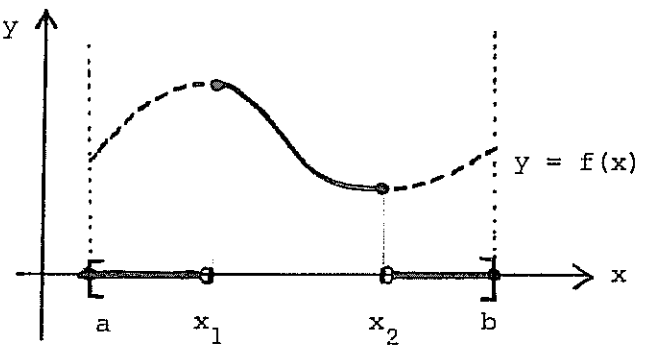
is adequate for most work in Applied (and Pure) Mathematics.

We conclude with two results which should already be familiar to you.

We begin with some terminology.

Given any $f \in Reg[a,b]$ and x_1, x_2 with $a \le x_1 \le x_2 \le b$, we first observe that the point wise product $f \cdot \chi_{[x_1,x_2]}$ is in Reg[a,b]. [If (s_n) is a sequence of step functions converging uniformly to f on [a,b], then $(s_n \cdot \chi_{[x_1,x_2]})$ is also a sequence of <u>step functions</u> which converges to $f \cdot \chi_{[x_1,x_2]}$]



graph of $f \cdot \chi_{[x_1,x_2]}$

The definite integral from a to b of $f \cdot \chi_{[x_1,x_2]}$ will be denoted by $\int_{x_1}^{x_2} f$; that is

$$\int_{x_1}^{x_2} f \ \underset{\text{definition}}{=} \ \int_a^b (f \cdot \chi_{[x_1,x_2]}) \ .$$

since $f \cdot \chi_{[x_1,x_2]} = f \cdot \chi_{[x_1,x_3]} + f \cdot \chi_{[x_3,x_2]}$ where $a \le x_1 \le x_3 \le x_2 \le b$, we clearly have

$$\int_{x_1}^{x_2} f = \int_{x_1}^{x_3} f + \int_{x_3}^{x_2} f \quad .$$

To extend this identity to all x_1, x_2, $x_3 \in [a,b]$ irrespective of their order, for $x_2 < x_1$ we define

$$\int_{x_1}^{x_2} f \quad \text{to be} \quad -\int_{x_2}^{x_1} f \ .$$

INTEGRAL MEAN VALUE THEOREM:

If $f \in Reg[a,b]$ *is such that, for some* $m, M \in \mathbb{R}$ *we have* $m \le f(x) \le M$ *for all* $x \in [x_1, x_2]$ *where* $a \le x_1 \le x_2 \le b$, *then*

$$m(x_2 - x_1) \le \int_{x_1}^{x_2} f \le M(x_2 - x_1) .$$

PROOF: Since $m \leq f(x)$ for all $x \in [x_1, x_2]$ we see that

$$f\cdot\chi_{[x_1,x_2]} - m\chi_{[x_1,x_2]}$$

is a positive function on $[a,b]$ and so, by property 3),

$$0 \leq \int_a^b (f\cdot\chi_{[x_1,x_2]} - m\chi_{[x_1,x_2]})$$

$$= \int_{x_1}^{x_2} f - m(x_2 - x_1).$$

This gives, $m(x_2 - x_1) \leq \int_{x_1}^{x_2} f$. The upper inequality is established similarly (do so). ∎

The function

$$F: [a,b] \to \mathbb{R}: s \mapsto \int_a^x f$$

is termed the *primitive* (indefinite integral) of $f \in \text{Reg}[a,b]$.

FUNDAMENTAL THEOREM OF CALCULUS:

If $f \in \text{Reg}[a,b]$ *is continuous at the point* $x_0 \in [a,b]$, *then the primitive* F *of* f *is differentiable at* x_0 *and*

$$F'(x_0) = f(x_0).$$

PROOF: For $h > 0$,

$$\frac{F(x_0 + h) - F(x_0)}{h} = \frac{1}{h}\left(\int_a^{x_0+h} f - \int_a^{x_0} f\right)$$

$$= \frac{1}{h}\int_{x_0}^{x_0+h} f$$

and so, by the integral mean value theorem,

$$\underset{x_0 \leq x \leq x_0+h}{\text{infemum}} f(x) \leq \frac{F(x_0 + h) - F(x_0)}{h} \leq \underset{x_0 \leq x \leq x_0+h}{\text{supremum}} f(x).$$

Now f is continuous at x_0 and so, as $h \to 0$ we have

$$\underset{x_0 \le x \le x_0+h}{\text{infemum}} f(x) \to f(x_0) \quad \text{and} \quad \underset{x_0 \le x \le x_0+h}{\text{supremum}} (f) \to f(x_0)$$

(Prove this).

We therefore have

$$\underset{h \to 0^+}{\text{limit}} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0).$$

A similar argument for $h < 0$ (give it) establishes that

$$\underset{h \to 0-}{\text{limit}} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0)$$

and so we have $F'(x_0)$ exists and equals $f(x_0)$. ∎

REMARKS: 1) From the last result we can go on to develop the usual integral calculus:

If F is an anti-derivative of f (that is $F' = f$), then $\int_{x_1}^{x_2} f = F(x_2) - F(x_1)$; "change of variable" formulae for integration; integration by parts; theory of improper integrals, etc. All this should be well known to you and we will not persue it further here.

2) We have established the integrability of a class of functions, Reg[a,b], which is adequate for most applications. However, while Reg[a,b] is complete with respect to $\|\cdot\|_\infty$ it is not complete with respect to the norms

$$\|f\|_1 = \int_a^b |f|$$

or

$$\|f\|_2 = \left(\int_a^b f^2 \right)^{\frac{1}{2}}$$

which we can define using the integral developed. This is an impediment to the application of many theorems of abstract analysis. To obtain complete spaces with respect to "these" norms we must further extend the "definite integral" to a larger class of functions, L[a,b], whose elements are the uniform limit of a sequence of step functions except on a set of points in [a,b] with "zero length". That is, except for a set of points which can be contained in a countable union of intervals the sum of whose lengths is arbitrarily small. This leads to Lebesgue's theory of integration - see for example the book by A.J. Weir "Lebesgue Integration and Measure" (Cambridge University Press).

APPENDIX

Suprema and Infema

Let S be a non-empty subset of the real numbers $\mathbb{R}$. We say S is bounded above if there exists a real number α such that $s \leq \alpha$ for all $s \in S$ and refer to α as an upper bound for S.

The supremum of S, denoted by $\operatorname{Sup} S$, is the "least upper bound" for S. That is $M = \operatorname{Sup} S$ if and only if

(i) $s \leq M$ for all $s \in S$ (M is an upper bound for S)

and

(ii) if α is any upper bound for S, then $M \leq \alpha$ (M is the smallest upper bound for S)

The supremum axiom for $\mathbb{R}$ ensures that every non-empty subset S which is bounded above has a supremum.

Note: The supremum of S need not belong to S. For example $\operatorname{Sup}(0,1) = 1$ but $1 \notin (0,1)$. If it happens that $\operatorname{Sup} S$ is a member of S we usually refer to it as the maximum of S and denote it by $\operatorname{Max} S$.

It will be convenient to write $\operatorname{Sup} S = \infty$ in case S is not bounded above.

The following simple result is assumed frequently in these notes.

PROPOSITION: *For* $\phi \neq S \subseteq \mathbb{R}$ *there exists a sequence* (s_n) *of points of* S *with* $s_n \to \operatorname{Sup} S$. [Note: In case $\operatorname{Sup} S = \infty$, this must be interpreted as s_n "diverges to $+\infty$"; that is, given any real number r, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow s_n > r$.]

Proof. In case $M = \operatorname{Sup} S < \infty$, for each $n \in \mathbb{N}$ there must exist $s_n \in S$ with $M - \frac{1}{n} \leq s_n$ (otherwise $M - \frac{1}{n}$ would be an upper bound for S, contradicting the fact that M is the smallest upper bound). But then $M - \frac{1}{n} \leq s_n \leq M$ (M is an upper bound) and so as $n \to \infty$ we have $s_n \to M$. The proof in case $\operatorname{Sup} S = \infty$ is similar and is left as an exercise. □

The infemum (or greatest lower bound) of S may be defined and analysed similarly. Alternatively questions concerning infema may be converted into questions about suprema by noting that

$$\inf S = -\operatorname{Sup}\{-s : s \in S\}.$$

PROJECT

In many situations a metric is defined "locally" in terms of differentials.

For example; the Poincaré metric on the (open) upper-half plane $\{\underset{\sim}{x} = (x,y) \in \mathcal{R}^2 : y > 0$ has as the infinitesimal element of "distance"

$$dS = \frac{\| d\underset{\sim}{x} \|_2}{y} . \tag{1}$$

The distance from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$ is then taken to be

$$\begin{aligned} d(\underset{\sim}{x}_0, \underset{\sim}{x}_1) &= \min_f \int_{x_2}^{x_1} \frac{\| d(x, f(x)) \|_2}{f(x)} \\ &= \min_f \int_{x_0}^{x_1} \frac{\sqrt{1 + f'(x)^2}}{f(x)} \, dx \end{aligned} \tag{2}$$

where the minimum is taken over all differentiable curves lying in the upper-half plane which join $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$; that is, functions f such that

$$f(x_1) = y_1, \quad f(x_2) = y_2 \quad \text{and} \quad f(x) > 0 \quad \text{for all} \quad x \in (x_0, x_1).$$

<u>Exercise</u>: Show that $d(\underset{\sim}{x}_0, \underset{\sim}{x}_1)$ as defined above is indeed a metric function.

If $F(x, f, f') = \frac{\sqrt{1 + f'^2}}{f}$ denotes the integrand in (2), then from the calculus of variations the minimum is achieved at an f satisfying the Euler-Lagrange equation:

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) = 0.$$

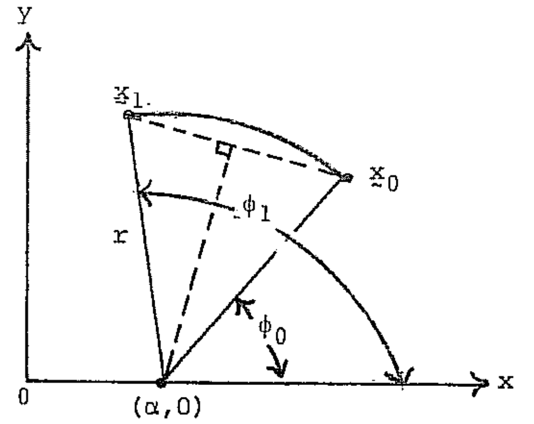
Show that this leads to the equation

$$1 + ff'' + f'^2 = 0$$

of

$$1 + \frac{1}{2}(f^2)'' = 0.$$

Hence conclude that the curve for which the minimum in (2) must be achieved is the circular arc joining $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$ with centre on the x-axis and lying in the upper-half plane.



This curve is the "geodesic" (path of shortest length) joining $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$. From this deduce that (2) leads to the expression

$$d(\underset{\sim}{x}_0, \underset{\sim}{x}_1) = \left| \mathrm{Log}_e \left| \frac{\tan\ (\phi_0/2)}{\tan\ (\phi_1/2)} \right| \right|$$

(For points vertically above one another this is to be interpreted as the limiting expression $|\log_e |y_0/y_1||$.)

Here ϕ_0 and ϕ_1 are the angles illustrated above.

[Hint: use the substitutions $x = \alpha + r \cos \phi$, $y = r \sin \phi$.]

The upper-half plane with this metric is a model for a non-euclidean (hyperbolic) geometry. [For a fuller discussion see for example, Siegel "Topics in Complex Function Theory" Volume II pp14 - 29.]

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