Proof. Since the complement of a closed set is open, $M = G \cap (X \setminus \overline{E})$ is an open subset of X. Further, if M were empty we would have $G \subseteq \overline{E}$, that is, the closure of E would contain the non-empty open set G and so it would have non-empty interior, contradicting the fact that E is nowhere dense. Thus, M is a non-empty open subset of X. Choose any point $x \in M$, then x is an interior point of M and so there exists $r > 0$ such that $B_r(x) \subseteq M$.

Since by the definition of M we have $M \subseteq G$ and $M \cap E = \emptyset$, we see that $B_r(x) \subseteq G$ and $B_r(x) \cap E = \emptyset$ proving the result. □

EXERCISE: Prove the converse of lemma 1, that is: If E is such that, for every non-empty open subset G of (X,d) there exists a point $x \in G$ and $r > 0$ such that $B_r(x) \subseteq G$ and $B_r(x) \cap E = \emptyset$, then E is a nowhere dense subset of (X,d).

[Hint: Assume E is not nowhere dense and consider $G = \text{int}\,\overline{E}$.]

Before proceeding to the next stage in the construction of Baire Categories we must

RECALL: Following Cantor, a set A is said to be *countable* (or *denumerable*) if there exists a one-to-one (1-1) function from A into the natural numbers - intuitively; "A has no more elements than there are natural numbers".

For example, the following sets are countable.

(a) *The set* $\mathbb{Z}$ *of all integers.*

Consider the function $f: \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -(2n+1) & \text{if } n < 0. \end{cases}$$

$\mathbb{Z}$	0,	−1,	1,	−2,	2,	−3,	3, ...
	↓	↓	↓	↓	↓	↓	↓
$\mathbb{N}$	0	1	2	3	4	5	6, ...

(b) *The set* $\mathbb{Q}$ *of rational numbers*, that is numbers which may be uniquely represented in the form $\frac{p}{q}$ where p and q are integers with $q > 0$ and p, q having no common factors. In this case, consider the function

$$f: \mathbb{Q} \to \mathbb{N}: \frac{p}{q} \to \begin{cases} 2^q\, 3^p & \text{if } p \geq 0 \\ 2^q\, 5^{-p} & \text{if } p < 0\,. \end{cases}$$

(Verify that f is 1-1.)

The following result is of subsequent importance.

PROPOSITION 3: *A finite union of countable sets is itself a countable set,* that is, if $A_1, A_2, \ldots, A_n$ are a finite number of sets each of which is countable, then $A = A_1 \cup A_2 \cup \ldots \cup A_n$ (also denoted by $\bigcup_{i=1}^{n} A_i$, or $\cup\{A_i : i = 1,2,\ldots,n\}$) is a countable set.

Proof. (Outline) A 1-1 function f from the union A into $\mathbb{N}$ may be defined as follows. For each $a \in A$

$$f(a) = 2^{i_a} 3^{f_{i_a}(a)},$$

where i_a is the smallest of the indices $1,2,3,\ldots,n$ for which $a \in A_{i_a}$ and f_{i_a} is a 1-1 function from A_{i_a} into $\mathbb{N}$, which exists since A_{i_a} is assumed to be countable. □

NOTE: Whenever a collection of objects (which may themselves be sets) form a countable set, we say that we have a countable "number" of objects. Any collection of objects which are arranged in a sequence $E_1, E_2, E_3, \ldots$ form a countable set (consider the function $f: E_n \mapsto n$). Conversely, if a collection of objects form a countable set A we can always arrange the objects into a sequence. Let f be a 1-1 function from A into $\mathbb{N}$, then take as the first object E_1 the element of A for which the value of f is smallest. For the second object E_2 select that element of A for which f assumes its second smallest value, etc.

Henceforth we will assume that any countable number of objects have been arranged into such a sequence.

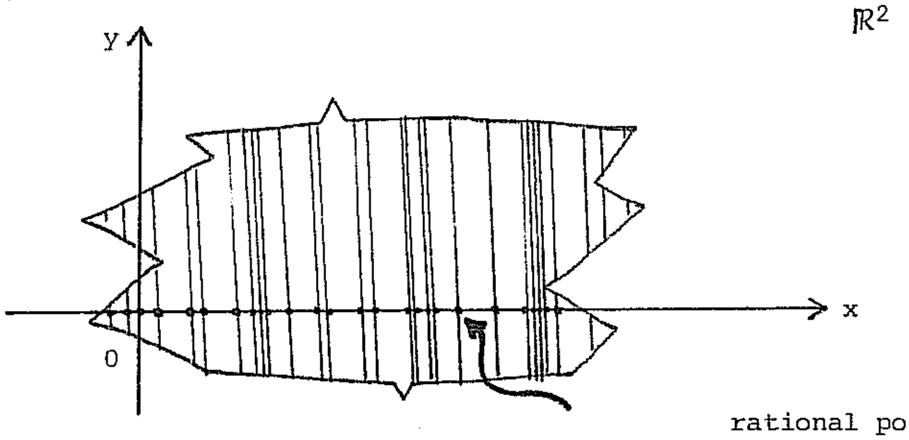
We are now ready to state

DEFINITION 4: A subset M of a metric space is <u>*meagre*</u> if it is the union of a countable number of nowhere dense sets. Thus M is meagre if and only if there exist nowhere dense sets $E_1, E_2, E_3, \ldots$ such that $M = E_1 \cup E_2 \cup E_3 \cup \ldots$, such a countable union will also be denoted by writing $M = \bigcup_{n=1}^{\infty} E_n$, $M = \bigcup_{n\in\mathbb{N}} E_n$ or $M = \cup\{E_n : n \in \mathbb{N}\}$.

EXAMPLES: 1. If $\left\{\frac{p}{q}\right\}$ denotes the set whose only element is the rational number $\frac{p}{q}$, then $\left\{\frac{p}{q}\right\}$ is nowhere dense in $\mathbb{R}$ with the usual metric. Since the set $\mathbb{Q}$ of rational numbers is countable we therefore have that $\mathbb{Q} = \bigcup_{\frac{p}{q}\in\mathbb{Q}} \left\{\frac{p}{q}\right\}$ is a countable union of nowhere dense sets and so conclude that *the set of rational numbers is a meagre subset of* $\mathbb{R}$ *with the usual metric.*

[Note: This example shows that while a meagre set is "small" it may nonetheless be dense. Thus, while a finite union of nowhere dense sets is readily seen to be itself nowhere dense, a countable union of such sets, by definition meagre, need not be nowhere dense.]

2. Since, in $\mathbb{R}^2$ with the euclidean metric, each line is a nowhere dense set, we see that the set of all "vertical" lines which intersect the x-axis in a rational point, being a countable union of nowhere dense sets, is itself a meagre set.



The next proposition is of importance for applications.

PROPOSITION 5: *The union of a finite* (indeed, countable) *number of meagre sets is itself a meagre set.*

Proof. Let $M = M_1 \cup M_2 \cup \ldots \cup M_n$ where each of the n sets $M_1, M_2, \ldots, M_n$ is meagre, that is for each $i = 1,2,\ldots,n$ there are a countable number of nowhere dense sets $E_{i1}, E_{i2}, \ldots, E_{in}, \ldots$ such that $M_i = \bigcup_{m=1}^{\infty} E_{im}$.

Now, let $\mathcal{E}$ denote the collection of all the E_{im} for $i = 1,2,\ldots,n$ and $m = 1,2,3,\ldots$. By proposition 3,

$$\mathcal{E} = \{E_{11},E_{12},E_{13},\ldots\} \cup \{E_{21},E_{22},E_{23},\ldots\} \cup \ldots \cup \{E_{n1},E_{n2},E_{n3},\ldots\}$$

being a finite union of countable sets is itself countable.

So,

$$M = \bigcup_{i=1}^{n} \bigcup_{m=1}^{\infty} E_{im} = \bigcup_{E\in\mathcal{E}} E$$

is a countable union of nowhere dense sets, and we conclude that M is a meagre set, as required. □

We are now in a position to give an important theorem due to Baire and Hausdorff.

THEOREM 6: *If* (X,d) *is a* <u>*complete*</u> *metric space and* M *is a meagre subset of* X, *then the complement* $X\backslash M$ *is a dense subset of* X.

In proving theorem 6 it is useful to extract the following lemma, which is of importance in its own right.

LEMMA 7 (Cantor's Intersection Theorem):

If (X,d) *is a complete metric space and* $(F_n)_{n=1}^{\infty}$ *is a nested sequence of non-empty closed sets whose diameters tend to zero* (that is, $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots \supseteq F_n \supseteq \dots$ and

$$\operatorname{diam} F_n \overset{\text{def}^n}{=\!=} \operatorname{Sup}\{d(x,y): x,y \in F_n\} \to 0 \quad \text{as} \quad n \to \infty),$$

then the intersection of the F_n's, $\bigcap_{n=1}^{\infty} F_n$, *contains exactly one point,* in particular it is non-empty.

Proof: For each $n \in \mathbb{N}$ choose an element x_n from F_n. Note, that since the sets are nested we have $x_n \in F_m$ for all $m \leqslant n$.

<u>We first show the sequence</u> (x_n) <u>is a Cauchy sequence</u>. To this end, given any $\varepsilon > 0$, since $\operatorname{diam} F_n \to 0$, we may choose an $N \in \mathbb{N}$ such that $\operatorname{diam} F_N < \varepsilon$. Now, for $n, m > N$ we have (by the above "note") that $x_n, x_m \in F_N$ and so $d(x_n,x_m) \leqslant \operatorname{diam} F_N < \varepsilon$. Thus (x_n) is a Cauchy sequence and so, since (X,d) is assumed complete, (x_n) converges to some point $x \in X$.

We next <u>establish that this limit x belongs to the intersection of the</u> F_n's. For any $n \in \mathbb{N}$ we have that the subsequence $x_n, x_{n+1}, x_{n+2}, \dots$ still converges to x and consists entirely of points in F_n. Since F_n is closed it follows that $x \in F_n$. Thus, $x \in F_n$ for each n and so $x \in \bigcap_{n=1}^{\infty} F_n$.

To see that <u>x is the only point in the intersection</u> note that if y is any point of the intersection, then both x and y belong to each of the sets F_n. Thus $0 \leqslant d(x,y) \leqslant \operatorname{diam} F_n$ for every n. Since, $\operatorname{diam} F_n \to 0$ we have that $d(x,y) = 0$ or $y = x$. □

EXERCISES: 1. Show that the conclusion of lemma 7 may fail to hold if either of the conditions

(a) each F_n is closed

(b) $\operatorname{diam} F_n \to 0$

is dropped, by giving, in each case, an example of a nested family of non-empty sets with empty intersection.

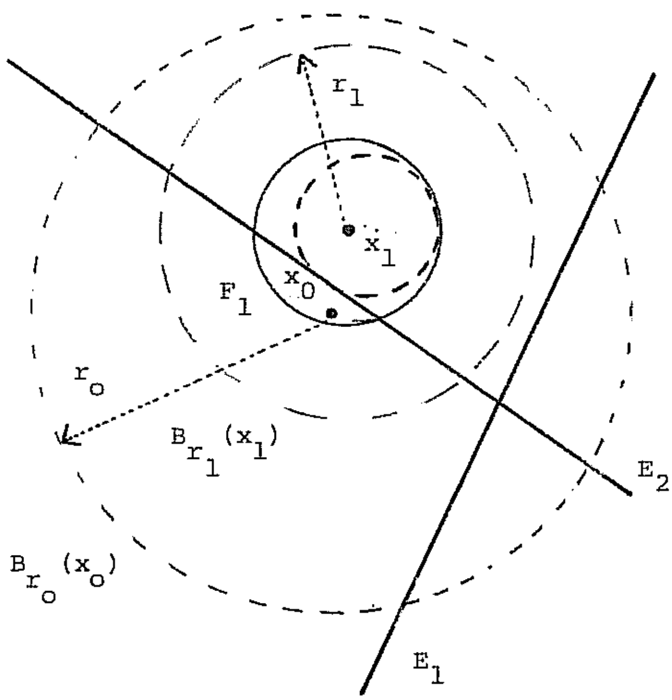
*2. Prove the converse of lemma 7, that is: If in (X,d) every

family of nested non-empty closed sets whose diameters tend to zero has exactly one point in its intersection, then (X,d) is complete.
[Hint: Given any Cauchy sequence (x_n) in (X,d), let $F_n = \overline{\{x_k : k \geqslant n\}}$.]

We now return to the

proof of theorem 6. To prove the density of $X\backslash M$ in X we show that for any $x_0 \in X$ and $r_0 > 0$, the intersection $B_{r_0}(x_0) \cap (X\backslash M)$ is non-empty. Since M is meagre, $M = \bigcup_{n=1}^{\infty} E_n$ for some collection $E_1, E_2, E_3, \ldots$ of nowhere dense sets.

By lemma 1 with $G = B_{r_0}(x_0)$, there exists x_1 and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq B_{r_0}(x_0)$ and $B_{r_1}(x_1) \cap E_1 = \emptyset$. Let F_1 be the closed ball $B_{r_1'}[x_1]$ where $r_1' = \text{Min}\{\tfrac{1}{2}r_1, \tfrac{1}{2}\}$, then F_1 is a non-emply closed set of diameter at most 1 such that $F_1 \subseteq B_{r_0}(x_0)$ and $F_1 \cap E_1 = \emptyset$.



Now apply lemma 1 with $G = B_{r_1'}(x_1)$, to obtain $x_2, r_2 > 0$ such that $B_{r_2}(x_2) \subseteq B_{r_1'}(x_1)$ and $B_{r_2}(x_2) \cap E_2 = \emptyset$. Form $F_2 = B_{r_2'}[x_2]$ where $r_2' = \text{Min}\{\tfrac{1}{2}r_2, \frac{1}{2^2}\}$, then F_2 is a closed non-empty set of diameter at most $\tfrac{1}{2}$ such that $F_2 \subseteq F_1$ and $F_2 \cap E_2 = \emptyset$.

Next apply lemma 1 with $G = B_{r_2'}(x_2)$, to obtain $x_3, r_3 > 0$ such that ... etc.

Continuing in this way we construct a sequence of closed sets $F_1, F_2, F_3, \ldots, F_n, \ldots$ such that $B_{r_0}(x_0) \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$, $\text{diam } F_n \leqslant \frac{1}{2^{n-1}}$ and $F_n \cap E_n = \emptyset$ for all n.

By lemma 7, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point, x say. Since for each n $x \in F_n$ we have that $x \in B_{r_0}(x_0)$ and $x \notin E_n$ (as $F_n \cap E_n = \emptyset$). Thus $x \in B_{r_0}(x_0) \cap (X\backslash M)$ and the proof is complete. □

Corollary 8 (Baire's Category Theorem): *The compliment of a meagre subset in a complete metric space is non-meagre.*

Proof. Let (X,d) be a complete metric space and M_1 a meagre subset of X. Assume that $M_2 = X\backslash M_1$ is also a meagre subset of X, then by proposition 5 we have $M = M_1 \cup M_2$ is a meagre subset of X. By Theorem 6 $X\backslash M$ is dense and so certainly not empty, but by definition

$$\begin{aligned} X\backslash M &= X\backslash (M_1 \cup M_2) \\ &= X\backslash (M_1 \cup (X\backslash M_1)) \\ &= X\backslash X = \emptyset, \end{aligned}$$

a contradiction, establishing that $X\backslash M_1$ must be non-meagre. □

Before presenting exercises we illustrate the use of the above result by considering one application the details of which are slightly more involved than those called for in the subsequent exercises.

3. Illustrative Application - *the existence of continuous nowhere differentiable functions*

We take as our metric space X the set of all continuous real valued functions on some non-empty closed bounded interval $[a,b]$ equipped with the uniform metric $d_\infty(f,g) \overset{\text{def}^n}{=\!=} \underset{a\leqslant x\leqslant b}{\text{Max}} |f(x) - g(x)|$.

That is, X denotes $(C[a,b], d_\infty)$. By the appendix to Part I of this course, we know that X is a complete metric space, so theorem 6 and corollary 8 apply in X.

For each $n \in \mathbb{N}$, let E_n be the set of all functions f in X for which there exists *some* point t_0 in $[a,b]$ such that

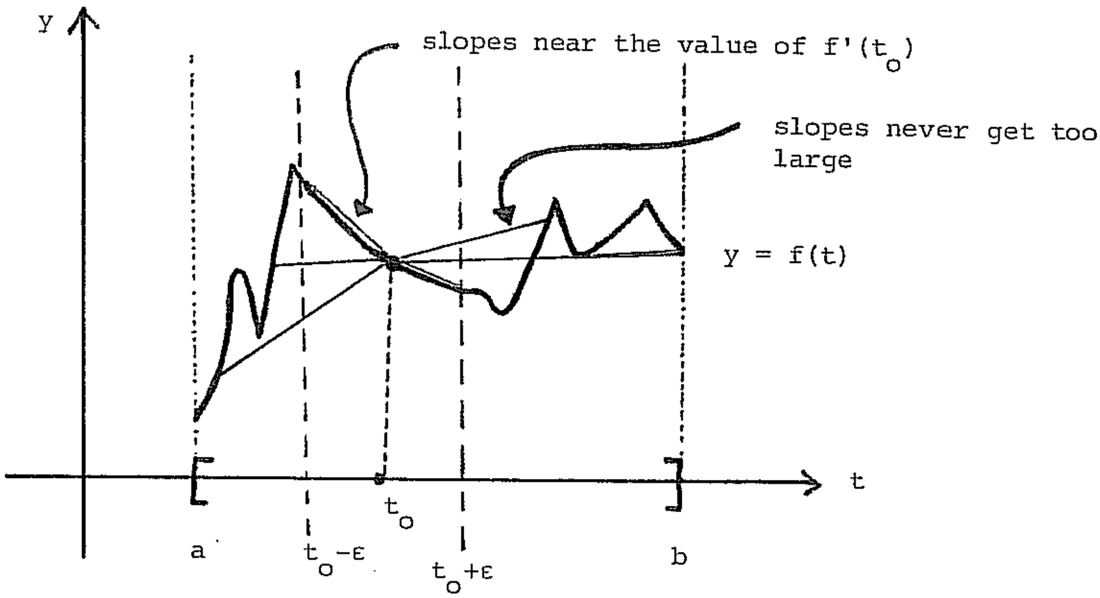
$$\left|\frac{f(t) - f(t_0)}{t - t_0}\right| \leqslant n$$

for all $t \in [a,b]$, $t \neq t_0$.

To see the relevance of the sets E_n to our problem, note that if f is differentiable at *some* point t_0 in $[a,b]$, that is $f'(t_0) = \underset{\substack{t\to t_0 \\ t\in[a,b]}}{\text{limit}} \frac{f(t)-f(t_0)}{t-t_0}$

exists, then, by the meaning of this limit, there exists a $\delta > 0$ such that $\left|\frac{f(t) - f(t_0)}{t - t_0} - f'(t_0)\right| < \frac{1}{2}$ for $t \in [a,b]$, $|t - t_0| < \delta$ and $t \neq t_0$. So for such t we have $\left|\frac{f(t) - f(t_0)}{t - t_0}\right| < \frac{1}{2} + |f'(t_0)|$, while for other $t \in [a,b]$ we have $|t - t_0| \geqslant \delta$ and so for these t,

$$\left|\frac{f(t) - f(t_0)}{t - t_0}\right| \le \frac{|f(t) - f(t_0)|}{\delta} \le \frac{f\ max - f\ min}{\delta}$$



where f_{max} and f_{min} are respectively the maximum and minimum value of f on [a,b] (which exist since f is continuous and [a,b] is compact).

Therefore, if f is differentiable at $t_0 \in [a,b]$ we have for all $t \in [a,b]$, $t \neq t_0$ that

$$\left|\frac{f(t) - f(t_0)}{t - t_0}\right| \le \text{Max}\left\{\tfrac{1}{2} + |f'(t_0)|\ ,\ \frac{f_{max} - f_{min}}{\delta}\right\}$$

and so $f \in E_n$ for some sufficiently large n.

Thus $D = \bigcup_{n=1}^{\infty} E_n$ contains all functions in X which are differentiable at at least one point of [a,b].

The proof is completed by showing that each E_n is nowhere dense, and so D is meagre. To establish that E_n is nowhere dense we first prove that E_n is closed so $\bar{E}_n = E_n$, and then show int $E_n = \phi$.

a) E_n is Closed.

Let f_m be a sequence of functions in E_n which converge to f. For each f_m there exists a point $t_m \in [a,b]$ such that $\left|\frac{f_m(t) - f_m(t_m)}{t - t_m}\right| \le n$ for all $t \in [a,b]$, $t \neq t_m$. Since [a,b] is *compact*, there exists a subsequence t_{m_k} convergent to some point t_o of [a,b].

Further, for each $k \in \mathbb{N}$ we have

$$\left|\frac{f(t) - f(t_{m_k})}{t - t_{m_k}}\right| = \left|\frac{f_{m_k}(t) - f_{m_k}(t_{m_k}) + (f(t)-f_{m_k}(t)) - (f(t_{m_k}) - f_{m_k}(t_{m_k}))}{t - t_{m_k}}\right|$$

$$\leq \left|\frac{f_{m_k}(t) - f_{m_k}(t_{m_k})}{t - t_{m_k}}\right| + \left|\frac{f(t) - f_{m_k}(t)}{t - t_{m_k}}\right| + \left|\frac{f(t_{m_k}) - f_{m_k}(t_{m_k})}{t - t_{m_k}}\right|$$

$$\leq n + \frac{2\, d_\infty(f, f_{m_k})}{|t - t_{m_k}|}$$

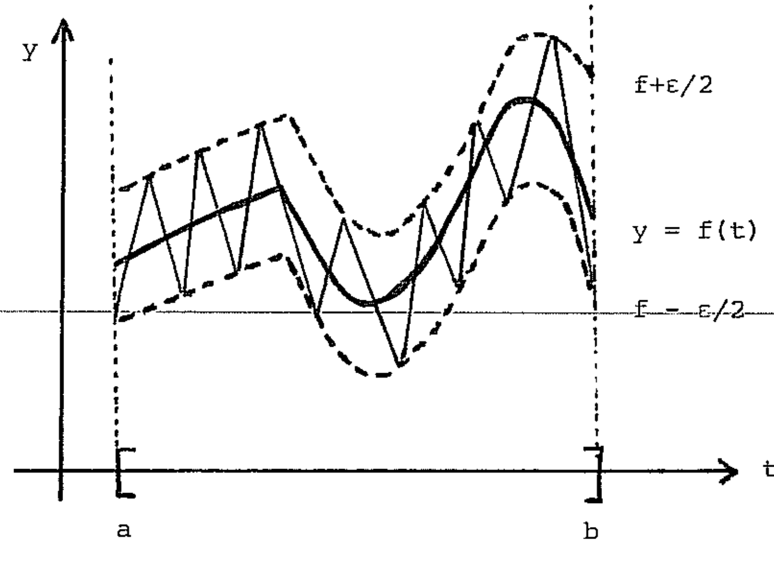
Now, $f(t_{m_k}) \to f(t_0)$, as f is continuous, and $d_\infty(f, f_{m_k}) \to 0$, so for $t \neq t_0$, taking the limit as $k \to \infty$, we have

$$\left|\frac{f(t) - f(t_0)}{t - t_0}\right| \leq n + 0$$

and conclude that $f \in E_n$.

b) Int $E_n = \phi$. For *any* $f \in E_n$ and $\varepsilon > 0$ it suffices to show that the ball $B_\varepsilon(f)$ contains a function g not in E_n (for then, f is not an interior point of E_n). To see that this is so consider the function obtained by inscribing in the strip $\{(x,y): f(x)-\varepsilon/2 \leq y \leq f(x) + \varepsilon/2\}$ a zigzagging graph consisting of straight line segments of slopes greater than $2n$ or less than $-2n$.

[Remark: that this construction for g is possible follows from the fact that the set of "step functions" is dense in $(C[a,b], d_\infty)$ - see elsewhere, or alterentively it may be proved directly using arguments similar to those used to establish this last result.]



EXERCISES

1. (a) Using the completeness of $\mathbb{R}$ with the usual metric, deduce that the set of irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ is non-meagre. [This provides a pure existence proof of the existence of irrational numbers, and should be constrasted with the constructive proof found in most school books - what is this proof?]

   (b) In $\mathbb{R}$ with the usual metric, show that any countable subset of real numbers is a meagre set. [Hint: refer to the proof that $\mathbb{Q}$ is a meagre subset of $\mathbb{R}$.]

   (c) Cantor defined a set to be *uncountable* if it is not countable. One of Cantor's major achievements in the theory of "infinite sets" was to establish the existence of uncountable sets. In particular he proved that the set of all real numbers $\mathbb{R}$ and the set of irrational numbers are both uncountable. Deduce this last pair of results.

   (d) (Optional). A real number is said to be algebraic if it is the root of some polynomial with integer coefficients.
   For example: any rational number p/q is the root of $qx - p = 0$ and so is algebraic; $\sqrt{2}$ is a root of $x^2 = 2$, so $\sqrt{2}$ is algebraic, though it is not rational. A number which is not algebraic is said to be transcendental. Let A_m denote the set of algebraic numbers arising as the root of a polynomial of degree less than or equal to m with integer coefficients each of absolute value less than or equal to m.

   i) Show that A_m is finite. Indeed there are at most $(2m+1)^{m+1}$ such polynomials each of which can have at most m roots, so $\#(A_m) \le m(2m+1)^{m+1}$]

   ii) Show that $A = A_1 \cup A_2 \cup A_3 \cup \ldots$, the set of all algebraic numbers is a meagre subset of $\mathbb{R}$ with the usual metric [indeed A is a countable set].

   iii) Deduce that the transcendental numbers are a non-meagre (and hence uncountable) subset of $\mathbb{R}$ with the usual metric and so conclude that transcendental numbers exist.
   [The proofs that certain numbers are transcendental. e.g. e(Hermite, 1873) and π(Lindemann 1882) represented major advances in the theory of numbers. Today it is still unknown whether Euler's constant $\gamma = \lim_{n\to\infty} [(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}) - \ln n]$, or $\pi + e$ are transcendental. Indeed, even their irrationality has not been proved.]

2. (a) In a metric space (X,d) show that a subset $A \subseteq X$ is nowhere dense if and only if the complement of its closure, $X \backslash \bar{A}$ is dense.

   (b) If G_1, G_2, G_3, ..., are a countable number of dense open sets in the complete metric space (X,d) show that their intersection $A = G_1 \cap G_2 \cap G_3 \cap \ldots$ is a dense subset of X.

   [A set such as A above which is the intersection of a countable number of open sets is termed a G_δ -set. *Show that the intersection of two dense G_δ- subsets of a complete metric space is itself a dense G_δ- subset. Give an example to show that if the assumption that the sets are G_δ- sets is dropped then the result need not be valid. That is show that the intersection of two dense subsets need not itself be dense.]

3. (a) If a complete metric space (X,d) is the union of a countable number of closed subsets (that is, $X = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed set), show that one of the subsets has a non-empty interior.

   *(b) Let $\mathcal{F}$ be a "point-wise bounded" family of functions in $C[a,b]$: that is, for each point $t \in [a,b]$ there exists a constant M_t such that $|f(t)| \le M_t$ for all $f \in \mathcal{F}$. Prove that there is a subinterval $[c,d]$ of $[a,b]$ with $c < d$ on which the family of functions is unformly bounded; that is, there exists a constant M (independent of t) such that $|f(t)| \le M$ for all $t \in [c,d]$.

   [Hint: Let $F_n = \{t \in [a,b]: |f(t)| \le n \text{ for all } f \in T\}$.]

*4. Prove the following result.

If $f: [0,1] \to \mathbb{R}$ is continuous on a dense set of points J, then the set of points in $[0,1]$ at which f is discontinuous must form a meagre set.

[Hint: Note that the set of points of discontinuity for f equals $\bigcup_{n=1}^{\infty} E_n$, where E_n is the set of points at which the oscillation of f is at least $\frac{1}{n}$, that is $t \in E_n$ if and only if there exists a sequence of points (t_m) in $[0,1]$ with $t_m \to t$ and $|f(t_m) - f(t)| \ge \frac{1}{n}$ for all m.]

[Remark: As a consequence of this and exercise 1 above it is impossible to have a real valued function which is continuous at the rational points in $[0,1]$, but discontinuous at all irrational points. The result does not of course preclude the possibility of a function which is continuous at the irrational points but discontinuous at every rational point. Indeed such functions do exist: for example, the so called "ruler function"

$$f(t) = \begin{cases} 0 \text{ if } t \text{ is irrational} \\ \frac{1}{q} \text{ if } t = \frac{p}{q} \text{ where } p,\ q \in \mathbb{N} \text{ have no common factors.} \end{cases}$$

(Can you see why this function might be termed the "ruler function"? *Can you establish the continuity properties claimed for it?)]

5. A metric space is *compact* if every sequence of points in X has a convergent subsequence.

   (a) If $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ is a nested family of non-empty closed subsets in a compact metric space, show that $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

   (b) Let (X,d) be a compact metric space and $A_1, A_2, A_3, \ldots$ be a family of closed sets with the "finite intersection property:" that is, the intersection of any finite number of the A_n's is non-empty, show that $\bigcap_{n=1}^{\infty} A_n \neq \phi$.

   [Hint: Let $F_N = \bigcap_{n=1}^{N} A_n$.]

   [Remark: The converse of (b) is also true. That is, if every family of closed sets with the finite intersection property has a non-empty intersection, then the space is compact. *Can you prove this?]

6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions and suppose $f_n(x) \to f(x)$ (as $n \to \infty$) for each $x \in \mathbb{R}$ (i.e., f_n converges pointwise to f). Prove that f is continuous except at a meagre set of points in $\mathbb{R}$.

   [Hint: Define the countable family of sets

$$F_{m,n} = \{x \in \mathbb{R} : |f_n(n) - f_k(x)| < \frac{1}{m},\ \text{for all } k \geq n \quad (m,\ n \in N)$$

   and consider

$$M = \bigcup_{m,n \in N} (F_{m,n} \setminus \text{int } F_{m,n})$$
$$= R \setminus \bigcap_{m,n} \text{int } F_{m,n}.]$$

An excellent account of some of the applications of Baire's Category theory to the theory of real functions may be found in: Boas, R.P. Jr. "A Primer of Real Functions", Wiley.

SECTION 4. *INTEGRATION THEORY - UTILITY GRADE*

As a preliminary to our main business: defining an integral for a suitably large class of functions; we study the notion of "uniform continuity" in metric spaces.

§1. *Uniform Continuity*

DEFINITION: Let $(X,d),(Y,d')$ be two metric spaces. The mapping $f\colon X \to Y$ is uniformly continuous on $A \subseteq X$ if, given $\epsilon > 0$ there exists $\delta > 0$ such that $d'(f(x_1),f(x_2)) < \varepsilon$ whenever $x_1, x_2 \in A$ and $d(x_1, x_2) < \delta$.
Clearly, this is equivalent to requiring

$$f(A \cap B_\delta(x)) \subseteq B_\varepsilon(f(x)) \quad \text{for all } x \in A.$$

This definition should be contrasted with that for the (global) continuity of f:

f is *continuous* on A if, given $\varepsilon > 0$, for each $x_0 \in A$ there exists a $\delta_0 > 0$ [here the value of δ_0 may vary when the point x_0 is changed; that is, $\delta_0 \equiv \delta_0(x_0)$] such that

$$d'(f(x),f(x_0)) < \varepsilon \text{ whenever } x \in A \text{ and } d(x, x_0) < \delta_0,$$

or equivalently

$$f(A \cap B_{\delta_0}(x_0)) \subseteq B_\varepsilon(f(x_0)).$$

Given the $\varepsilon > 0$, in continuity, for each point x_0 there is a $\delta_0 \equiv \delta_0(x_0)$ which 'works', on the other hand, in uniform continuity there is one δ which 'works' for every x. This last requirement is equivalent to having

$$(\delta =) \inf_{x_0 \in A}\text{emum}\ \delta_0(x_0) \gneq 0.$$

From the definitions it should be clear that

(f *uniformly continuous* on A) $\Rightarrow$ (f *continuous* on A)

The converse of this is false as the following example shows.

EXAMPLE: Let (X,d) be the open interval $(0,1)$ with metric $d(x_1, x_2) = |x_1 - x_2|$ and let (Y,d) be the real numbers $\mathbb{R}$ with the same metric $d(y_1, y_2) = |y_1 - y_2|$. Then, the function $f\colon X \to Y\colon x \mapsto {}^1/_x$ is continuous on X (indeed f is differentiable at every point of $(0,1)$). f is not however uniformly continuous on X. To prove this, assume f were uniformly continuous; that is, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x_1 - x_2| < \delta \Rightarrow \left|\frac{1}{x_1} - \frac{1}{x_2}\right| < \varepsilon$$

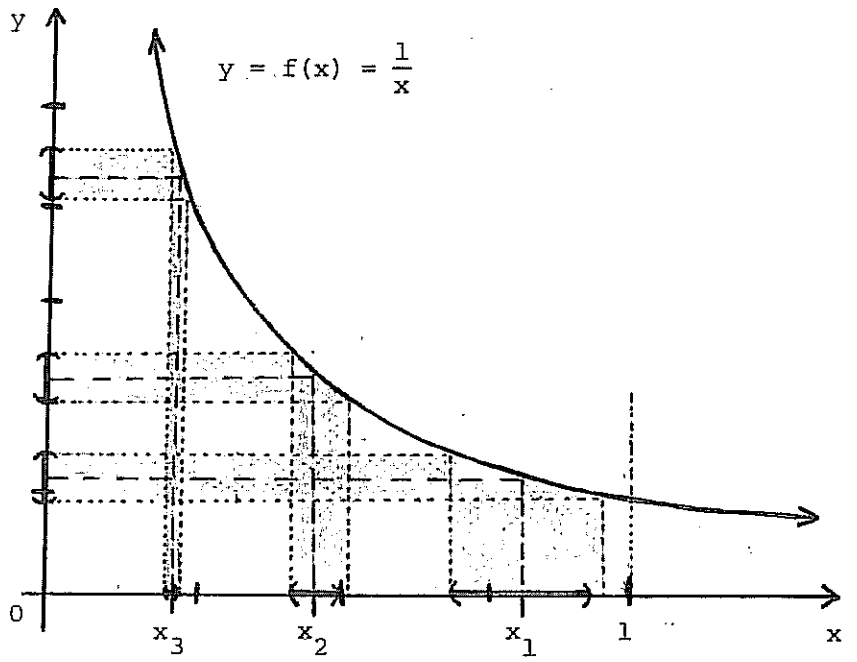
~~Now choose,~~

$$x_1 = \text{Min}\ \{\delta, \frac{1}{2\varepsilon}\} \quad \text{and let } x_2 = \tfrac{1}{2}x_1, \quad \text{then}$$

$|x_1 - x_2| = \frac{1}{2}x_1 \leq \frac{\delta}{2} < \delta,$ but

$$\left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \frac{1}{2x_1} \geq 2\varepsilon \not< \varepsilon.$$

A contradiction, showing that f cannot be uniformly continuous.



Given a $\varepsilon > 0$, the δ's applicable at $x_1, x_2, x_3, \ldots$ grow progressively smaller (indeed tend to 0) as $x_n \to 0$.

Sequential Characterization of Uniform Continuity

Recall that the continuity of $f: A \subseteq (X,d) \to (Y,d')$ may be sequentially characterized as follows. f is continuous if and only if for each $x_o \in A$ whenever $(x_n) \subseteq A$ and $x_n \overset{d}{\to} x_o$ we have $f(x_n) \overset{d'}{\to} f(x_o)$.

LEMMA 1: $f: A \subseteq (X,d) \to (Y,d')$ *is uniformly continuous on* A *if and only if whenever* (x_n) *and* (t_n) *are two sequences in* A *such that* $d(x_n,t_n) \to 0$ *as* $n \to \infty$ *we have* $d'(f(x_n),f(t_n)) \to 0$ *as* $n \to \infty$.

Proof ($\Rightarrow$) Given $\varepsilon > 0$ let δ be such that, for $x, t \in A$,

$$d(x,t) < \delta \Rightarrow d'(f(x),f(t)) < \varepsilon \quad \text{(definition of uniform continuity)}$$

and let N be such that

~~$d(x_n,t_n) < \delta$ whenever $n > N$ (possible since $d(x_n,t_n) \to 0$)~~

then for $n > N$ we have $d'(f(x_n),f(t_n)) < \varepsilon$ and so $d'(f(x_n),f(t_n)) \to 0$ as $n \to \infty$.

($\Leftarrow$) We prove the logically equivalent *contrapositive*; that is, if f is not uniformly continuous on A, then there exists sequences (x_n) and (t_n) in A such that $d(x_n,t_n) \to 0$ but

$$d(f(x_n),f(t_n)) \not\to 0.$$

Now if f is not uniformly continuous then there is an $\varepsilon_0 > 0$ such that for each $\delta > 0$ there exists a pair of points x_δ, t_δ in A with $d(x_\delta,t_\delta) < \delta$ but $d'(f(x_\delta),f(t_\delta)) \geq \varepsilon_0$.

Thus for each $n \in \mathbb{N}$, by taking $\delta = \frac{1}{n}$, we have a pair of points x_n,t_n such that $d(x_n,t_n) < \frac{1}{n}$ but $d'(f(x_n),f(t_n)) \geq \varepsilon_0$.

The sequences (x_n) and (t_n) arrived at in this way are such that

$$d(x_n,t_n) < \frac{1}{n}, \quad \text{so} \quad d(x_n,t_n) \to 0$$

while

$$d'(f(x_n),f(t_n)) \geq \varepsilon_0 > 0 \quad \text{and so} \quad d'(f(x_n),f(t_n)) \not\to 0. \qquad \blacksquare$$

An argument similar to ($\Rightarrow$) above yields:

<u>LEMMA 2</u>: *If* $f: A \subseteq (X,d) \to (Y,d')$ *is uniformly continuous on* A *and* (x_n) *is a Cauchy sequence in* A *then* $(f(x_n))$ *is a Cauchy sequence in* Y. That is a uniformly continuous function maps Cauchy sequences to Cauchy sequences.

Proof: Given $\varepsilon > 0$ let $\delta > 0$ be such that, for $x, t \in A$,

$$d(x,t) < \delta \Rightarrow d'(f(x),f(t)) < \varepsilon \qquad \text{(definition of uniform continuity)}$$

and let $N \in \mathbb{N}$ be such that $n,m > N \Rightarrow d(x_n,x_m) < \delta$ (possible since (x_n) is a Cauchy sequence).

Then for $n,m > N$ we have

$$d'(f(x_n),f(x_m)) < \varepsilon$$

and so $(f(x_n))$ is a Cauchy sequence. $\blacksquare$

REMARK: The sequential characterization of continuity shows that a continuous function maps convergent sequences to convergent sequence, it need not however, preserve Cauchy sequences. <u>For example</u>, taking X,Y and f as in the previous example $(f(x) = \frac{1}{x})$ we see that, if $x_n = \frac{1}{n}$, then (x_n) is a Cauchy sequence in (0,1) however $(f(x_n))$ is not a Cauchy sequence, indeed $f(x_n) = n$.

<u>RECALL: A subset A of the metric space (X,d) is dense if $\bar{A} = X$; that is,</u> if every point of X is a limit point of A, or equivalently for every $x \in X$ and $\varepsilon > 0$ there exists $a \in A$ with $\|x - a\| < \varepsilon$.

THEOREM 3 (EXTENSION THEOREM)

Let (X,d) *be a metric space and let* (Y,d') *be a* <u>*complete*</u> *metric space.*

If A *is a dense subset of* X *and* $f: A \to Y$ *is a uniformly continuous function on* A, *then there exists a unique function* $\tilde{f}: X \to Y$ *which satisfies:*

i) $\tilde{f}$ *is uniformly continuous on* X

and

ii) $\tilde{f}(a) = f(a)$ *for all* $a \in A$ (that is, $\tilde{f}$ is an *extension* of f to X).

Proof. Each $x \in X$ is a limit point of A, so there exists a sequence (a_n) in A with $a_n \to x$. Since (a_n) is convergent it is a Cauchy sequence and so, by Lemma 2, $(f(a_n))$ is a Cauchy sequence in Y. Now Y is complete, so $(f(a_n))$ converges to some point in Y. Thus we may define $\tilde{f}$ by

$$\tilde{f}(x) = \lim_{n\to\infty} f(a_n), \quad \text{where } a_n \overset{d}{\to} x$$

We must show

<u>$\tilde{f}$ is well defined</u> - the value of $\tilde{f}$ at x is independent of the particular choice of sequence $a_n \to x$. That is, if $a_n \to x$ and $b_n \to x$, where (a_n) and (b_n) are both sequences in A, then

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$$

<u>Note</u>, once this is established, it will also follow that $\tilde{f}$ is an extension of f, since, for $a \in A$ the constant sequence $a, a, a, \ldots.$ converges to a so $\tilde{f}(a) = \lim_{n\to\infty} f(a) = f(a)$.

Now, let (a_n), (b_n) be sequences in A with $a_n \to x$ and $b_n \to x$, then, since $d(a_n,b_n) \le d(a_n,x) + d(x,b_n)$, we have $d(a_n,b_n) \to 0$ and so, by lemma 1, $d'(f(a_n),f(b_n)) \to 0$.

But then,

$$0 \le d'(\lim_{n\to\infty} f(a_n), \lim_{n\to\infty} f(b_n))$$

$$\le d'(\lim_{n\to\infty} f(a_n), f(a_m)) + d'(f(a_m),f(b_m)) + d'(f(b_m), \lim_{n\to\infty} f(b_n))$$

$$\to 0 \quad \text{as } m \to \infty$$

and so, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$ as required.

$\tilde{f}$ is uniformly continuous - Given any $\varepsilon > 0$, since f (the restirction of $\tilde{f}$ to A) is uniformly continuous on A, there exists $\delta > 0$ such that

$$d'(f(a_1),f(a_2)) < \frac{\varepsilon}{3} \quad \text{whenever } a_1,a_2 \in A \text{ and } d(a_1,a_2) < \delta.$$

Now, for any $x,t \in X$ with $d(x,t) < \frac{\delta}{2}$ there exists $a_1,a_2 \in A$ such that $d(a_1,x) < \frac{\delta}{4}$ and $d(a_2,t) < \frac{\delta}{4}$ (density of A) and also $d'(f(a_1),\tilde{f}(x))$ and $d'(f(a_2),\tilde{f}(t))$ are less than $\frac{\varepsilon}{3}$ (definition of f). Note this also implies that

$$\begin{aligned} d(a_1, a_2) &\le d(a_1,x) + d(x,t) + d(t,a_2) \\ &\le \frac{\delta}{4} + \frac{\delta}{2} + \frac{\delta}{4} \\ &= \delta \end{aligned}$$

and so $d'(\dot{f}(a_1),f(a_2)) < \frac{\varepsilon}{3}$.

Thus, for any $x,t \in X$ with $d(x,t) < \frac{\delta}{2}$ we have

$$\begin{aligned} d'(\tilde{f}(x), \tilde{f}(t)) &\le d'(\tilde{f}(x),\tilde{f}(a_1)) + d'(\tilde{f}(a_1),\tilde{f}(a_2)) + d'(\tilde{f}(a_2),\tilde{f}(t)) \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

so f is uniformly continuous.

$\tilde{f}$ is unique - assume $g: X \to Y$ is uniformly continuous and $g(x) = f(x)$ all $x \in A$. Then $g(x) = \tilde{f}(x)$ for all $x \in A$. Now for $x \in X\backslash A$, let $x_n \to x$ where $x_n \in A$ (all n), then

$$\begin{aligned} d'(\tilde{f}(x),g(x)) &\le d'(\tilde{f}(x),f(x_n)) + d'(f(x_n),g(x)) \\ &= d'(\tilde{f}(x),f(x_n)) + d'(g(x_n),g(x)) \\ &\to 0 \text{ (by the definition of } \tilde{f} \text{ and the uniform continuity of } g). \end{aligned}$$

Hence $\tilde{f}(x) = g(x)$ all $x \in X$ and so $\tilde{f}$ is unique. ∎

APPLICATION: In the elementary theory of real valued functions of a real variable it is often natural to construct a function first on $\mathbb{Q}$ the set of rational numbers (equipped with the metric $d(x,t) = |x - t|$) and then "extend" it to the whole real line. The above theorem guarantees this can be done provided the function constructed on $\mathbb{Q}$ is uniformly continuous.

EXAMPLE: Using the "laws of indices" we can define the function $x \mapsto a^x$ (for fixed $a \in [0,\infty)$) and all $x \in \mathbb{Q} \cap [0,1]$. An application of the theorem allows us to extend the domain of definition to $[0,1]$. In this way the function exp may be obtained $(a=e)$.

EXERCISE: Let $\mathbb{R}$ be equipped with the usual metric $d(x,t) = |x - t|$. Show that there is a unique continuous function

$f\colon \mathbb{R} \to \mathbb{R}$ satisfying the <u>functional equation</u>

$f(x + y) = f(x) + f(y)$ with $f(1) = 1$

The final two results, developed below, provide examples of uniformly continuous functions.

THEOREM 4 (HEINE'S THEOREM)

Let A *be a compact subset of the metric space* (X,d) - that is, every sequence of elements of A has a subsequence which converges to an element of A - *and let* $f\colon A \to (Y,d')$ *be continuous on* A. *Then,* f *is uniformly continuous.*

PROOF: Assume f is not uniformly continuous on A, then there exist sequences (x_n) and (t_n) in A such that $d(x_n,t_n) \to 0$ but $d'(f(x_n),f(t_n)) \geq \varepsilon_0$, for some $\varepsilon_0 > 0$ (see proof of lemma 1 ($\Leftarrow$)). Now A is compact so there exists a subsequence (x_{n_k}) convergent to some $x \in A$. Then, for the subsequence (t_{n_k}) we have

$$\begin{aligned} d(t_{n_k},x) &\leq d(t_{n_k},x_{n_k}) + d(x_{n_k},x) \\ &\to 0 \end{aligned}$$

so $t_{n_k} \to x$ also.

From the continuity of f we therefore have $f(x_{n_k}) \to f(x)$ and $f(t_{n_k}) \to f(x)$ and so

$$\begin{aligned} d'(f(x_{n_k}),f(t_{n_k})) &\leq d'(f(x_{n_k}),f(x)) + d'(f(x),f(t_{n_k})) \\ &\to 0 \end{aligned}$$

contradicting the fact that $d'(f(x_{n_k}),f(t_{n_k})) \geq \varepsilon_0 > 0$ for all k. ∎

Since for $-\infty < a < b < \infty$, the closed interval $[a,b]$ is compact we have:-

COROLLARY 5: *Every element (function) in* $C[a,b]$ *is uniformly continuous on* $[a,b]$

For the remainder of this section $(X,\|\cdot\|)$ and $(Y,\|\cdot\|')$ denote two normed linear spaces.

RECALL: A function $T\colon X \to Y$ is <u>linear</u> if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x,y \in X$ and all real numbers α,β.

We first prove

LEMMA 6: *For a linear mapping* $T\colon (X,\|\cdot\|) \to (Y,\|\cdot\|')$ the following are equivalent:

i) T *is continuous;*

ii) T *is continuous at the origin* 0;

iii) T *is bounded; that is, there exists some* $M > 0$ *such that* $\|Tx\|' \leq M\|x\|$ *for all* $x \in X$.

PROOF: That i) ⇒ ii) is immediate from the definitions. To see that ii) ⇒ iii), note that since T is continuous at 0. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|T(x) - T(0)\|' < \varepsilon \text{ whenever } \|x - 0\| < \delta .$$

Now, for a linear mapping $T(0) = 0$ and so the last line becomes

$$\|T(x)\|' < \varepsilon \text{ whenever } \|x\| < \delta.$$

Observing that for any $x \in X$ $(x \neq 0)$ $\frac{\delta}{2\|x\|} x$ has norm less than δ, we therefore have

$$\|T(\frac{\delta}{2\|x\|} x)\|' < \varepsilon \quad \text{for all } x \in X$$

or using the linearity of T and the properties of a norm,

$$\|Tx\|' \leq \frac{2\varepsilon}{\delta} \|x\|.$$

Thus T is bounded with $M = {}^{2\varepsilon}/_{\delta}$. The proof is completed by showing iii) ⇒ i). Now, if $x_n \to x$, then we have

$$\begin{aligned} \|T(x_n) - T(x)\|' &= \|T(x_n - x)\|' \\ &\leq M\|x_n - x\| \text{ (by iii)} \\ &\to 0 \end{aligned}$$

and so $T(x_n) \to T(x)$ establishing i). ∎

As a consequence we have

THEOREM 7: *If* $T:(X,\|\cdot\|) \to (Y,\|\cdot\|')$ *is a continuous (bounded) linear mapping, then* T *is uniformly continuous on* X.

PROOF: Let $M > 0$ be such that

$$\|T(x)\|' \leq M\|x\| \quad \text{for all } x \in X,$$

then given any $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{M}$ we have

$$\begin{aligned} \|T(x) - T(y)\|' &= \|T(x - y)\|' \\ &\leq M \|x - y\| \\ &< \varepsilon \text{ whenever } x,y \in X \text{ are such that} \end{aligned}$$

$\|x - y\| < \delta$, and so T is uniformly continuous. ∎

Combining Theorem 7 with Theorem 3 we have: -

LEMMA 8: *If* A *is a* <u>*dense subspace*</u> *of the normed linear space* $(X,\|\cdot\|)$ *and* $T: A \to (Y,\|\cdot\|')$ *is a linear mapping into the* <u>*complete*</u> *normed linear space* Y *then there is a unique extension* $\tilde{T}$ *of* T *to* X. *Further* $\tilde{T}$ *is also a linear mapping.*

PROOF: It is only the last remark that needs proof. To see that $\tilde{T}$ is linear, note that, if $x,y \in X$ there exists sequences (x_n), (y_n) in A with $x_n \to x$ and $y_n \to y$ (density of A) and then by the definition of $\tilde{T}$

$$\begin{aligned}
\tilde{T}(x+y) &= \lim_{n\to\infty} \tilde{T}(x_n + y_n) \quad (\text{as } x_n + y_n \to x + y) \\
&= \lim_{n\to\infty} T(x_n + y_n) \quad (\text{as } x_n + y_n \in A, \text{ since A is a subspace}) \\
&= \lim_{n\to\infty} (T(x_n) + T(y_n)) \quad (\text{as T is linear}) \\
&= \lim_{n\to\infty} T(x_n) + \lim_{n\to\infty} T(y_n) \\
&= \tilde{T}(x) + \tilde{T}(y).
\end{aligned}$$

■

§2 *Integration* on [a,b]

Throughout this final section a,b will denote a fixed pair of real numbers with $a < b$. I, J, I_1, I_2 etc. will denote intervals (closed, open, half-open) of real numbers contained in [a,b]. The "end-points" of I, for example, will be denoted by i, i^* and we will always assume that $a \le i \le i^* \le b$.
Thus I may be any one of the following

$$\begin{aligned}
(i,i^*) &= \{x: i < x < i^*\} \\
[i,i^*] &= \{x: i \le x \le i^*\} \\
[i,i^*) &= \{x: i \le x < i^*\} \\
(i,i^*] &= \{x: i < x \le i^*\}.
\end{aligned}$$

$F[a,b]$ will denote the set of all real valued functions defined on [a,b].

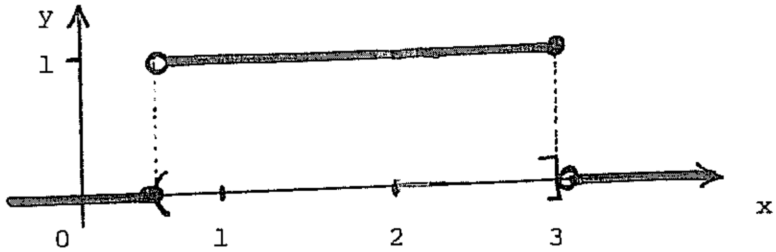
RECALL (Pure Mathematics 2, Linear Algebra), $F[a,b]$ is a vector (or linear) space with point-wise definitions of "vector" (function) addition and scalar multiplication.

A particularly simple type of function in $F[a,b]$ is the <u>characteristic function</u> of the interval $I(\subseteq [a,b])$, defined by

$$\chi_I(x) = \begin{cases} 1 \text{ if } x \in I \\ 0 \text{ if } x \notin I \end{cases}$$

Thus, for example, if $I = (\tfrac{1}{2},3]$ we have

$$\chi_I(x) = \begin{cases} 1 \text{ if } \tfrac{1}{2} < x \le 3 \\ 0 \text{ if } x \le \tfrac{1}{2} \text{ or } x > 3 \end{cases}$$



$\chi_{(\frac{1}{2},3]}$

The *subspace* of $F[a,b]$ spanned by the set of characteristic functions of intervals in $[a,b]$ is denoted by $St[a,b]$.

The elements of $St[a,b]$ are termed <u>step functions</u> and consist of (finite) linear combinations of characterictic functions of intervals in $[a,b]$.

Thus, s is a step function if and only if

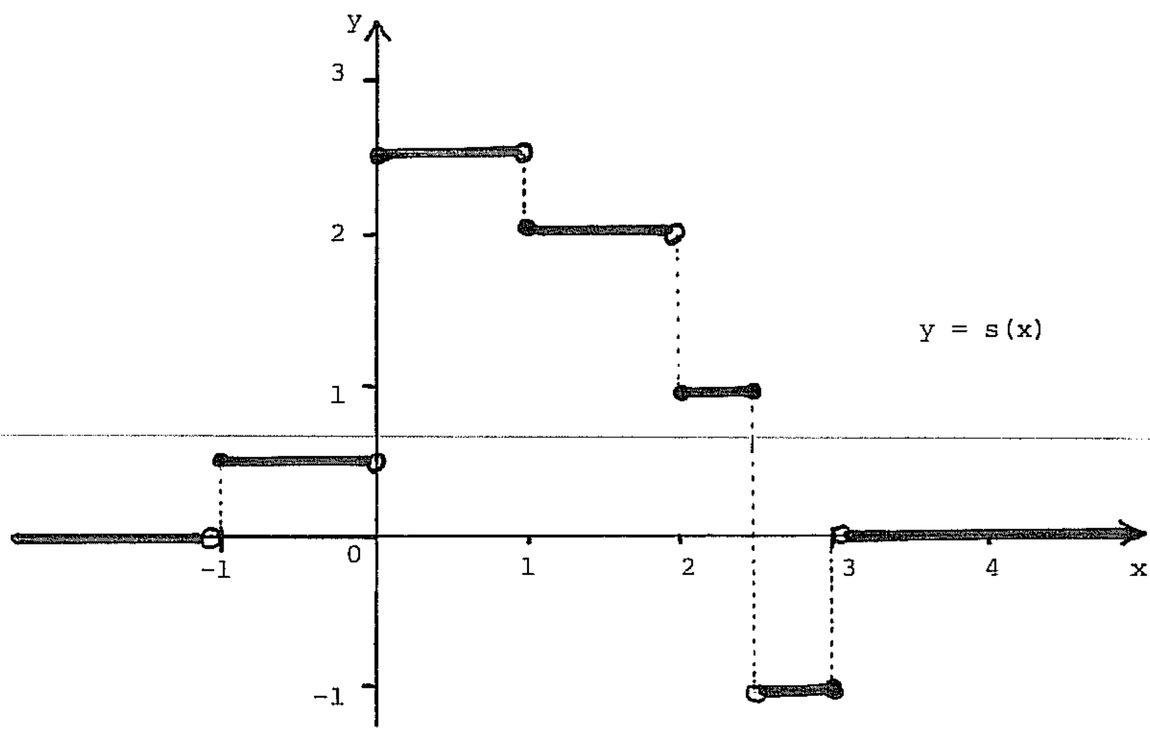
$$s = \sum_{k=1}^{n} s_k \chi_{I_k}$$

where $n \in \mathbb{N}$; $s_1, s_2, \ldots, s_n$ are real numbers and $I_1, I_2, \ldots, I_n$ are intervals (in $[a,b]$). For example,

$$s = \tfrac{1}{2}\chi_{[-1,1)} + 2\chi_{[0,2\frac{1}{2}]} - \chi_{[2,3]}$$

is a step function and we see that

$$s(x) = \begin{cases} \frac{1}{2} \text{ if } -1 \le x < 0 \\ 2\frac{1}{2} \text{ if } 0 \le x < 1 \\ 2 \text{ if } 1 \le x < 2 \\ 1 \text{ if } 2 \le x \le 2\frac{1}{2} \\ -1 \text{ if } 2\frac{1}{2} < x \le 3 \\ 0 \text{ if } x < -1 \text{ or } x > 3 \end{cases}$$



We now define the "definite integral from a to b" for any step function.

Let $s = \sum_{k=1}^{n} s_k \chi_{I_k} \in St[a,b]$

If our definition is to agree with what we usually understand by "integral" and relate to "area under the curve", it is clear that we must take

$$\int_a^b s = \sum_{k=1}^{n} s_k \, \mu(I_k)$$

where, for any interval I, $\mu(I)$ is the "length" of I and so

$$\mu(I) = i^* - i.$$

EXERCISE: for our previous example we have

$$\int_a^b (\tfrac{1}{2}\chi_{[-1,1)} + 2\chi_{[0,2\frac{1}{2}]} - \chi_{[2,3]})$$

$$= \tfrac{1}{2}(1 - (-1)) + 2(2\tfrac{1}{2} - 0) - 1(3 - 2).$$

Check that this is indeed the "area under the curve".

From this definition it is clear that the defintie integral from a to b has the following properties.

1) $\int_a^b (s_1 + s_2) = \int_a^b s_1 + \int_a^b s_2$ for all s_1 and s_2 in St[a,b]

2) $\int_a^b rs = r \int_a^b s$ for all $s \in St[a,b]$ and all real scalars r.

3) If s is a *positive* step function (that is $s(x) \geq 0$ for all $x \in [a,b]$), then

$$\int_a^b s \geq 0.$$

We will take these to be defining properties of a definite integral*.

* These, together with the *normalizing axiom;*

$$\int_a^b \chi_{[a,b]} = b - a,$$

and the *translation invariance axiom;*

if $I,J \subset [a,b]$ are such that $\mu(I) = \mu(J)$, then

$$\int_a^b \chi_I = \int_a^b \chi_J ,$$

completely determine the definite integral from a to b on St[a,b]. You might try

FOOTNOTE CONTINUED ON NEXT PAGE

REMARK: If we define the mapping $\int_a^b$ by

$$\int_a^b : \ St[a,b] \to \mathbb{R}: \ s \mapsto \int_a^b s$$

then 1) and 2) are equivalent to requiring $\int_a^b$ to be a *linear mapping*. [Such a mapping from a vector space into the real scalars is often termed a (linear) *functional* - thus 1) and 2) state: The "definite integral from a to b" defines a linear functional.]

Our aim is now to extend the domain of $\int_a^b$ from St[a,b] to a larger class of functions, while preserving the three properties listed above.

Let $\mathcal{B}[a,b]$ denote the set of bounded functions on [a,b]. That is, $f \in \mathcal{B}[a,b]$ if $|f(x)| \le M$ for all $x \in [a,b]$ and some $M > 0$. It is readily checked (do so) that $\mathcal{B}[a,b]$ is a (vector) subspace of $\mathcal{F}[a,b]$ which contains St[a,b] and that

$$\|f\|_\infty = \sup_{a\le x\le b} |f(x)|$$

defines a norm on $\mathcal{B}[a,b]$ with respect to which it is complete.

Thus, $(\mathcal{B}[a,b], \|\cdot\|_\infty)$ *is a Banach space.*

Let Reg[a,b] denote the closure of St[a,b] in $\mathcal{B}[a,b]$. That is, $f \in$ Reg[a,b] if and only if, there exists a sequence of step functions (s_n) with $\|s_n - f\|_\infty \to 0$, in which case we say f is a "uniform limit of step functions" (refer to the discussion of uniform convergence given in the proof of completeness of $\mathcal{C}[a,b]$).

The elements of Reg[a,b] are referred to as regulated functions*.

EXERCISE: Let $(X, \|\cdot\|)$ be a normed linear space and let M be a subspace of X. Show that the closure of M, $\bar{M}$, is also a subspace.

From the above exercise and the earlier one establishing that a closed subset of a complete metric space is itself complete, we see that

---

* This terminology and approach to "elementary" integration theory is adapted from that of the French Mathematician Jean Dieudonné (1906 -) - see his book: "Foundations of Modern Analysis" (Academic Press).

FOOTNOTE FROM PREVIOUS PAGE CONTINUED HERE

proving this as an exercise. That is, show that if $\int_a^b s$ satisfies 1), 2), 3) and the above two axioms, then

$$\int_a^b \sum_{k=1}^n s_k \chi_{I_k} = \sum_{k=1}^n s_k \mu(I_k).$$

$(\text{Reg}[a,b], \|\cdot\|_\infty)$ *is a Banach space* (St[a,b] is a subspace and so $\text{Reg}[a,b] = \overline{\text{St}[a,b]}$ is a closed subspace of the complete space $\mathcal{B}[a,b]$).

We now show that the "defintie integral", $\int_a^b$, may be extended to Reg[a,b]. We first need

LEMMA: $\int_a^b : \text{St}[a,b] \to \mathbb{R} : s \mapsto \int_a^b s$ *is a bounded (continuous) linear mapping (functional) from* $(\text{St}[a,b], \|\cdot\|_\infty)$ *to* $\mathbb{R}$ *with the usual metric.*

PROOF: Let $s \in \text{St}[a,b]$, then $\|s\|_\infty = \sup_{a\le x\le b} |s(x)|$ and so for each $x \in [a,b]$

$$-\|s\|_\infty \le s(x) \le \|s\|_\infty$$

or

$$0 \le s(x) + \|s\|_\infty \qquad \text{and} \qquad 0 \le \|s\|_\infty - s(x).$$

These two inequalities may be re-expressed as:

$$s + \|s\|_\infty \chi_{[a,b]} \quad \text{and} \quad \|s\|_\infty \chi_{[a,b]} - s$$

are both positive functions on [a,b], and so by property 3)

$$0 \le \int_a^b (s + \|s\|_\infty \chi_{[a,b]}) \qquad \text{and} \qquad 0 \le \int_a^b (\|s\|_\infty \chi_{[a,b]} - s).$$

Using the linearity of the integral and its definition, we therefore have

$$0 \le \int_a^b s + (b-a)\|s\|_\infty \qquad \text{and} \qquad 0 \le (b-a)\|s\|_\infty - \int_a^b s$$

or

$$\left| \int_a^b s \right| \le (b-a)\|s\|$$

and so $\int_a^b$ is a bounded linear mapping (see definition in §1, p. 7) as required. ∎

Now, by construction, $\text{Reg}[a,b] = \overline{\text{St}[a,b]}$ and so St[a,b] is a dense subspace of Reg[a,b]. Hence by the above lemma and lemma 8 of §1, p. 7) we have that $\int_a^b$ has a unique bounded linear extension from St[a,b] to Reg[a,b]. We will denote this extension by $\int_a^b$ (rather then the more formal $\tilde{\int_a^b}$) and for $f \in \text{Reg}[a,b]$ we will call the value of this extension, $\int_a^b f$, the "definite integral of f from a to b". Since the extension is linear properties 1) and 2) hold for it, we now establish that 3) is also satisfied.

First, let us note that from the proof of the extension theorem (Theorem 3 of §1, p. 4) we have the constructive result:

$$\int_a^b f = \lim_{n\to\infty} \int_a^b s_n$$ *where* (s_n) *is any sequence of step functions which converges uniformly to* f (that is, for which $\|s_n - f\|_\infty \to 0$).

Now let f $\in$ Reg[a,b] be a positive function. From the above observation and the density of St[a,b] in Reg[a,b], for any $\varepsilon > 0$ there exists a step function s such that

$$\left| \int_a^b f - \int_a^b s \right| \le \frac{\varepsilon}{2} \quad \text{and} \quad \|s - f\|_\infty \le \varepsilon' = \varepsilon/2(b-a)$$

From the second inequality and the definition of $\|\cdot\|_\infty$ we have

$$-\varepsilon' \le s(x) - f(x) \quad \text{for all } x \in [a,b],$$

and so, since f is a positive function,

$$0 \le \varepsilon' + s(x)$$

or

$$\varepsilon' \chi_{[a,b]} + s \quad \text{is a positive step function.}$$

Thus $\quad 0 \le \int_a^b (\varepsilon' \chi_{[a,b]} + s) \quad$ (by property 3) for step functions)

or

$$0 \le \varepsilon'(b-a) + \int_a^b s = \frac{\varepsilon}{2} + \int_a^b s$$

and so $\quad -\frac{\varepsilon}{2} \le \int_1^b s$.

Combining this with the first inequality, $\left| \int_a^b f - \int_a^b s \right| \le \frac{\varepsilon}{2}$, we therefore have

$$\int_a^b f \ge \int_a^b s - \frac{\varepsilon}{2} \ge -\varepsilon, \quad \text{for all } \varepsilon > 0$$

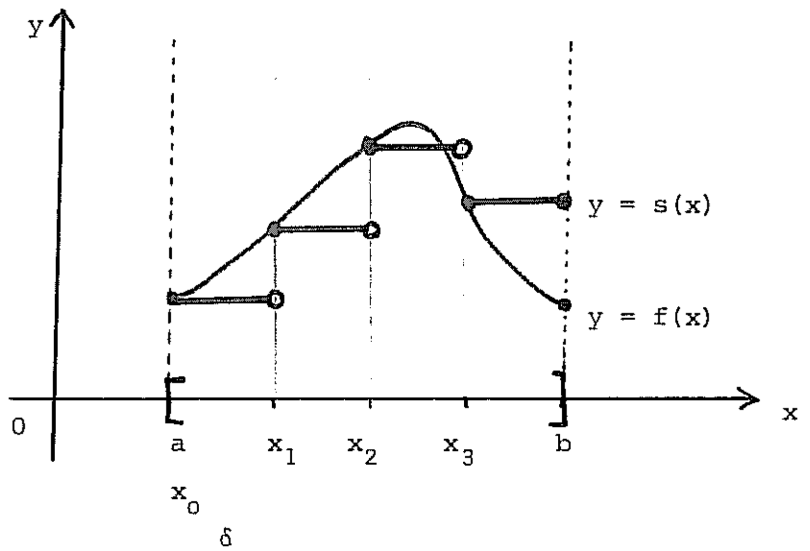
and so we conclude that $\int_a^b f \ge 0$ *whenever f is a positive function in* Reg[a,b].

Having extended the "definite integral from a to b" to Reg[a,b], we now establish that Reg[a,b] is a 'large' enough class of functions to be of interest. We do this by showing that C[a,b] - the set of all continuous functions on [a,b] - is a subspace of Reg[a,b]. That is, we show that every continuous function f on [a,b] is a uniform limit of step functions, or equivalently, given any $\varepsilon > 0$ there exists s $\in$ St[a,b] with $\|f - s\|_\infty < \varepsilon$.

THEOREM: *$C[a,b]$ is a subspace of* $\text{Reg}[a,b]$.

PROOF: Let $f \in C[a,b]$, then by Corollary 5 of §1, p. 6, f is uniformly continuous on $[a,b]$ and so, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x,y \in [a,b]$ are such that $|x - y| < \delta$.

Now choose $N \in \mathbb{N}$ such that $(b - a)/N < \delta$, let $x_0 = a$, $x_1 = a + \frac{(b-a)}{N}$, $x_2 = a + \frac{2(b-a)}{N}$, ..., $x_N = b$ and let $I_1 = [x_0, x_1)$, $I_2 = [x_1, x_2)$, ..., $I_N = [x_{N-1}, x_N]$



Define the step function s by

$$s = \sum_{k=1}^{N} f(x_{k-1})\, \chi_{I_k}$$

The proof is completed by showing $\|f - s\|_\infty < \varepsilon$.
To see this, note that for $x \in [a,b]$, x belongs to precisely one of the disjoint intervals $I_1, I_2, \ldots I_N$, say $x \in I_k$, then $s(x) = f(x_{k-1})$ and $|x_{k-1} - x| \le \frac{b-a}{N} < \delta$

so

$$|f(x) - s(x)| = |f(x) - f(x_{k-1})| < \varepsilon .$$

We therefore have

$$\|f - s\|_\infty = \sup_{a \le x \le b} |f(x) - s(x)| < \varepsilon$$

and the result is established.

■

REMARK: The above theorem shows that every continuous function on $[a,b]$ is in $\text{Reg}[a,b]$ and is therefore integrable according to our theory. In fact $\text{Reg}[a,b]$ contains most functions of practical importance and so our theory of integration