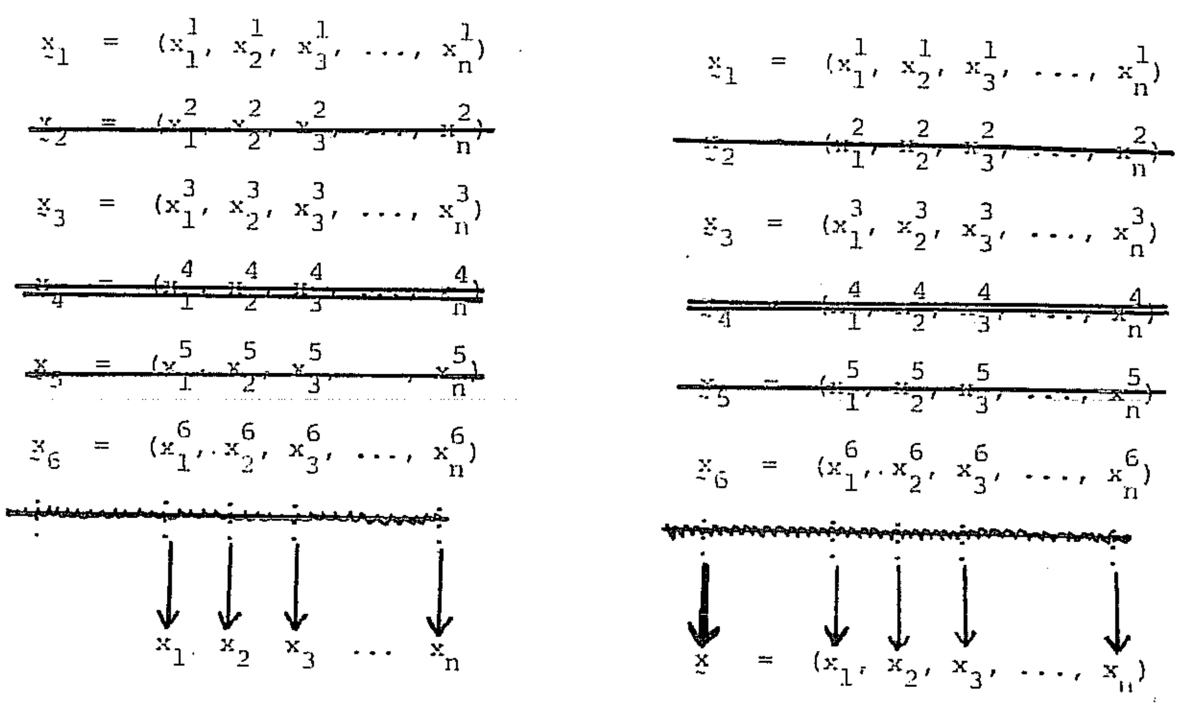
Proof. Let $\underset{\sim}{x}_m = (x_1^m, x_2^m, x_3^m, \ldots, x_n^m)$

be a sequence of points in B, <u>i.e.</u> $|x_j^m| \leq 1$ for all $m \in \mathbb{N}$ and $j \in \{1,2,\ldots,n\}$.

Now $x_1^1, x_1^2, x_1^3, \ldots, x_1^m, \ldots$ (the sequence of first components) is a sequence in $[-1, 1]$ and so by Theorem 5 above there is a subsequence $x_1^{n_1}, x_1^{n_2}, x_1^{n_3}, \ldots$ and a point $x_1 \in [-1, 1]$ with $x_1^{n_k} \to x_1$ as $k \to \infty$.

Similarly, the second components of the <u>subsequence</u> $\underset{\sim}{x}_{n_1}, \underset{\sim}{x}_{n_2}, \underset{\sim}{x}_{n_3}, \ldots,$





viz, $x_2^{n_1}$, $x_2^{n_2}$, $x_2^{n_3}$,... is a sequence in [-1, 1] and so there is a subsequence $x_2^{n_{m_1}}$, $x_2^{n_{m_2}}$, $x_2^{n_{m_3}}$, ... with $x_2^{n_{m_k}} \to x_2 \in [-1, 1]$. We therefore have a subsequence $(\underset{\sim}{x}_{n_{m_k}})$ of $(\underset{\sim}{x}_n)$ for which the sequences of first and second components converge to x_1 and x_2 respectively. Continuing in this manner we will arrive at a subsequence of $(\underset{\sim}{x}_n)$, which for notational convenience we will denote by $\underset{\sim}{x}_{s_1}$, $\underset{\sim}{x}_{s_2}$, ..., $\underset{\sim}{x}_{s_k}$, ... each of whose component sequences converge, <u>i.e</u>. for each $j \in \{1,2,...,n\}$

$$x_j^{s_k} \to x_j \in [-1, 1] \text{ as } k \to \infty.$$

Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, then $\underset{\sim}{x} \in B$ (as $x_j \in [-1, 1]$, $1 \leqslant j \leqslant n$) and

$$\|\underset{\sim}{x}_{s_k} - \underset{\sim}{x}\|_\infty = \max_{1\leqslant j\leqslant n} |x_j^{s_k} - x_j| \to 0 \text{ as } k \to \infty \text{ since } x_j^{s_k} \to x_j \text{ as } k \to \infty.$$

Thus $(\underset{\sim}{x}_n)$ has a subsequence convergent to $\underset{\sim}{x} \in B$ and so B is compact. ■

Corollary 7. (A version of the Heine-Borel Theorem): *Let X be as in Lemma 6 above, then every closed and bounded subset is compact.*

Proof. Let F be a closed, bounded subset of X, then $R = \text{Sup}\{\|x\|: x \in F\} < \infty$ so F is a closed subset of $B_R = \{x \in X: \|x\| \leqslant R\}$ so by lemma 4 it suffices to show B_R is compact. Now $B_R = f(B)$ where B is an as lemma 6 above and f is the function $f(x) = Rx$ which is easily seen to be continuous and so since B is compact by lemma 6 we have by lemma 3 that B_R is compact as required. ■

We now characterize the compact subsets of any finite dimensional space.

Theorem 8. *Let* $(X, \|\cdot\|)$ *be a normed linear space of finite dimension n. A subset A of X is compact if and only if A is closed and bounded.*

Remark: *The assumption X is finite dimensional is over strong, all we need is that A be a subset of a finite dimensional subspace of X.*

Proof. (⇒) has already been established in lemmas 1 and 2.

(⇐) Let $\{b_1, b_2, \ldots, b_n\}$ be a basis of X, then each $x \in X$ may be expressed uniquely as

$x = \sum_{i=1}^{n} x_i b_i$ ($x_i \in \mathbb{R}$, all i) and so we can define the mapping

$T: X \to \mathbb{R}^n$ by $T(x) = (x_1, x_2, \ldots, x_n)$.

It is an elementary exercise in linear algebra to show T is 1-1 and onto and consequently invertible with inverse $T^{-1}: \mathbb{R}^n \to X: (x_1, x_2, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i b_i$. Further, if $\mathbb{R}^n$ has norm $\|\underset{\sim}{x}\|_\infty = \underset{1\leqslant i\leqslant n}{\text{Max}} |x_i|$, where $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ then T^{-1} (and indeed T also) is a continuous function because

$$\|T^{-1}(\underset{\sim}{x}) - T^{-1}(\underset{\sim}{y})\| = \|\sum_{i=1}^{n} x_i b_i - \sum_{i=1}^{n} y_i b_i\| = \|\sum_{i=1}^{n} (x_i - y_i) b_i\|$$

$$\leqslant \sum_{i=1}^{n} |x_i - y_i| \|b_i\| \text{ by the triangle inequality and (N3)}$$

$$\leqslant \sum_{i=1}^{n} \|\underset{\sim}{x}-\underset{\sim}{y}\|_\infty \|b_i\| \text{ as } \|\underset{\sim}{x}-\underset{\sim}{y}\|_\infty = \underset{1\leqslant i\leqslant n}{\text{Max}} |x_i - y_i|$$

$$= \left(\sum_{i=1}^{n} \|b_i\|\right) \|\underset{\sim}{x}-\underset{\sim}{y}\|_\infty$$

and so $T^{-1}(\underset{\sim}{y}) \to T^{-1}(\underset{\sim}{x})$ as $\underset{\sim}{y} \to \underset{\sim}{x}$.

Thus since $A = T^{-1}(T(A))$ it suffices to show $T(A)$ is a closed bounded subset of $\mathbb{R}^n$, for then by lemma 6 it is compact and so A is the image under T^{-1} of a compact set and so by lemma 3 is itself compact.

It therefore only remains to prove:

(a) T(A) is closed. Thus let $(\underset{\sim}{x}_n)$ be a sequence of T(A) convergent to $\underset{\sim}{x}$ then $T^{-1}(\underset{\sim}{x}_n) \to T^{-1}(\underset{\sim}{x})$ by the continuity of T^{-1}, but $T^{-1}(\underset{\sim}{x}_n) \in A$, which is closed, and so $T^{-1}(\underset{\sim}{x}) \in A$ whence $\underset{\sim}{x} = T(T^{-1}(\underset{\sim}{x})) \in T(A)$. So T(A) is closed.

(b) T(A) is bounded. Let $S = \{\underset{\sim}{y} \in \mathbb{R}^n : \|\underset{\sim}{y}\|_\infty = 1\}$. It suffices to show $m = \underset{\underset{\sim}{y}\in S}{\text{infimum}} \|T^{-1}(\underset{\sim}{y})\|$ is non-zero, for then, if $0 \neq \underset{\sim}{x} \in T(A)$, $\dfrac{\underset{\sim}{x}}{\|\underset{\sim}{x}\|_\infty} \in S$ so $m \leqslant \|T^{-1}(\underset{\sim}{x}/\|\underset{\sim}{x}\|)\| = \|T^{-1}(\underset{\sim}{x})\|/\|\underset{\sim}{x}\|$ or $\|\underset{\sim}{x}\|_\infty \leqslant \frac{1}{m} \|T^{-1}(\underset{\sim}{x})\|$ and the result follows since $T^{-1}(\underset{\sim}{x}) \in A$ and A is bounded.

Now, S is a closed bounded subset of $\mathbb{R}^n$ and therefore, by the above lemma, compact. Hence, by theorem 1, the continuous mapping $\underset{\sim}{y} \mapsto \|T^{-1}(\underset{\sim}{y})\|$ achieves its infimum m on S at some (necessarily non zero) point $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n) \in S$, and so $m \neq 0$, for otherwise $T^{-1}(\underset{\sim}{a}) = \sum_{i=1}^{n} a_i b_i = 0$ contradicting the linear independence of the basis $\{b_i\}$.

∎

Corollary 9. *In the notation of the last theorem, the mapping T is a homeomorphism.* [*This shows that any normed linear space of finite dimension n is topologically equivalent to* ℓ^n_∞ *- see the remark on page 64*].

Proof. We have already established the continuity of T^{-1}. That T is itself continuous follows from (b) of the above proof. For any $x \neq 0$ we have $Tx \neq 0$ (as T is 1-1) and so $\frac{Tx}{\|Tx\|_\infty} \in S$, but then by (b) above $m \leqslant \left\|T^{-1}\left(\frac{Tx}{\|Tx\|}\right)\right\|$

or $m \leqslant \frac{1}{\|Tx\|_\infty}\|(T^{-1}(Tx))\| = \frac{\|x\|}{\|Tx\|_\infty}$.

Rearranging gives

$$\|Tx\|_\infty \leqslant \frac{1}{m}\|x\|$$

and so T is bounded (and hence continuous) as $m \neq 0$. ∎

*EXERCISE: Prove that any linear mapping between two finite dimensional normed linear spaces is continuous.

Our final result is the promised demonstration that all finite dimensional normed linear spaces are complete.

Corollary 10. *Any finite dimensional normed linear space is complete.*

Proof. Let (x_n) be a Cauchy sequence of the finite dimensional normed linear space $(X, \|\cdot\|)$, then (x_n) is bounded, <u>i.e.</u> there exists $R > 0$ with $\|x_n\| \leqslant R$ for all n. Thus (x_n) is a sequence of the closed bounded set $\{x \in X: \|x\| \leqslant R\}$, which by the above theorem is compact, so (x_n) has a subsequence (x_{n_k}) convergent to x. But

$$\|x_k - x\| \leqslant \|x_k - x_{n_k}\| + \|x_{n_k} - x\| \quad \text{and so} \quad \|x_k - x\| \to 0 \text{ as } k \to \infty,$$

<u>i.e.</u> (x_n) is convergent to x. ∎

EXERCISES: 1) Show that any finite dimensional subspace of a normed linear space is closed.

2) By considering the sequence $(\underset{\sim}{e}_n)$ where

$$\underset{\sim}{e}_n = (0,0,\ldots,0,\underset{\substack{\uparrow \\ n\text{'th place}}}{1},0,\ldots.)$$

show that the unit ball of ℓ_∞ is not compact.

[*This shows that the conclusion of Theorem 8 is not valid in the infinite dimensional space* ℓ_∞. *Indeed it can be proved that the unit ball of a normed linear space is compact if and only if the space is finite dimensional. This is a consequence of Riesz' Lemma, which honours students might like to try and prove.*

(Riesz' Lemma). Let M *be a proper closed subspace of the normed linear space* $(X,\|\cdot\|)$. *For each* $\delta \in (0,1)$ *show there exists a point* $x_0 \in X$ *such that* $\|x_0\| = 1$ *and* $d(x_0, M) = \inf_{m\in M}\|x_0 - m\| \geqslant \delta$.]