B SIMS

Introduction

These notes are based on lecture courses given to IV'th year honours and post-graduate students at the University of New England over the last few years. They introduce that area of functional analysis which has become known as the "Geometric theory of Banach spaces". There has been a resurgence of interest in Banach space theory following a number of important developments over the last decade or so:

Geometric aspects of the theory of vector valued measures, particularly spaces with the Radon-Nikodým property;

The study of general convex functions and Monotone Mappings, as in the theory of (weak) Asplund spaces;

The theory of weakly compactly generated spaces and attendant renorming results;

Super-Properties;

The theory of Banach space valued random variables;

Considerable advances in the theory of the "classical Banach spaces".

These developments are only hinted at in the current notes, the "classical" problem of reflexivity being the main application considered. Also, some of the more specialized concepts of current interest in Banach space geometry (for example uniform Gâteaux differentiability, uniform rotundity in directions, Vlasov's local compact uniform rotundity) have been omitted. None-the-less many ideas derived from recent work have been included. The course work is meant to provide the sound background in elementary Banach space geometry necessary for the study of these new and exciting areas.

The reader is assumed to have a working knowledge of general functional analysis and topology (as contained in the books by Simmons and Rudin, for example). Many of the tools commonly required in the geometric theory of Banach spaces have been summarized in §0. Because of the selective nature of the course there are however exceptions. The most notable omissions are:

The Krěin-Mil'man Theorem: *every compact convex subset of a locally convex linear topological space is the closed convex hull of its extreme points*, and its improvement to Choquet type theorems.

Many of the results associated with the name of Baire. (The Baire category theorem has in fact been assumed in the course, but belongs more properly to the general theory of metric spaces and so is not included in §0.)

I wish to thank my colleagues who assisted in the development of the course, particularly those students who wittingly or unwittingly served as guinea pigs. Special thanks are due to Mrs. Ferraro who transformed long scrolls of blotchy manuscript into very readable typescript.

B. Sims
April, 1979

CONTENTS

REFERENCES

I know of no single reference covering most of this course. Each of the following books contain relevant material, which will be referred to during the course.

Mahlon M. Day "Normed Linear Spaces" 3rd Edition Springer-Verlag, 1973.

Joseph Diestel "Geometry of Banach Spaces - Selected Topics" Springer, Lecture Notes in Mathematics, No.485, 1975.

N. Dunford and J.T. Schwartz "Linear Operators" Volume I, Interscience, 1957.

Richard B. Holmes "Geometric Functional Analysis and Applications" Springer, Graduate texts in Mathematics, No.24, 1975.

John L. Kelley "General Topology" Springer graduate texts in Mathematics, No.27.

John L. Kelley, Isaac Namioka et al "Linear Topological Spaces" Springer, Graduate texts in Mathematics, No.36.

Gottfried Köthe "Topological Vector Spaces I" Springer, Die Grundlehren der Mathematischen Wissenschaften, Band 159, 1969.

Walter Rudin "Functional Analysis" Mc Graw-Hill, 1973.

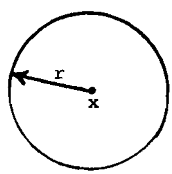
Albert Wilansky "Functional Analysis" Blaisdell, 1964.

§0 Preliminaries, Including Basic Notation

In this section we list those results from the General Theory of Banach spaces which are necessary tools for our work. You should have seen most of them before. Proofs are only included for a few results which may be unfamiliar to you. The order in which results appear is not necessarily the normal order in which they would be proved. It is not essential that you know the proofs, but you must understand what each result is saying and so be able to apply it in a variety of circumstances. None-the-less, this section should not be "learnt" as part of the course, but rather treated as reference material.

Although much of our theory remains valid in normed linear spaces, or with obvious modifications, in spaces over the complex field of scalars, we will restrict ourselves to real Banach spaces.

Unless otherwise stated X or $(X, \|\cdot\|)$ will denote a real (infinite dimensional) Banach space. $B[X] = \{x \in X: \|x\| \le 1\}$ is the <u>unit ball</u> of X. The boundary of $B[X]$, $S(X) = \{x \in X: \|x\| = 1\}$ is the <u>unit sphere</u>. In general $B_r(x)$ denotes the <u>open ball</u>, centre x and radius r that is, $B_r(x) = \{y \in X: \|x - y\| < r\}$, while $B_r[x]$ is the <u>closed ball</u> $\{y \in X: \|x - y\| \le r\}$. Thus $B[X] = B_1[0]$.



$B_r[x]$

A linear mapping $T: X_1 \to X_2$ where $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are Banach spaces is <u>bounded = (norm) continuous</u> if $\|T\| = \operatorname{Sup}\{\|Tx\|_2: x \in S(X_1)\} < \infty$, and then $\|Tx\|_2 \le \|T\|\|x\|_1$ for all $x \in X_1$.

CLOSED GRAPH THEOREM: *A linear map* $T: X_1 \to X_2$ *is continuous if and only if whenever the sequence* $(x_n) \subset X$ *is such that* $x_n \to x$ *and* $Tx_n \to y$ *we have* $y = Tx$.

NOTE: While the condition here resembles the sequential characterization of continuity: $x_n \to x \Rightarrow Tx_n \to Tx$, it is much weaker. By the closed graph theorem we do not need to establish the convergence of (Tx_n) from that of (x_n). It is sufficient to assume both sequences converge and then show $y = Tx$.

For our purposes the important consequence of the OPEN MAPPING THEOREM is: *If a continuous linear mapping* $T: X_1 \to X_2$, *where* X_1 *and* X_2 *are both Banach spaces, is 1-1 and onto then* T *is invertible and its inverse* T^{-1} *is also a continuous linear map.*

A linear mapping T satisfies these conditions if and only if there exists scalars $m, M > 0$ such that $m\|x\| \le \|Tx\| \le M\|x\|$ for all $x \in X_1$. Since a mapping is continuous if and only if the inverse images of open sets (unions of open balls) are open, we have: Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent Banach space norms for X (that is, both give rise to the same open sets) if and only if the identity map $I: X \to X$ is a continuous map from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$ and this happens if and only if there exists scalars $m, M > 0$ such that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$ for all $x \in X$.

[Since finite dimensional normed linear spaces are always complete and linear mappings between finite dimensional spaces are always continuous, this shows that, all norms on a finite dimensional space are equivalent.]

Of particular importance is the set of all continuous linear functionals from X to $\mathcal{R}$, denoted by X^*. With point-wise definitions of addition and scalar multiplication and norm defined by

$$\|f\| = \operatorname{Sup}\{|f(x)| : x \in S(X)\} \quad \text{all } f \in X^*,$$

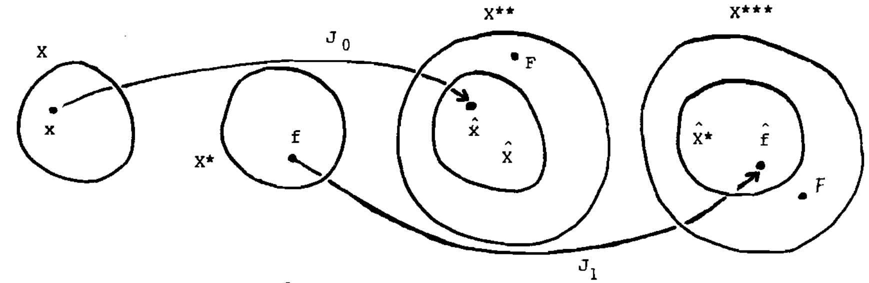
X^* is a Banach space, the dual space of X.

We will write X^{**} for $(X^*)^*$ the dual of the dual, X^{***} for $(X^{**})^*$ etc. Typically x, f, F, $\mathfrak{F}$ will be elements of X, X^*, X^{**}, X^{***} respectively.

For $x \in X$ the evaluation functional $\hat{x}$ is defined by

$$\hat{x}(f) = f(x) \quad \text{for all } f \in X^*.$$

The mapping $J_0: x \mapsto \hat{x}$ is an isometry (that is, $\|\hat{x}\| = \|x\|$) from X into X^{**}. The range of J_0, denoted by $\hat{X}$ is the natural embedding of X in X^{**}. Since X is a Banach space, $\hat{X}$ is a closed subspace of X^{**}. If $\hat{X} = X^{**}$ we say X is reflexive.



A linear functional $f: X \to \mathcal{R}$ is continuous (i.e., belongs to X^*) if and only if its kernel ($\operatorname{Ker} f = \{x \in X: f(x) = 0\}$) is a *closed* subspace of X. If $f \in X^*$ is not identically zero and $x_0 \in X$ is such that $f(x_0) \neq 0$, then for any $x \in X$ we note that $x = \lambda x_0 + k$ where $\lambda = f(x)/f(x_0)$ and $k = (x - \lambda x_0) \in \operatorname{Ker} f$. Thus, $X = \langle x_0\rangle \oplus \operatorname{Ker} f$, the direct sum of the one dimensional subspace spanned by x_0 and $\operatorname{Ker} f$, and so $\operatorname{Ker} f$ has co-dimension one in X. The converse is also true: If M is a subspace of co-dimension one in X, i.e. $X = \langle x_0\rangle \oplus M$ for some $x_0 \in X$, $x_0 \neq 0$, then $M = \operatorname{Ker} f$ for some $f \in X^*$ [any $x \in X$ may be written uniquely as $x = \lambda x_0 + m$ for some $\lambda \in \mathcal{R}$ and $m \in M$, define $f(x) = \lambda$.]

HAHN-BANACH THEOREM: *Let* p *be a semi-norm on* X. *Let* M *be a subspace of* X *and* f *a linear functional from* M *to* $\mathcal{R}$ *such that* $|f(m)| \le p(m)$ *for all* $m \in M$. *Then there exists a linear functional* $\tilde{f}$ *on* X *such that*

i) $\tilde{f}|_M = f$

and ii) $|\tilde{f}(x)| \le p(x)$ *for all* $x \in X$.

[$\tilde{f}$ is usually referred to as a Hahn-Banach extension of f].

We note the following consequences of this theorem.

1) (SUPPORT THEOREM) *For each* $x_0 \in S(X)$ *there exists an* $f_0 \in S(X^*)$ *with* $f_0(x_0) = \|f_0\|\ \|x_0\| = 1$.

2) X^* *is total over* X that is, if $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$, or if $x \neq y$ then there exists $f \in X^*$ with $f(x) \neq f(y)$.

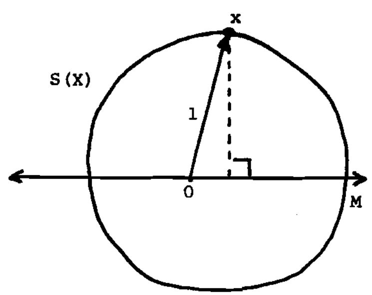
[Note: almost by definition, $\hat{X}$ is total over X^*.]

3) *If* M *is a closed subspace of* X *and* $x_0 \notin M$ *there exists* $f \in B[X^*]$ *with* $f(x_0) = \text{dist}(x_0,M) \neq 0$ *and* $f|_M = 0$ $(M \leq \text{Ker } f)$.

[$p(x) = \text{dist}(x,M) = \inf\{\|x - m\|: m \in M\}$ is a semi-norm on X.]

4) REISZ' LEMMA: *For any* r *with* $0 < r < 1$ *and any proper closed subspace* M *of* X, *there exists* $x \in S(X)$ *with* $\text{dist}(x,M) > r$.

Proof. By 3) there exists $f \in X^*$ with $f(m) = 0$ for all $m \in M$ but $f \neq 0$. Hence there exists $(x_n) \subset S(X)$ with $\frac{f}{\|f\|}(x_n) \to 1$. *But* then



$$\|x_n - m\| \geq \left|\frac{f}{\|f\|}(x_n) - \frac{f}{\|f\|}(m)\right|$$
$$= \left|\frac{f}{\|f_n\|}(x_n)\right| \to 1.$$

So $\inf_m \|x_n - m\| \to 1$. □

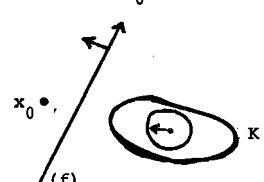
[Note: As a result of 4) the f in 3) has $\|f\| = 1$.]

5) SEPARATION THEOREMS.

(a) Mazur's Separation Theorem (special case; see Day p. 23 .)

Let K *be a convex set with* $\text{interior}(K) \neq \emptyset$. *If* $x_0 \notin \text{interior}(K)$, *then there exists* $f \in X^*$ *with* $f(x_0) > f(k)$ *for all* $k \in \text{interior}(K)$.

We say f separates x_0 from K.



Proof. By a translation we may without loss of generality suppose $0 \in \text{interior}(K)$, i.e., for some $r > 0$,

$B_r(0) \subseteq K$.

Let p be the *minkowski gauge functional* of K,

$$p(x) = \inf\{\lambda: x/\lambda \in K,\ \lambda > 0\}.$$

p is a semi-norm on X; $\text{interior}(K) = \{x \in X: p(x) < 1\}$ and clearly, $p(x) < (\tfrac{1}{2}r^{-1})\|x\|$.

2

Define f_0 on $\langle x_0 \rangle$ by $f_0(\lambda x_0) = \lambda$, then since $x_0 \notin \text{interior}(K)$ we have $f_0(x_0) = 1 \leq p(x_0)$ and so $|f_0(\lambda x_0)| \leq p(\lambda x_0)$ for all λ. Let f be a Hahn-Banach extension of f_0 from $\langle x_0 \rangle$ to X, then $|f(x)| \leq p(x) \leq (\tfrac{1}{2}r^{-1})\|x\|$ for all $x \in X$, so $f \in X^*$. Further, $f(k) \leq p(k) < 1 = f(x_0)$ for all $k \in$ interior (K) as required. □

(b) Eidelheit separation Theorem: *Let* K_1 *and* K_2 *be convex sets with* $\text{interior}(K_1) \neq \emptyset$ *and* $K_2 \cap \text{interior}(K_1) = \emptyset$, *then there exists* $f \in S(X^*)$ *such that* $\text{Sup } f(K_2) \leq \inf f(K_1)$.

Proof. Let $K = K_2 - K_1$, then K has interior points and 0 is not one of them. So, by (a) there exists $f \in S(X^*)$ such that $0 > f(k)$ all $k \in K$. Thus if $k_1 \in K_1$ and $k_2 \in K_2$ we have $f(k_2 - k_1) < 0$ or $f(k_2) < f(k_1)$. □

(c) (Basic Separation Theorem). *If* K *is a closed convex subset of* X *and* $x_0 \notin K$ *then there exists* $f \in S(X^*)$ *such that* $f(x_0) > \text{Sup } f(K)$.

Proof. Since K is closed, there exists $r > 0$ such that $B_r(x_0) \cap K = \emptyset$. Apply (b) with $K_1 = B_r(x_0)$, $K_2 = K$ and observe that $\inf f(B_r(x_0)) \neq f(x_0)$. □

REMARK. The single point x_0 in (c) can be replaced by any compact convex subset of X disjoint from K. (Can you prove this?)

TOPOLOGIES ON X

So far we have only considered the norm (strong) topology on X. This is the topology on X generated by the metric $d(x,y) = \|x - y\|$. The set of open balls $\{B_r(x): r > 0 \text{ and } x \in X\}$ is a *base* for this topology. Indeed, for each point $x \in X$ $\{B_q(x): q$ is a strictly positive rational number$\}$ is a countable *open base at* x. Further, the norm topology is a linear space topology that is, if $\mathcal{N}$ is an open base at 0, then for any $\lambda \in \mathbb{R}$ $\lambda\mathcal{N} = \{\lambda N: N \in \mathcal{N}\}$ is also an open base at 0 and for any $x \in X$ $x + \mathcal{N} = \{x + N: N \in \mathcal{N}\}$ is an open base at x. Thus, the operations of addition and scalar multiplication are continuous with respect to this topology.

Two other linear space topologies are the weak and weak* topologies defined respectively on a space and on its dual.

The weak (w) topology on X, sometimes denoted by $\sigma(X,X^*)$ is the weakest topology on X with respect to which the elements of X^* are continuous. Since a linear functional f will be continuous if and only if

$f^{-1}(-\varepsilon,\varepsilon) = f^{-1}(-\infty,\varepsilon) \cap f^{-1}(-\varepsilon,\infty)$ is an open subset of X for each $\varepsilon > 0$, we see that a *subbase* for the w topology at 0 consists of sets of the form

$$N(f,\varepsilon) = \{x \in X: f(x) < \varepsilon\} \quad \text{for } \varepsilon > 0 \text{ and } f \in X^*.$$

Not only is every functional in X^* continuous when X is equipped with the w topology but the elements of X^* are the only linear functionals continuous with respect to this topology. Thus *a linear functional is w-continuous if and only if it is norm continuous.* (Note: the same is not true of operators).

MAZUR'S THEOREM: *The w-closed convex hull and the norm closed convex hull of any set $S \subset X$ coincide.*

Proof: Let $\overline{co}\, S$ denote the norm closed convex hull of S (equal to the norm closure of the convex hull of S) and let $\overline{co}^w S$ denote the w-closed convex

hull of S. Since a w-open set is norm open $\overline{co}\, S \subseteq \overline{co}^w S$. If $x \notin \overline{co}\, S$, then there exists $f \in X^*$ such that $f(x) > \operatorname{Sup} f(\overline{co}\, S)$. Now $\{y: f(y) \le \operatorname{Sup} f(\overline{co}\, S)\}$ is a w-closed set (as f is also w-continuous) which does not contain x but contains $\overline{co}^w S$, so $x \notin \overline{co}^w S$. □

The weak* (w*) topology on X^*, sometimes denoted by $\sigma(X^*,X)$ is the weakest topology on X^* with respect to which the elements of $\hat{X}$ are continuous. A subbase for this topology at 0 consists of sets of the form

$$\begin{aligned} N(x,\varepsilon) &= \{f \in X^*: \hat{x}(f) < \varepsilon\} \\ &= \{f \in X^*: f(x) < \varepsilon\} \quad \text{for } \varepsilon > 0 \text{ and } x \in X. \end{aligned}$$

A linear functional F on X^* is continuous with respect to the w topology if and only if $F = \hat{x}$ for some $x \in X$.*

Thus, unless X is reflexive, the w* topology on X^* is strictly weaker than the w topology on X^* with respect to which every element of X^{**} is continuous.

BANACH ALAOGLU THEOREM: $B[X^*]$ *is compact in the w* topology.*

[Note: No similar result holds for the w topology. Indeed $B[X]$ is w compact if and only if X is reflexive.]

The Separation Theorems 5) page 4, remain true in other linear space topologies. For example, *if K is a w*-closed convex subset of X^* and $f \notin K$, then there exists a w* continuous linear functional $\hat{x} \in \hat{X}$ such that* $\hat{x}(f) > \operatorname{Sup} \hat{x}(K)$ *or* $f(x) > \operatorname{Sup} \{k(x): k \in K\}$.

We note the following Corollaries.

1) **GOLDSTINE'S THEOREM:** $\hat{B}[X] = \{\hat{x} \in \hat{X}: \|\hat{x}\| \le 1\}$ *is w* dense in* $B[X^{**}]$, that is the w* closure of $\hat{B}[X]$ equals $B[X^{**}]$.

Proof: Let K denote the w* closure of $\hat{B}[X]$. Since $\hat{B}[X] \subset B[X^{**}]$ and $B[X^{**}]$ is w* compact and so certainly w*-closed we have $K \subseteq B[X^{**}]$.

Now, assume there exists $F \in B[X^{**}]$ but $F \notin K$. Since $\hat{B}[X]$ is convex so too is K, thus there exists $f \in X^*$ with $\hat{f}(F) > \operatorname{Sup} \hat{f}(K)$ and so $\|f\| = \|\hat{f}\| \geq \hat{f}(F) > \operatorname{Sup} \hat{f}(K)$. However, $\operatorname{Sup} \hat{f}(K) \geq \operatorname{Sup} \hat{f}(\hat{B}[X]) = \operatorname{Sup} \{f(x) : x \in B[X]\} = \|f\|$, a contradiction. □

2) $S(X^*)$ *is* w^* *dense in* $B[X^*]$.

[Notes: This is only true for X infinite dimensional. A similar result is true in the w topology on X.]

Proof: We need to show that for any $f \in B[X^*]$ and w^*-neighbourhood N_f^* of f there exists $g \in S(X^*) \cap N_f^*$.

Now N_f^* has the form

$$N_f^* = \{g \in X^* : |(g - f)(x_i)| < \epsilon\}$$

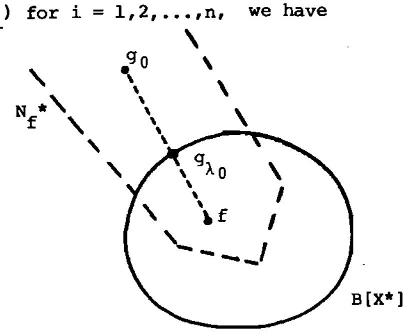
for some $\epsilon > 0$ and finite set $x_1, x_2, \ldots, x_n \in X$.

Let $M = \langle x_1, x_2, \ldots, x_n \rangle$ (the linear span of $x_1, x_2, \ldots, x_n$), Since X is infinite dimensional we may choose an $x_0 \in S(X) \backslash M$ and form $M' = \langle x_0, x_1, \ldots, x_n \rangle = \langle x_0 \rangle \oplus M$. Any element of M' has the form $x = \alpha x_0 + m$ for some unique $\alpha \in \mathcal{R}$ and $m \in M$.

Define g_0 on M' by

$$g_0(\alpha x_0 + m) = \alpha + f(m),$$

and extend g_0 to all of X by the Hahn-Banach Theorem. Then, $\|g_0\| \geq 1$ as $g_0(x_0) = 1$, further, since $g_0(x_i) = f(x_i)$ for $i = 1,2,\ldots,n$, we have $g_0 \in N_f^*$



Now N_f^* is convex, so for any $\lambda \in [0,1]$

$$g_\lambda \equiv \lambda f + (1-\lambda) g_0 \in N_f^*$$

and, since $\|f\| \leq 1$, $\|g_0\| \geq 1$ and the norm function is continuous there exists a $\lambda_0 \in [0,1]$ with $\|g_{\lambda_0}\| = 1$.

g_{λ_0} is the required norm 1 element of N_f^*. □

A sometimes useful result is

The w^* *topology on* $B[X^*]$ *is a metric topology if and only if* X *is separable.*

Similarly

The w *topology on* $B[X]$ *is a metric topology if and only if* X^* *is seperable.*

Neither topology is a metric topology on the whole of the space unless X is finite dimensional.

Nets and *Sequences*

For a metric topology, in which there is a COUNTABLE open base at each point, many topological properties are usefully characterized in terms of sequences. For example: *When* X *and* Y *are metric spaces,* $f : X \to Y$ *is continuous if and only if* $x_n \to x \Rightarrow f(x_n) \to f(x)$;

A subset C *of a metric space is compact if and only if every sequence of points from* C *has a subsequence converging to some element of* C.

For other linear space topologies such as the w or w^* topologies sequences are not enough. [Examples demonstrating this are to found in Kelley, B on p.76 or Wilansky, problem 44 on p.142.] This inadquacy is overcome by the notion of a *net* (or Moore-Smith convergence) which generalizes the idea of sequence, but retains much of our intuition about sequences. The elements of a sequence are indexed by the Natural Numbers N. In considerations of convergence the natural ordering of N is important. The idea behind nets is to allow index sets more general than N.

DEFINITION: By a <u>directed set</u> Λ we understand a set Λ on which an order relation $\geq$ is defined such that:

i) If $\alpha, \beta, \gamma \in \Lambda$ and $\alpha \geq \beta$, $\beta \geq \gamma$ then $\alpha \geq \gamma$;

ii) $\alpha \geq \alpha$ for $\alpha \in \Lambda$;

iii) If $\alpha, \beta \in \Lambda$ then there exists $\gamma \in \Lambda$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$

$\mathbb{N}$ with the natural ordering is a directed set. In a topological space, the family of sets in an open base at any point is directed by $\subseteq$.

DEFINITION: A <u>net</u> in X is a function x from some directed set Λ into X. (c.f. the definition of sequence.)

As with sequences, we will write x_α for $x(\alpha)$ and denote the net by $(x_\alpha)_{\alpha\in\Lambda}$ or simply (x_α).

Let X be a topological space, we say the net (x_α) in X converges to x if given any neighbourhood N of x there exists $\alpha_0 \in \Lambda$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in N$.

To make things work out, the definition of *subnet* is somewhat more general than work with sequences might suggest.

Let $x: \Lambda \to X$ be a net in X.

Let B be any other ordered set with a mapping $\underline{\alpha}: B \to \Lambda$ having the property, that for any given $\alpha_0 \in \Lambda$ there exists $\beta_0 \in B$ such that $\underline{\alpha}(\beta) \geq \alpha_0$ whenever $\beta \geq \beta_0$ (intuitively, "the values $\underline{\alpha}(\beta)$ become arbitrarily large as β increases). The composite $x \circ \underline{\alpha}: B \to X$ is a <u>subnet</u> of x.

By analogy with sequences, we will write (x_{α_β}) to indicate a subnet of (x_α).

"Fortunately", we rarely need to use these details. For most of our applications it is sufficient to know: *The net* (x_α) *converges to* x *if and only if every subnet converges to* x and *if* (x_α) *does not converge to* x *then there exists a neighbourhood* N *of* x *and a subnet* (x_{α_β}) *with* $x_{\alpha_\beta} \notin N$ *for any* β.

In terms of Nets we have:

1) *A subset* A *of* X *is closed if and only if no net in* A *converges to a point outside of* A.

2) f: $X \to Y$ *is continuous if and only if for each net* (x_α) *in* X *which converges to a point* x, *the net* $f(x_\alpha) \to f(x)$.

3) *A subset* A *of* X *is compact if and only if every net in* A *has a subnet converging to some point of* A.

Note: $(x_\alpha) \subset X$ is such that $x_\alpha \xrightarrow{w} x$ if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$. Similarly, $f_\alpha \xrightarrow{w^*} f$ if and only if $f_\alpha(x) \to f(x)$ for all $x \in X$.

<u>EXAMPLES</u>

The following specific spaces may be used to illustrate our theory.

Hilbert spaces. Inner-product denoted by $(\cdot,\cdot)$.

The sequence spaces

Let $x = (x_1, x_2, \ldots, x_n, \ldots)$ denote an infinite sequence of real numbers. The set of all such sequences $\mathbb{R}^\infty$ is a linear space under "component-wise" definitions of addition and scalar multiplication.

A linear functional $\underline{f}: \mathbb{R}^\infty \to \mathbb{R}$ has the form $\underline{f}(x) = \sum_{i=1}^{\infty} f_i x_i$ for some set of scalars $f_1, f_2, \ldots$. Thus $\underline{f}$ may itself be identified with an element of $\mathbb{R}^\infty$, $f = (f_1, f_2, \ldots)$ and we can write $\underline{f}(x) = f\cdot x$ where $\cdot$ stands for the usual "dot" product of vectors.

From $\mathbb{R}^\infty$ we can extract a number of important Banach spaces.

ℓ_∞: the subspace of all bounded sequences with norm defined by $\|x\|_\infty = \sup_n |x_n|$.

c_0: the subspace of ℓ_∞ consisting of all sequences convergent to 0.

ℓ_1: the subspace of all absolutely summable sequences with norm defined by $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$.

ℓ_p for $1 < p < \infty$: the subspace of all sequences x for which $\sum_{n=1}^{\infty} |x_n|^p < \infty$, with norm defined by $\|x\|_p = \sqrt[p]{\sum_{n=1}^{\infty} |x_n|^p}$.

NOTES: 1) The notations ℓ_1, ℓ_p and ℓ_∞ are consistent. Clearly ℓ_1 is the result of setting $p = 1$ in the definition of ℓ_p, while $\lim_{p\to\infty} \|x\|_p = \sup_n |x_n| = \|x\|_\infty$.

2) For $p > q$

$$\|x\|_p \le \|x\|_q \qquad \text{and so}$$

$$\ell_1 \subseteq \ldots \subseteq \ell_2 \subseteq \ldots \subseteq \ell_q \subseteq \ldots \subseteq \ell_p \subseteq \ldots \subseteq \ell_\infty$$

$\uparrow$ Hilbert space.

3) $\ell_1 = c_0{}^*$, $\ell_\infty = \ell_1{}^* = c_0{}^{**}$. (Thus ℓ_0, ℓ_1 and ℓ_∞ are non-reflexive.)

For $1 < p < \infty$ $\ell_p{}^* = \ell_q$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

(Thus, for $1 < p < \infty$ ℓ_p is reflexive.)

4) With the exception of ℓ_∞ all these spaces are separable that is, they have countable dense subsets.

In each case the set of sequences with only finitely many non zero components is a countable dense subset.

In ℓ_∞, the set of sequences with components either 0 or 1 is in correspondence with the binary representation of real numbers in [0,1] and so is uncountable. Any pair of distinct elements from this set are distance 1 apart. Since any dense set must have elements arbitrarily close to each of these sequences it cannot be countable.

Continuous function spaces

$C[a,b]$ the set of all continuous functions mapping the closed (bounded) interval $[a,b]$ into $\mathbb{R}$ is a Banach space with addition and scalar multiplication defined point-wise and norm defined by

$$\|f\|_\infty = \text{Max } \{|f(x)| : x \in [a,b]\}$$

[Note: [a,b] could be replaced by any compact topological space.]

The set of polynomials (with rational coefficients) is a countable set which by Weierstrass' Theorem is dense in $C[a,b]$. Thus $C[a,b]$ is separable. $C[a,b]$ is not reflexive.

The Lebesgue Function Spaces

Let μ be Lebesgue measure on $\Omega = [0,1]$, or more generally on any <u>finite</u> measure space (Ω, Σ, μ).

For any Lebesgue integrable function $f: \Omega \to \mathbb{R}$ let $\underline{f}$ denote the equivalence class $\{g: \int_\Omega |f - g|\, d\mu = 0\} = \{g: f - g = 0 \text{ almost everywhere}\}$.

Then, since $\underline{f + g} = \underline{f} + \underline{g}$, $\underline{\lambda f} = \lambda \underline{f}$ and f is Lebesgue integrable if and only if $|f|$ is, the space of all such equivalence classes $\mathcal{L}_1(\Omega,\mu)$ is a normed linear space with $\|\underline{f}\|_1 = \int_\Omega |f|\, d\mu$. Indeed $\mathcal{L}_1(\Omega,\mu)$ is a Banach space.

For $1 \le p \le \infty$ we can construct a Banach space $\mathcal{L}_p(\Omega,\mu)$ with elements those $\underline{f}$ for which $\int_\Omega |f|^p\, d\mu < \infty$ and norm defined by $\underline{f}_p = \sqrt[p]{\int_\Omega |f|^p\, d\mu}$.

We have

$$\mathcal{L}_p{}^*(\Omega,\mu) = \mathcal{L}_q(\Omega,\mu) \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

and for $p > q$

$$\|\underline{f}\|_p \le \|\underline{f}\|_q$$

Eg $x^{-1/2} \in \mathcal{L}_1([0,1], \text{Leb})$
$\notin \mathcal{L}_2(\ \text{"}\ \ \text{"}\ )$.

Further, as sets, we have by Hölder's inequality

$$\mathcal{L}_\infty(\Omega,\mu) \subsetneq \ldots \subsetneq \mathcal{L}_p(\Omega,\mu) \subsetneq \ldots \subsetneq \mathcal{L}_q(\Omega,\mu) \subsetneq \ldots \subsetneq \mathcal{L}_2(\Omega,\mu) \subsetneq \ldots \subsetneq \mathcal{L}_1(\Omega,\mu) .$$

$\uparrow$ Hilbert space

$\longleftarrow$ reflexive $\longrightarrow$

$\longleftarrow$ separable $\longrightarrow$

§1 SUPPORT FUNCTIONALS

By a hyperplane in X we mean a translate of a subspace of co-dimension 1 in X. Thus H is a hyperplane if and only if $H = x_0 + M$ for some $x_0 \in X$ and subspace M of X with $X = \langle y_0 \rangle \oplus M$ for some $y_0 \neq 0$. By the discussion on p.3 $M = \operatorname{Ker} f$ for some $f \in S(X^*)$. This leads to

PROPOSITION 1. *$H \subset X$ is a hyperplane if and only if for some $f \in S(X^*)$ and $c \in \mathbb{R}$ we have* $H = f^{-1}(c) = \{x \in X: f(x) = c\}$.

Proof. ($\Rightarrow$) If H is a hyperplane, then $H = x_0 + \operatorname{Ker} f$ for some $x_0 \in X$ and $f \in S(X^*)$. So $h \in H$ if and only if $h = x_0 + m$ where $f(m) = 0$. Thus for all $h \in H$ we have $f(h) = c$ where $c = f(x_0)$. Conversely, if $f(h) = c$, then $h = x_0 + (h - x_0)$ and $f(h - x_0) = 0$, so $h \in H$.

($\Leftarrow$) If $H = \{x \in X: f(x) = c\}$ for some $f \in S(X^*)$ and $c \in \mathbb{R}$, then choosing any $x_0 \in H$ we have for any $h \in H$ that $h = x_0 + (h - x_0)$, where $f(h - x_0) = 0$. So H is contained in $x_0 + \operatorname{Ker} f$. Conversely, if $x \in x_0 + \operatorname{Ker} f$ then $f(x) = f(x_0) = c$ and so $x \in H$. Thus $H = x_0 + \operatorname{Ker} f$ and is a hyperplane. □

REMARK: This correspondence between Hyperplanes in X and points in X^* is reminiscent of the duality between lines and points in projective geometry and partly explains the term *dual* space.

A hyperplane $H = f^{-1}(c)$ divides X into two closed "half-spaces"*;

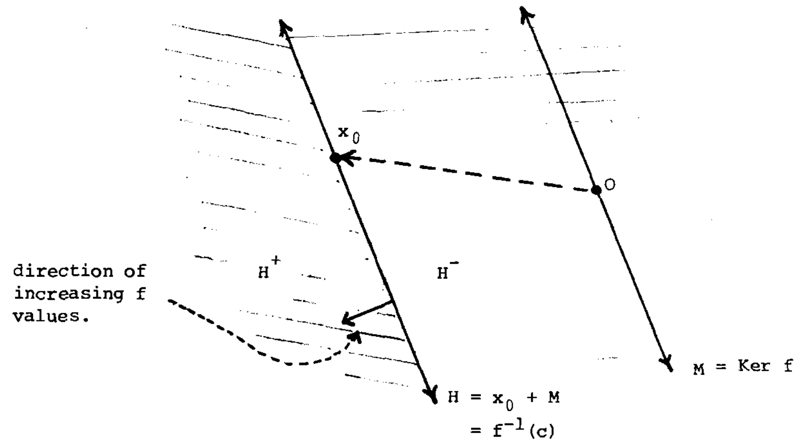
$$H^+ = \{x \in X: f(x) \geq c\}$$

and

$$H^- = \{x \in X: f(x) \leq c\}$$

where $H^+ \cap H^- = H$.

* A half-space is a convex set whose complement is also convex.



We say the Hyperplane H supports $B[X]$ at $x \in S(X)$ if $x \in H$ and $B[X] \subset H^-$. Intuitively, H is a "tangent plane" to $B[X]$ at x.

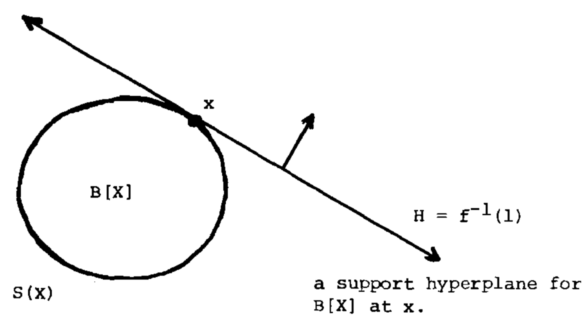
PROPOSITION 2. *H supports $B[X]$ at $x \in S(X)$ if and only if $H = f^{-1}(1)$ for some $f \in S(X^*)$ with $f(x) = 1$.*

Proof. ($\Rightarrow$) Let $H = f^{-1}(c)$ where $f \in S(X^*)$, then

$$c = f(x) \leq \sup_{y \in S(X)} f(y) = \|f\| = 1$$

and, since $B[X] \subset H^-$, $f(y) \leq c$ for all $y \in S(X)$, so $1 = \sup_{y \in S(X)} f(y) \leq c$.

($\Leftarrow$) If $f \in S(X^*)$ is such that $f(x) = 1$ then $x \in H = f^{-1}(1)$. Further, if $y \in B[X]$ then $f(y) \leq \|f\| = 1$ and $y \in H^-$. Thus $x \in H$ and $B[X] \subset H^-$, so H supports $B[X]$ at x. □



This proposition shows that there is a one-to-one correspondence between support hyperplanes for $B[X]$ and linear functionals in $S(X^*)$ which attain their norms. A functional $f \in S(X^*)$ <u>attains its norm</u> if there exists $x \in S(X)$ with $f(x) = \sup_{y \in S(X)} f(y) = \|f\| = 1$. Such a functional is referred to as a <u>support functional</u> for $B[X]$ at x. The support theorem 1) on page 4 can now be rephrased as:

3. *There exists a support functional for* $B[X]$ *at every* $x \in S(X)$

OR

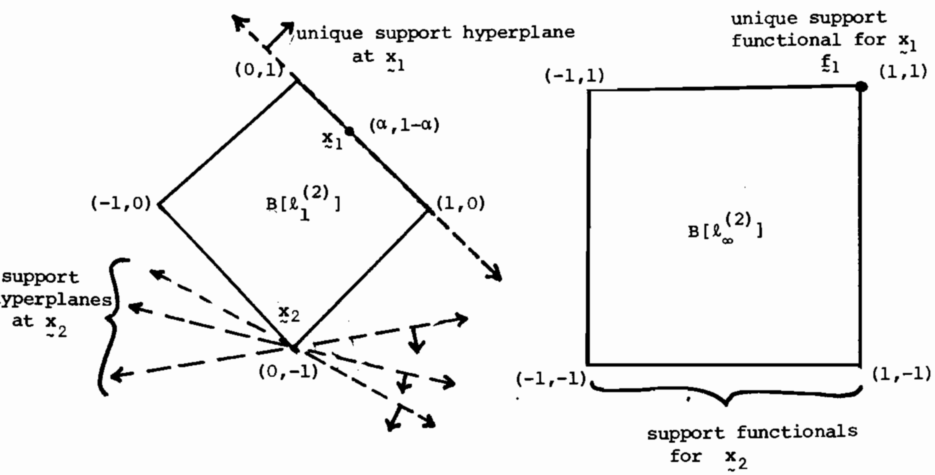
For each $x \in S(X)$ *there exists a Hyperplane which supports* $B[X]$ *at* x.

[Intuitively, there is a "tangent" at every point of $S(X)$.]

<u>EXAMPLES</u>

1) Let $X = \ell_1^{(2)}$ the space of ordered pairs of real numbers with $\|\underset{\sim}{x}\|_1 = \|(x_1,x_2)\|_1 = |x_1| + |x_2|$, then $\ell_1^{(2)*} = \ell_\infty^{(2)}$ the space of ordered pairs of real numbers with $\|\underset{\sim}{f}\|_\infty = \|(f_1,f_2)\|_\infty = \text{Max } |f_1|,|f_2|$ and $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{f}\cdot\underset{\sim}{x} = f_1x_1 + f_2x_2$.

The unit balls of $\ell_1^{(2)}$ and $\ell_\infty^{(2)}$ are illustrated below.



For any point $\underset{\sim}{x}_1$ on the segment of $S(\ell_1^{(2)})$ joining (0,1) and (1,0) we have $\underset{\sim}{x}_1 = (\alpha,1-\alpha)$ for some $\alpha \in [0,1]$. If $\underset{\sim}{f}_1$ is a support functional for $\underset{\sim}{x}_1$ we have $\underset{\sim}{f}_1 = (f_1,f_2)$ where

$$\|\underset{\sim}{f}_1\|_\infty = \text{Max } |f_1|,|f_2| = 1$$

and $\underset{\sim}{f}_1(\underset{\sim}{x}) = \alpha f_1 + (1-\alpha)f_2 = 1$.

The only solution of which is $\underset{\sim}{f}_1 = (1,1)$.

Similarly, any functional of the form $(\alpha,-1)$ with $\alpha \in [-1,1]$ supports $B[\ell_1^{(2)}]$ at $\underset{\sim}{x}_2 = (0,-1)$.

2. *For* X *infinite dimensional every* $f \in S(X^*)$ *is not necessarily a support functional.*

Let $X = c_0$, then $X^* = \ell_1$. Let $f = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots)$, then $\|f\|_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. For $x = (x_1, x_2, \ldots) \in S(X)$ we have $|x_n| \le \sup_m |x_m| = \|x\|_\infty = 1$ and $f(x) = \sum_{n=1}^{\infty} f_n x_n = \sum_{n=1}^{\infty} x_n/2^n$, for this last sum to equal 1 we must have $x_n = 1$ for all n. This is impossible as $(1,1,1,\ldots) \notin c_0$.

EXERCISES: 1) Identify at least one support functional for each $x \in S(c_0)$. At the point $(1,0,0,\ldots) \in S(c_0)$, show there is only one support functional. Give an example of a point at which there are infinitely many support functionals.

2) Show the conclusion of Example 2 is false for reflexive spaces that is,

If X *is reflexive then every* $f \in S(X^*)$ *is a support functional for some* $x \in S(X)$.

The observations made in Example 2 and Exercise 2 are greatly strenghened by a rather recent, deep result of R.C. James [<u>Reflexivity and the Supremum of linear functionals</u>, Ann. of Math. 66 (1957) pp.157-169] which states:

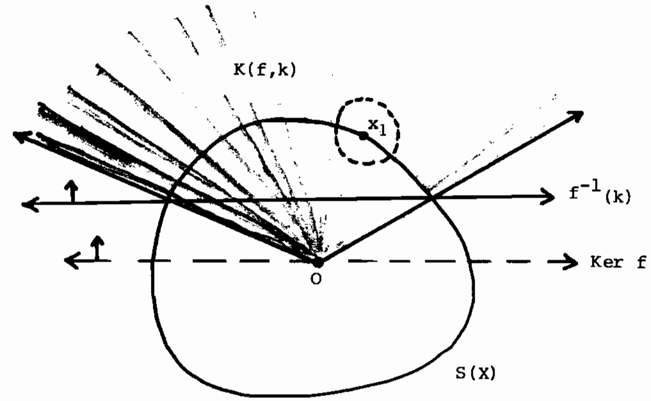
A Banach space X *is reflexive if and only if every element of* S(X*) *attains its norm on* S(X) (that is, is a support functional).

We will not prove this theorem [see Diestel or Holmes for a "simplified" proof], nor will we use it to develop theory. None-the-less, several of our results can be proved very simply if we assume James' Theorem. When this is the case, such a proof will be called for as an exercise.

A related, very useful, result was established by E. Bishop and R.R. Phelps [announced in Bull. A.M.S. 67 (1961) pp.97-98]. They show that in any Banach space the set of functionals in S(X*) which do attain their norm on S(X) is a norm dense subset of S(X*). For obvious reasons, this property of all Banach spaces is referred to as Subreflexivity. We will prove this result. While it is probably the deepest (hardest) proof which we shall encounter, given the right approach, the ideas underlying the proof are fairly simple.

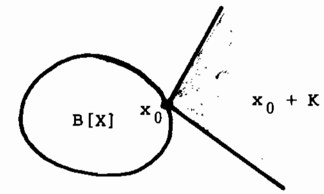
A subset K of X is a cone (vertex the origin) if, whenever $x \in K$ we have $\lambda x \in K$ for all $\lambda \geq 0$. A cone is completely determined by the norm one elements in it. Indeed, K consists of all the half-lines from O through norm one elements of K. [If $x \in K$, $x \neq 0$, then $x_1 = x/\|x\|$ is a norm one element of K. Further $x = \|x\| x_1 \in \{\lambda x_1 : \lambda \geq 0\}$ the half-line from O through x_1.]

For any $f \in S(X^*)$ and $k \in (0,1)$ define $K(f,k)$ to be $\{x \in X: k\|x\| \leq f(x)\}$. Clearly $K(f,k)$ *is a cone*. The norm one elements of $K(f,k)$ are precisely those points of S(X) which lie in the positive half-space determined by the hyperplane $f^{-1}(k)$, thus $K(f,k)$ is as illustrated below.



It is readily checked that $K(f,k)$ *is closed and convex* (do so). Further, as k approaches 0, $K(f,k)$ becomes more nearly a half-space.

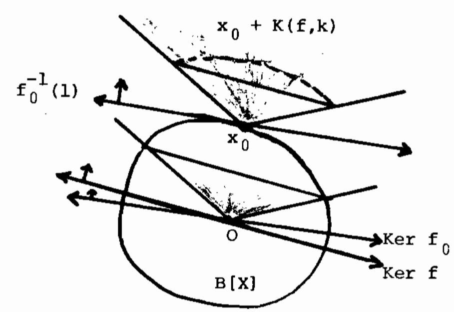
We will say that the cone K supports B[X] at x_0 if $(x_0+K) \cap B[X] = \{x_0\}$.



We now have the ingredients needed to outline the strategy of the proof. Given any $f \in S(X^*)$ and $k \in (0,1)$ we first show that there exists $x_0 \in S(X)$ at which $K(f,k)$ supports B[X]. Then since, B[X] has a non-empty interior, we may apply the Eidelheit separation theorem (p.5) to obtain $f_0 \in S(X^*)$ such that

$$1 = \sup_{x \in B[X]} f_0(x) = f_0(x_0) = \inf_{x \in K(f,k)} f_0(x),$$

(see the figure opposite).



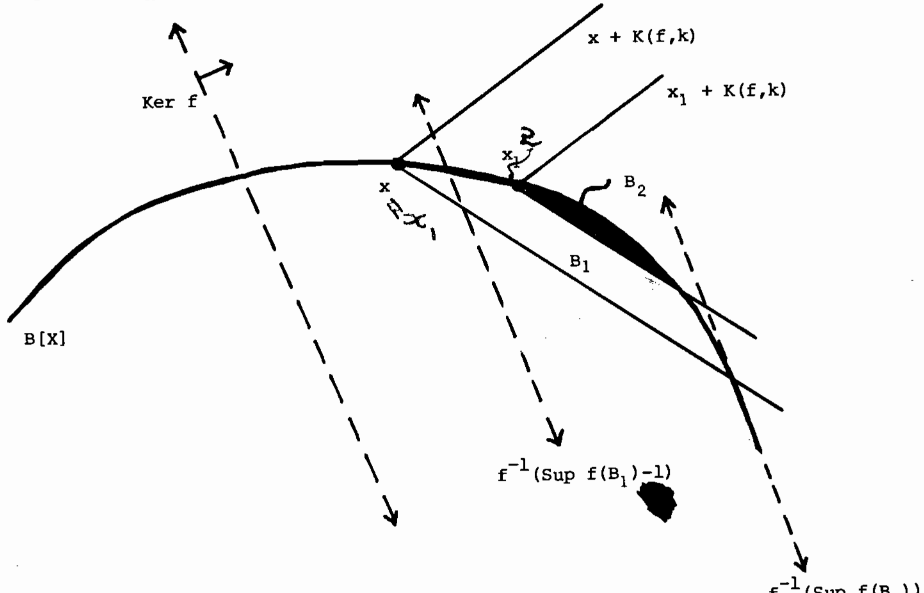
The remainder of the proof amounts to showing that, since the half-plane $\{x: f_0(x) \geq 0\}$ contains $K(f,k)$, which for k small, is itself nearly a half-plane, then the "directions" of f and f_0 are near to one another.

LEMMA 4 (Existence of Support points): *Let* $f \in S(X^*)$, $k \in (0,1)$ *and* $x \in B[X]$ *then there exists* $x_0 \in B[X]$ *such that*

i) $x_0 \in x + K(f,k)$

and ii) $K(f,k)$ *supports* $B[X]$ *at* x_0.

Proof: Let $B_n = B[X] \cap (x_n + K(f,k))$ where (x_n) is defined inductively by $x_1 = x$, $x_{n+1} \in B_n$ is such that $\operatorname{Sup} f(B_n) < f(x_{n+1}) + \frac{1}{n}$.



Clearly B_n is closed, and $B_n \neq \emptyset$ $(x_n \in B_n)$.

Also $x_{n+1} \in B_n \subset x_n + K(f,k)$

so $x_{n+1} + K(f,k) \subseteq x_n + K(f,k) + K(f,k))$

$= x_n + K(f,k)$, as $K(f,k)$ is a convex cone.

Thus $B_{n+1} \subseteq B_n$.

Further, if $y \in B_{n+1}$, then

$$\|y - x_{n+1}\| \le k^{-1} f(y - x_{n+1}), \quad \text{as } y - x_{n+1} \in K(f,k)$$

$$\le k^{-1} \operatorname{Sup} f(B_n) - k^{-1} f(x_{n+1}) \text{ as } y \in B_{n+1} \subset B_n .$$

$$\le (n\,k)^{-1} .$$

So, diameter $B_{n+1} < \frac{2}{kn} \to 0$.

Cantor's intersection theorem* now applies (since X is complete) and so $\bigcap_n B_n$ consists of a single point x_0.

Since $x_0 \in B_1$ we have: i) $x_0 \in x + K(f,k)$.

Finally, since $x_0 \in B_n = B[X] \cap (x_n + K(f,k)) \subset x_n + K(f,k)$, we have $x_0 + K(f,k) \subset x_n + K(f,k) + K(f,k)$

$= x_n + K(f,k)$, again $K(f,k)$ is a convex cone.

Thus $(x_0 + K(f,k)) \cap B[X] \subseteq B_n$ for all n and so:

ii) $(x_0 + K(f,k)) \cap B[X] = \{x_0\}$. $\square$

EXERCISE: In the above proof we twice used the observation that if ~~K~~ a cone K (vertex 0) is a convex ~~cone (vertex 0)~~ then $K + K = K$. Prove this property characterizes ~~convex~~ those cones which are convex.

REMARK: In applying the Cantor intersection theorem, the completeness of X is vital. Indeed it can be shown that the conclusion of Lemma 4 is false for incomplete spaces [E. Bishop and R.R. Phelps, *Support functionals of convex sets*, Proc. Symposia in Pure Math. AMS, 7 (1963) pp.27-35.]

* Cantor's Intersection Theorem states: *A metric space* (X,d) *is complete if and only if for every nested sequence of closed non-empty sets* $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ *such that diameter* $F_n \to 0$ *we have* $\bigcap_n F_n$ *contains exactly one point.*
If you are not familiar with this result you should attempt to prove it (at least the "only if" part). You should also give an example to show that the assumption 'diameter $F_n \to 0$' is necessary.

As a consequence of the above proof we also have

COROLLARY 5: *Given* $f \in S(X^*)$ *and* $k \in (0,1)$, *if* $x \in B[X]$ *is chosen so that* $f(x) > 1 - \varepsilon$, *then the* x_0 *of lemma 4 also satisfies* $\|x_0 - x\| < \frac{\varepsilon}{k}$.

Proof: Since $x_0 \in B_1$ we have $x_0 - x \in K(f,k)$ and so

$$\begin{aligned}\|x_0 - x\| &\le k^{-1} f(x_0 - x)\\ &< k^{-1}(1 - (1 - \varepsilon)), \text{ as } f(x_0) < \|f\|\,\|x_0\| = 1,\\ &= \frac{\varepsilon}{k}. \qquad \Box\end{aligned}$$

Having established their existence, we now investigate the consequences of having at x_0 a support cone to $B[X]$ of the form $K(f,k)$.

LEMMA 6: *Given* $f \in S(X^*)$ *and* $k \in (0,1)$, *there exists* $f_0 \in S(X^*)$ *such that* f_0 *attains its norm (at the* x_0 *of lemma 4) and* $K(f,k) \subseteq \{x \in X : f_0(x) \ge 0\}$.

Proof: By lemma 4 there exists $x_0 \in B[X]$ such that $(x_0 + K(f,k)) \cap B[X] = \{x_0\}$. Clearly $\|x_0\| = 1$, otherwise, for a sufficiently small $\delta > 0$ and $x \in K(f,k)$ we would have $x_0 + \delta x \in B[X] \cap (x_0 + K(f,k))$. Thus $B(X) = \{x \in X : \|x\| < 1\}$, the interior of $B[X]$, is disjoint from $x_0 + K(f,k)$ and so, since both $B[X]$ and $x_0 + K(f,k)$ are convex, the Eidelheit separation theorem applies to give $f_0 \in S(X^*)$ such that $\operatorname{Sup} f_0(B[X]) \le \inf f_0(x_0 + K(f,k))$.

But, then $f(x_0) \le 1 = \operatorname{Sup} f_0(B[X]) \le \inf f_0(x_0 + K(f,k))$

$$\le f_0(x_0), \text{ as } 0 \in K(f,k),$$

thus, $f_0(x_0) = 1$ (so f_0 attains its norm at x_0) and

$1 \le 1 + \inf f_0(K(f,k))$ or $0 \le \inf f_0(K(f,k))$ as required. $\Box$

LEMMA 7: *Let* $f \in S(X^*)$, $k \in (0,1)$ *and let* $f_0 \in S(X^*)$ *be such that* $K(f,k) \subseteq \{x \in X : f_0(x) \ge 0\}$. *If* $x \in S(X) \cap \operatorname{Ker} f_0$ *then* $|f(x)| \le k$.

Proof: Given any $\varepsilon > 0$, choose $y \in S(X)$ such that $\|x - y\| < \varepsilon$ and

$f_0(y) < 0$. Then $y \notin K(f,k)$ so $k = k\|y\| > f(y)$ and

$$\begin{aligned}f(x) &= f(y) + f(x - y)\\ &\le f(y) + \|x - y\|\\ &< k + \varepsilon.\end{aligned}$$

Thus $f(x) \le k$ for all $x \in S(X) \cap \operatorname{Ker} f_0$. Since $x \in S(X) \cap \operatorname{Ker} f_0$ implies $-x \in S(X) \cap \operatorname{Ker} f_0$, we have $\pm f(x) = f(\pm x) \le k$ or $|f(x)| \le k$. $\Box$

LEMMA 8: *Let* $k > 0$ *and* $f, f_0 \in S(X^*)$. *If* $|f(x)| \le k$ *whenever* $x \in S(X) \cap \operatorname{Ker} f_0$, *then either* $\|f_0 - f\| < 2k$ *or* $\|f_0 + f\| < 2k$.

Proof. Let g denote a Hahn-Banach extension of $f|_{\operatorname{Ker} f_0}$ from $\operatorname{Ker} f_0$ to X, then $\|g\| \le k$. Further, since $(f - g)(x) = 0$ for all $x \in \operatorname{Ker} f_0$ we have $\operatorname{Ker}(f - g) \supseteq \operatorname{Ker} f_0$. Thus, by the remarks on p.3, for any chosen $y_0 \notin \operatorname{Ker}(f - g)$ and any $x \in X$ we have

$$x = \frac{f_0(x)}{f_0(y_0)}\, y_0 + k$$

where $k \in \operatorname{Ker} f_0$ and also

$$x = \frac{(f - g)(x)}{(f - g)(y_0)}\, y_0 + k.$$

Since these decompositions are unique,

$$\frac{f_0(x)}{f_0(y_0)} = \frac{(f - g)(x)}{(f - g)(y_0)} \quad \text{for all } x \in X,$$

or $(f - g) = \alpha f_0$ (where $\alpha = (f - g)(y_0)/f(y_0)$).

[Note: we have just proved a particular case of the general result; "If $\operatorname{Ker} f_1 \supseteq \operatorname{Ker} f_2$ then f_1 and f_2 are linearly dependent".]

Now,

$$\big|1 - |\alpha|\big| = \big|\|f\| - \|f - g\|\big| \le \|g\| \le k \quad .$$

Thus, if $\alpha \geq 0$, then

$$\|f_0 - f\| = \|(1-\alpha)f_0 - g\| \leq |1-\alpha| + \|g\|$$
$$= |1-|\alpha|| + \|g\| \leq 2k.$$

Similarly, if $\alpha < 0$, then

$$\|f_0 + f\| = \|(1+\alpha)f_0 - g\| \leq |1+\alpha| + \|g\|$$
$$= |1-|\alpha|| + \|g\| \leq 2k. \quad \square$$

REMARK: At this point we could establish *subreflexivity*, however since a slightly stronger conclusion [observed by Béla Bollobás, Bull. London Math. Soc. 2 (1970) pp.181-182.] is easily within our grasp, we will continue.

EXERCISE: Given any $f \in S(X^*)$ and $\varepsilon > 0$, show that by setting $k = \frac{\varepsilon}{2}$ in the sequence of lemmas 4, 6, 7 and 8 we obtain a functional $f_0 \in S(X^*)$ which attains its norm and is such that either $\|f - f_0\| < \varepsilon$ or $\|f + f_0\| < \varepsilon$. Hence show that there is a support functional within distance ε of f and so conclude that X is subreflexive. [Hint. Show that if f_0 attains its norm, so does $-f_0$.]

LEMMA 9: *Let* $f \in S(X^*)$, $k \in (0,1)$ *and let* $f_0 \in S(X^*)$ *be such that* $K(f,k) \subseteq \{x \in X\colon f_0(x) \geq 0\}$, *then either* $\|f - f_0\| < 2k$ *or* $\|f + f\| < 2k$. *Further, if* $k < \frac{1}{2}$ *then the last case is impossible, that is,* $\|f - f_0\| < 2k$.

Proof. The first part of the lemma follows immediately from lemmas 7 and 8.

Now, assume $k < \frac{1}{2}$, then there exists $x \in S(X)$ such that $f(x) > 2k$ ($\|f\| = \operatorname{Sup} f(S(X^*)) = 1.$), then $f(x) > k\|x\|$ so

$x \in K(f,k)$ and $f_0(x) \geq 0$. We therefore have

$$\|f + f_0\| \geq |(f + f_0)(x)| \geq f(x) > 2k. \quad \square$$

Combining lemmas 4, 6 and 9 together with the corollary 5 we have:

THEOREM 10 ("Phelps-Brondsted-Rockafellar"): *Given* $\varepsilon > 0$, $k \in (0,\frac{1}{2})$, $f \in S(X^*)$ *and* $x \in S(X)$ *such that* $f(x) > 1 - \varepsilon$, *there exists* $f_0 \in S(X^*)$ *and* $x_0 \in S(X)$ *such that:*

i) $f_0(x_0) = 1$;

ii) $\|x - x_0\| < \varepsilon/k$;

and iii) $\|f - f_0\| < 2k$.

REMARK: The Phelps-Brondsted-Rockafellar proof [see Holmes pp.165 and 166] is shorter and sharper, but I believe less "transparent", than ours. They obtain the conclusion with $k \in (0,1)$ and iii) replaced by $\|f - f_0\| < k$, though f_0 is not guaranteed to be of norm 1 and so i) is replaced by $f_0(x_0) = \|f_0\|$.

As a corollary of Theorem 10 we have

THEOREM 11: *Given* $\varepsilon > 0$ *and* $f \in S(X^*)$, $x \in S(X)$ *with* $f(x) > 1 - \varepsilon^2/2$, *there exists* $f_0 \in S(X^*)$ *and* $x_0 \in S(X)$ *such that:*

i) $f_0(x_0) = 1$ (<u>ie</u>, f_0 attains its norm at x_0);

ii) $\|x - x_0\| < \varepsilon$;

and iii) $\|f - f_0\| < \varepsilon$.

In particular every Banach space is subreflexive.

Proof. Replacing ε in theorem 10 by $\varepsilon^2/2$ and taking $k = \varepsilon/2$ (which, without loss of generality we may assume is less than $\frac{1}{2}$), we have:

there exists $f_0 \in S(X^*)$ and $x_0 \in S(X)$ such that

i) $f_0(x_0) = 1$;

ii) $\|x - x_0\| < \varepsilon^2/2k = (\varepsilon^2/2)/(\varepsilon/2) = \varepsilon$;

and iii) $\|f - f_0\| < 2k = 2\varepsilon/2 = \varepsilon$, as required. □

Some final REMARKS.

With some obvious modifications in definitions, lemma 4, corollary 5 and Theorem 10 remain valid if $B[X]$ is replaced by any closed bounded convex set B in X. The proofs remain essentially the same, except that, since B may have an empty interior, to apply the Eidelheit separation theorem, we must first show $K(f,k)$ has non-empty interior (do so). In this more general setting corollary 5 assumes real significance. It establishes the density in the boundary of B of points at which there exist support hyperplanes to B (ie points $x_0 \in B$ for which there exists $f_0 \in S(X^*)$ with $f_0(x_0) = \text{Sup}\, f_0(B)$). Prior to these results it was an open question whether an arbitrary closed bounded convex subset necessarily had any such points.

For these generalizations and some alternative proofs to the ones given here see both Diestel and Holmes.

§2 The Duality Map and Support Mappings

In this brief section we introduce two fundamental concepts and several miscellaneous ideas which will be of use later.

For $x \in S(X)$ let $\mathcal{D}(x)$ denote the set of support functionals for $B[X]$ at x. That is, $\mathcal{D}(x) = \{f \in S(X^*): f(x) = 1\}$. By the support theorem, 3) on p.16, $\mathcal{D}(x)$ is non-empty.

EXERCISE: Show that for any $x \in S(X)$, $\mathcal{D}(x)$ *is convex*. Also show that $\mathcal{D}(x)$ is w*-closed and so conclude that $\mathcal{D}(x)$ *is w*-compact*. [Hint: Use the Banach-Alaoglu Theorem of p.7.]

$\mathcal{D}: x \mapsto \mathcal{D}(x)$ defines a set valued mapping from $S(X)$ into the non-empty subsets of $S(X^*)$. We will refer to $\mathcal{D}$ as the Duality Map for X.

A selector for $\mathcal{D}$, that is, any mapping $\phi: S(X) \to S(X^*)$ where $\phi(x) \in \mathcal{D}(x)$, is a support mapping for X.

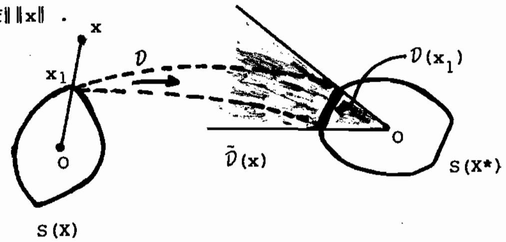
NOTATION: We will denote a typical element of $\mathcal{D}(x)$ by f_x. Further, when it is clear that a particular support mapping is being considered we will write f_x in place of $\phi(x)$.

REMARKS: 1) The Duality Map is analogous with the spherical image map introduced into differential geometry by Gauss. Indeed, when $\mathcal{D}$ was first introduced, by D.F. Cudia [Geometry of Banach Spaces. Smoothness, Trans. A. Math. Soc., 110 (1964) pp.284-314], it was termed the "Gaussian Spherical image map". Support mappings were introduced and extensively studied by J.R. Giles following earlier work by G. Lumer.

2) The duality map may be extended to all of X by defining

$$\tilde{\mathcal{D}}(x) = \begin{cases} X & \text{if } x = 0 \\ \|x\|\, \mathcal{D}(x/\|x\|) & \text{if } x \neq 0. \end{cases}$$

In this case it is more usual to extend the range of $\tilde{\mathcal{D}}$ also by replacing $\tilde{\mathcal{D}}(x)$ with the cone generated by it, $\{\lambda\, \tilde{\mathcal{D}}(x) : \lambda \geq 0\}$. The defining relationship then becomes $f \in \tilde{\mathcal{D}}(x)$ if and only if $f(x) = \|f\|\|x\|$.



Similarly, a support mapping ϕ may be extended to X by imposing the requirement that it be positive scalar homogeneous, $\phi(\lambda x) = \lambda\phi(x)$ for all $\lambda \geq 0$. Clearly, such an extended support mapping is a selector for $\tilde{\mathcal{D}}$, however not every selector for $\tilde{\mathcal{D}}$ is such a support mapping.

LEMMA 1: *Let* $x \mapsto f_x$ *and* $x \mapsto g_x$ *be two support mappings for* X, *then for* $\lambda > 0$ *and* $y \in X$ *we have*

$$g_x(y) \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(y)$$

Proof. $$g_x(y) = \frac{g_x(x + \lambda y) - g_x(x)}{\lambda}$$

$$\leq \frac{\|x + \lambda y\| - \|x\|}{\lambda}, \quad \text{as } \|g_x\| = 1 = \|x\| = g_x(x).$$

$$\leq \frac{\|x + \lambda y\| - f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(x)}{\lambda}, \quad \text{as } f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(x) \leq 1.$$

$$= \frac{f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(x + \lambda y) - f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(x)}{\lambda}, \quad \text{as } f_{\frac{x+\lambda y}{\|x+\lambda y\|}}\left(\frac{x+\lambda y}{\|x+\lambda y\|}\right) = 1.$$

$$= f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(y).$$

□

COROLLARY 2: *For* $\lambda < 0$ *we have*

$$f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(y) \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq g_x(y).$$

Proof. Replace λ by $-\lambda$ and y by $-y$ in the above lemma and multiply throughout by -1. □

EXERCISE 1) If $x \mapsto f_x$ and $x \mapsto g_x$ are two support mappings and $\lambda > 0$, deduce that, for each $y \in X$

$$f_{\frac{x-\lambda y}{\|x-\lambda y\|}}(y) \leq g_x(y) \leq f_{\frac{x+\lambda y}{\|x+\lambda y\|}}(y).$$

2) (Optional) Given any support mapping ϕ, $x \in S(X)$ and $\delta > 0$, show that

$$\mathcal{D}(B_\delta(x) \cap S(X)) \subseteq \overline{co}^{w^*}\phi(B_\delta(x) \cap S(X)).$$

[Argue as follows. Suppose there exists $x_0 \in B_\delta(x) \cap S(X)$ and $f_{x_0} \in \mathcal{D}(x_0)$ such that $f_{x_0} \notin \overline{co}^{w^*}\phi(B_\delta(x) \cap S(X))$. Use the separation theorem to obtain a $z \in S(X)$ with

$$f_{x_0}(z) > \sup\{\phi(y)(z) : y \in B_\delta(x) \cap S(X)\}.$$

From Exercise 1) deduce that, for $\lambda > 0$,

$f_{x_0}(z) \leq \phi\left(\frac{x_0 + \lambda z}{\|x_0 + \lambda z\|}\right)(z)$, and show that for λ small enough this leads to a contradication.]

The relationship between support mappings and the difference quotient in lemma 1 and Corollary 2 suggests a connection between support functionals and differentiability properties of the norm function. This connection is made more precise in what follows and is an observation basic to much of our subsequent theory.

LEMMA 3: *For fixed* $x \in S(X)$ and $y \in X$, $y \neq 0$, *the function* $\frac{\|x + \lambda y\| - \|x\|}{\lambda}$ *is increasing for* $\lambda > 0$.

Proof. Let $0 < \mu < \lambda$, then

$$\begin{aligned}\|x + \mu y\| - 1 &= \left\|\frac{\mu}{\lambda} x + \frac{\mu}{\lambda}\lambda y + \frac{\lambda-\mu}{\lambda} x\right\| - 1 \\ &\leq \frac{\mu}{\lambda} \|x + \lambda y\| + \left(\frac{\lambda-\mu}{\lambda}\right) - 1 \\ &= \frac{\mu}{\lambda}[\|x + \lambda y\| - 1]. \quad \square\end{aligned}$$

THEOREM 4: *For* $x \in S(X)$ *and* $y \in X$, $y \neq 0$,

$$g^+(x,y) = \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \quad \textit{exists.}$$

Proof. By lemma 3, $\frac{\|x + \lambda y\| - \|x\|}{\lambda}$ is decreasing as $\lambda \to 0^+$, thus to establish the existence of the limit it suffices to show that $\frac{\|x + \lambda y\| - \|x\|}{\lambda}$ is bounded below for small λ. To see this, observe that, for $\lambda < 1/\|y\|$ we have

$$\begin{aligned}\frac{\|x + \lambda y\| - \|x\|}{\lambda} &\geq \frac{|\|x\| - \lambda\|y\|| - 1}{\lambda} \\ &= \frac{1 - \lambda\|y\| - 1}{\lambda} \\ &= -\|y\| . \quad \square\end{aligned}$$

$g^+(x,y)$ is known as the <u>upper gâteaux derivative</u> of the norm at x in the direction y.

REMARK: With a little more attention to detail, the last two results can be established with any convex function in place of the norm (see, Holmes Ch.1, §7D, p.28).

EXERCISE: Show that

i) $|g^+(x,y)| \leq \|y\|$ for each $x \in S(X)$ and all $y \in X$.

ii) $g^+(x,\alpha x) = \alpha$ for all $\alpha \in \mathbb{R}$

LEMMA 5: *For fixed* $x \in S(X)$, $y \in X$, $\alpha \in \mathbb{R}$ *and* $\beta > 0$ *we have* $g^+(x,\alpha x+\beta y) = \alpha + \beta g^+(x,y)$.

Proof. For any $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}g^+(x,\alpha x+\beta y) &= \lim_{\lambda\to 0^+} \frac{\|x + \lambda(\alpha x + \beta) y\| - \|x\|}{\lambda} \\ &= \lim_{\lambda\to 0^+} \frac{(1 + \alpha\lambda)\left\|x + \frac{\beta\lambda}{1+\alpha\lambda} y\right\| - \|x\|}{\lambda}, \text{ as } |\alpha\lambda| < 1 \text{ for } \lambda \text{ small.} \\ &= \lim_{\lambda\to 0^+} \alpha\left\|x + \frac{\beta\lambda}{1+\alpha\lambda} y\right\| + \lim_{\lambda\to 0^+} \frac{\beta}{1+\alpha\lambda} \frac{\left[\left\|x+ \frac{\beta\lambda}{1+\alpha\lambda} y\right\| - \|x\|\right]}{\left[\frac{\beta\lambda}{1+\alpha\lambda}\right]} \\ &= \alpha + \beta\, g^+(x,y), \text{ since } \frac{\beta\lambda}{1+\alpha\lambda} \to 0^+ \text{ as } \lambda \to 0^+ . \quad \square\end{aligned}$$

By lemma 1, we have for any $f_x \in \mathcal{D}(x)$ that $f_x(y) \leq g^+(x,y)$. We now show that $\max\{f_x(y) : f_x \in \mathcal{D}(x)\} = g^+(x,y)$.

THEOREM 6: *Given* $x \in S(X)$ *and* $y \in X$ *there exists* $f_x \in \mathcal{D}(x)$ *such that* $f_x(y) = g^+(x,y)$.

Proof. If $y = \alpha x$, then $g^+(x,y) = \alpha$ (by Exercise ii) above.)

$= f_x(y)$ for any $f_x \in \mathcal{D}(x)$.

Now, assume x and y are linearly independent and let M be the subspace spanned by x and y.

Define f on M by

$$f(\alpha x+\beta y) = \alpha + \beta\, g^+(x,y) \quad , \text{ then}$$

$f(x) = 1$, so $\|f\| \geq 1$. It suffices to show $\|f\| = 1$, for then any Hahn-Banach extension $\tilde{f}$ of f from M to X will be an element of $\mathcal{D}(x)$ with $\tilde{f}(y) = f(y) = g^+(x,y)$. Now, for $\beta > 0$ we have

$$f(\alpha x + \beta y) = \alpha + \beta g^+(x,y) = g^+(x,\alpha x+\beta y) \quad , \text{ by lemma 5.}$$

So $|f(\alpha x + \beta y)| = |g^+(x,\alpha x+\beta y)| \leq \|\alpha x + \beta y\|$, by Exercise i) above.

On the other hand, if $\beta < 0$

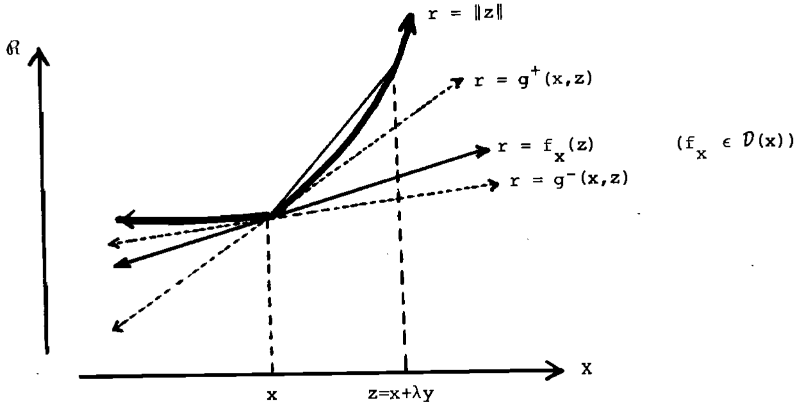
$f(\alpha x + \beta y) \;=\; -(-\alpha - \beta g^{+}(x,y)) \;=\; -g^{+}(x,-\alpha x-\beta y)$, again by lemma 5.

So $|f(\alpha x + \beta y)| \;=\; |g^{+}(x,-\alpha x-\beta y)| \;\le\; \|-\alpha x - \beta y\| \;=\; \|\alpha x + \beta y\|$.

Thus, $\|f\| \le 1$ as required. $\square$

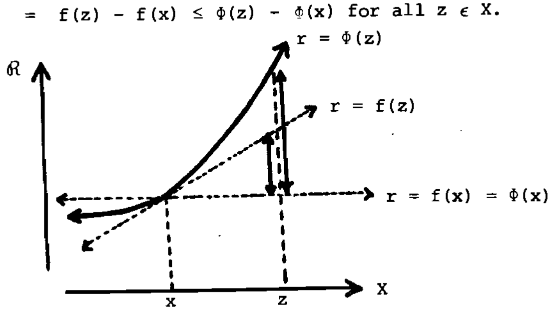
7: *The lower gâteaux derivative* $g^{-}(x,y) = \lim\limits_{\lambda \to 0^{-}} \dfrac{\|x + \lambda y\| - \|x\|}{\lambda}$ *exists and equals* $\mathrm{Min}\{f_x(y) : f_x \in \mathcal{D}(x)\}$. To see this, note that from their respective definitions $g^{-}(x,y) = -g^{+}(x,-y)$.

The conclusions of lemma 1 and Theorem 6 may be represented diagramatically as follows.



We conclude this section with an assortment of results which are not vital for subsequent work. They do however provide additional background.

A] (*Sub-gradients*) For any convex function $\phi: X \to \mathbb{R}$, we say $f \in S(X^*)$ is a sub-gradient (sub-differential) to ϕ at $x \in S(X)$ if $f(z - x) \;=\; f(z) - f(x) \le \phi(z) - \phi(x)$ for all $z \in X$.



A comparison of the last two diagrams suggests that any element of $\mathcal{D}(x)$ is a sub-gradient of the norm function at x. Indeed we have:

PROBLEM: Show that $f \in \mathcal{D}(x)$ if and only if f is a subgradient of the norm function at x.

B] PROBLEM: For any extended support mapping $x \mapsto f_x$, show that the following *primitive parallelogram law* holds.

$$(f_x + f_y)(x + y) + (f_x - f_y)(x - y) = 2\|x\|^2 + 2\|y\|^2.$$

Hence deduce that $\mathcal{D}$ *is a monotone* map, that is, for any $x, y \in S(X)$ and $f_x \in \mathcal{D}(x)$, $f_y \in \mathcal{D}(y)$ we have $(f_x - f_y)(x - y) \ge 0$.

C] (*Hilbert space*) Let H be a Hilbert space, show that for $x, y \in S(H)$ (x,y) [illegible] $= 1$ if and only if $x = y$.

Recalling, RIESZ' REPRESENTATION THEOREM: $f \in H^*$ *if and only if* $f(y) = (y,x)$ *for all* $y \in H$ *and some* $x \in H$, show that $\mathcal{D}(x)$ is a singleton set for each $x \in S(H)$. Hence conclude that there is a unique support map for H, which is given by $x \mapsto f_x = (-,x)$.

D] (*semi-inner-products*) G. Lumer [Semi-inner-product spaces, Trans. Amer. Maths. Soc. 100 (1961) pp.29-43] introduced the notion of a semi-inner-product for arbitrary Banach spaces, that is a mapping $(-,-): X \times X \to \mathbb{R}$ which satisfies:

i) $|(x,y)| \le \|x\|\|y\|$ and $(x,x) = \|x\|^2$

ii) $(x+y,z) = (x,z) + (y,z)$

iii) $(\lambda x,y) = \lambda(x,y)$.

PROBLEM: Show that $\phi: X \to X^*$ is an extended support mapping if and only if $(x,y) = \phi(y)(x)$ is a semi-inner-product for X. This shows that every Banach space admits a semi-inner-product.

E] (*orthogonality*) In a Hilbert space we say two elements x, y are orthogonal (written $x \perp y$) if $(x,y) = 0$. Show that $(x,y) = 0$ if and only if $\|x\| \le \|x + \lambda y\|$ for all $\lambda \in \mathcal{R}$.

G. Birkhoff [Orthogonality in linear metric spaces, Duke Maths. J. 1 (1935) pp.169-172] used the relationship $\|x\| \le \|x + \lambda y\|$ for all $\lambda \in \mathcal{R}$ as a definition of "$x \perp y$" in arbitrary Banach spaces. This notion of orthogonality was subsequently studied by R.C. James [Orthogonality in normed linear spaces, Duke Maths. J. 12 (1945) pp.291-302] and has become known as James' Orthogonality (see Diestel).

PROBLEM: For $x, y \in S(X)$, show that $x \perp y$ (in the sense of James') if and only if there exists $f_x \in \mathcal{D}(x)$ with $f_x(y) = 0$.
Hence conclude that $x \perp y$ if and only if for some semi-inner-product on X we have $(y,x) = 0$.
Give an example to show that $x \perp y$ need not imply $y \perp x$. (Hint: try in $\ell_\infty^{(2)}$.)

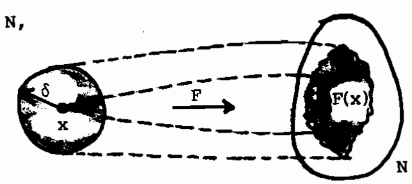
F] (*Upper-semi-continuity*) If $\mathcal{T}$ is any linear space topology on X^* and F is any set valued mapping from $S(X)$ to $S(X^*)$ we say F is upper-upper-semi-continuous norm to $\mathcal{T}$ at x, if given any $\mathcal{T}$-neighbourhood, N, with $F(x) \subseteq N$ there exists $\delta > 0$ such that

$$F(B_\delta(x) \cap S(X)) \subseteq N,$$

or equivalently if

$x_n \xrightarrow{\|\cdot\|} x$, then eventually

$F(x_n) \subseteq N$.



PROBLEM: Show that the duality map $\mathcal{D}$ is upper-semi-continuous norm to w^* at every $x \in S(X)$.

§3 Smoothness

DEFINITION: $x_0 \in S(X)$ is a smooth point if there is a unique supporting hyperplane to $B[X]$ at x_0. That is, if $\mathcal{D}(x_0)$ has only one element or, if all support mappings coincide at x_0.

If every point of $S(X)$ is a smooth point, then we say X is a smooth Banach space.

EXAMPLES: 1) From C] p.33, *every Hilbert space is smooth.*

2) Example 1) p.16 shows that $(0,-1)$ is not a smooth point of $\ell_1^{(2)}$, while all points of the form $(\alpha,1-\alpha)$ with $0 < \alpha < 1$ are smooth points.

3) The argument used in Example 2) p.17 shows that $(1,1,1,\ldots) \in S(\ell_\infty)$ is the only possible support functional for $(\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots) \in S(\ell_1)$. Thus $(\frac{1}{2},\frac{1}{4},\ldots,\frac{1}{2}n,\ldots)$ is a smooth point of ℓ_1.

Similarly $(1,0,0,0,\ldots)$ is a smooth point of c_0 [see, Exercise 1) p.17.].

EXERCISE: Characterize the smooth points of c_0 and ℓ_1.

Smoothness and related concepts were extensively investigated by V.L. Šmul'yan [see for example; Sur la derivabilité de la norme dans l'espace de Banach, Doklady (CR Acd. Sci. U.S.S.R.), 27 (1940), pp.643-648.]. We begin with one of his characterizations.

THEOREM 1: $x_0 \in S(X)$ *is a smooth point if and only if whenever* $(f_n) \subset S(X^*)$ *has* $f_n(x_0) \to 1$ *we have that* (f_n) *is* w^**-convergent.*

Proof. ($\Leftarrow$) Let $f, g \in \mathcal{D}(x_0)$, then the sequence $(h_n) = (f,g,f,g,f,\ldots)$ is such that $h_n(x_0) \to 1$ (indeed $h_n(x_0) = 1$ for all n). Thus $h_n \xrightarrow{w^*} h$ for some h, which implies $f(y) = g(y)$ for all $y \in X$, that is, $f = g$ and so, x_0 is a smooth point.

($\Rightarrow$) Let $f_n(x_0) \to 1$ where x_0 is a smooth point, then if $f_n \overset{w^*}{\to} f$ we have $f(x_0) = \underset{n}{\text{Limit}}\, f_n(x_0) = 1$, so $f = f_0$ the unique support functional at x_0.

Now, assume $f_n \overset{w^*}{\not\to} f_0$, then there exists an open w^*-neighbourhood N of f_0 and a subnet (f_{n_α}) of (f_n) with $f_{n_\alpha} \notin N$ for any α. Since $B[X^*]$ is w^*-compact, there exists a subnet $(f_{n_{\alpha_\beta}})$ of (f_{n_α}) with $f_{n_{\alpha_\beta}} \overset{w^*}{\to} f$ for some $f \in B[X^*]$. Further, since $X^* \setminus N$ is w^*-closed, $f \notin N$. *But*, $f(x_0) = \underset{\beta}{\text{Limit}}\, f_{n_{\alpha_\beta}}(x_0) = 1$, so $f = f_0 \in N$, a contradiction. □

In the next theorem we collect together a number of important equivalences to smoothness. Several of these characterizations can be deduced easily from results of the last section (see Exercise 1 immediately following the proof). The proof given below does not require any of the work in §2 beyond lemma 1 and its corollary.

THEOREM 2. *Let $x_0 \in S(X)$, then the following are equivalent.*

i) *x_0 is a smooth point.*

ii) *All support mappings are continuous norm to w^* at x_0.*

iii) *There exists a support mapping which is norm to w^* continuous at x_0.*

iv) *The norm function is Gâteaux differentiable at x_0,* that is $g^+(x_0,y) = g^-(x_0,y)$ for all $y \in X$ or the Gâteaux derivative at x_0 in the direction y, $g(x_0,y) = \underset{\lambda\to 0}{\text{limit}} \dfrac{\|x_0+\lambda y\| - \|x_0\|}{\lambda}$, exists for each $y \in X$.

Proof i) $\Rightarrow$ ii) Let $x \mapsto f_x$ be any support mapping, then if $\|x_n - x_0\| \to 0$ we have

$$\begin{aligned} |f_{x_n}(x_0) - 1| &= |f_{x_n}(x_0) - f_{x_n}(x_n)| \\ &\le \|f_{x_n}\|\,\|x_n - x_0\| \\ &= \|x_n - x_0\| \to 0. \end{aligned}$$

Thus, by Theorem 1, $f_{x_n} \overset{x^*}{\to} f_{x_0}$, establishing continuity.

ii) $\Rightarrow$ iii) Obvious.

iii) $\Rightarrow$ iv) For any $y \in X$ $\dfrac{x_0+\lambda y}{\|x_0+\lambda y\|} \to x_0$ as $\lambda \to 0^+$.

Thus, if the support mapping $x \mapsto f_x$ is continuous norm to w^* we have

$$f_{\frac{x_0+\lambda y}{\|x_0+\lambda y\|}} \overset{w^*}{\to} f_{x_0} \qquad \text{as } \lambda \to 0^+ .$$

So, $f_{x_0}(y) = \underset{\lambda\to 0^+}{\text{Limit}}\, f_{\frac{x_0+\lambda y}{\|x_0+\lambda y\|}}(y)$ and applying lemma 1 of §2, p.28, with $g_x = f_x$, yields

$$f_{x_0}(y) = \underset{\lambda\to 0^+}{\text{Limit}} \frac{\|x_0+\lambda y\| - \|x_0\|}{\lambda} = g^+(x_0,y) ,$$

but then,

$$g^-(x_0,y) = -g^+(x_0,-y) = -f_{x_0}(-y) = f_{x_0}(y) = g^+(x_0,y) .$$

iv) $\Rightarrow$ i) Let $\lambda > 0$, then if $x \mapsto g_x$ is any support mapping

$g(x_0,y) = g^+(x_0,y) \ge g_{x_0}(y)$ by lemma 1 of §2, and

$$g(x_0,y) = \underset{\lambda\to 0}{\text{limit}} \frac{\|x_0-\lambda y\| - \|x_0\|}{\lambda} \le g_{x_0}(y) ,$$

by Corollary 2 of §2.

Thus for any support mapping $x \mapsto g_x$ and $y \in X$, $g_{x_0}(y) = g(x_0,y)$ and so all support mappings coincide at x_0. □

EXERCISES: 1) a) Deduce the equivalence of i) and iv) in the above theorem from Theorem 6 and 7] of §2 (p.31 and 32).

b) Using F], p.34 of §2, deduce that i) $\Rightarrow$ ii)

*c) Using Exercise 2 on p.29 of §2 deduce that iii) $\Rightarrow$ ii).

2) Show that the norm is Gâteaux differentiable at x_0 in every direction if and only if $g^+(x_0,y)$ is a linear function of y.

rive as Corollary to:
$f \in S(X^*)$ sm. pt. iff whenever $(x_n) \subset S(X)$ is s.t. $f(x_n) \to 1$, then (x_n) is w-Cauchy.
(proof in back)

A sometimes useful result is a "pre-dual" characterization for smooth points of X^* analogous to Theorem 1 and also due to Šmul'yan.

THEOREM 3: *Let* f *be a support functional for* $B[X]$, *that is* $f \in \mathcal{D}(S(X))$. *Then* f *is a smooth point of* $S(X^*)$ *if and only if whenever* $(x_n) \subset S(X)$ *has* $f(x_n) \to 1$ *we have that* (x_n) *is w-convergent.*

Proof. Observe that since $f \in \mathcal{D}(S(X))$, there exists $x \in S(X)$ such that $f(x) = 1$, thus $\hat{x} \in \mathcal{D}(f)$.

($\Rightarrow$) If f is a smooth point $\hat{x}$ is the unique element of $\mathcal{D}(f)$. Now, if $f(x_n) \to 1$, then $\hat{x}_n(f) \to 1$ and so by Theorem 1, $\hat{x}_n \overset{w^*}{\to} \hat{x}$ or $x_n \overset{w}{\to} x$.

($\Leftarrow$) Assume f is not a smooth point, so there exists $F \in \mathcal{D}(f)$, $F \neq \hat{x}$. Since X^* separates points of X^{**} we may choose a $g \in X^*$ and $k \in \mathbb{R}$ such that $F(g) > k > \hat{x}(g) = g(x)$. For each n let

$$U_n = \{G \in X^{**}: G(g) > k \quad \text{and} \quad G(f) > 1 - \tfrac{1}{n}\},$$

then $U_n \neq \emptyset$, as $F \in U_n$, and U_n is w^*-open. By Goldsteine's Theorem (p.7), there exists $x_n \in B[X]$ such that $\hat{x}_n \in U_n$. The sequence $(\hat{x}_n)$ is therefore such that $\hat{x}_n(f) > 1 - \frac{1}{n}$ and so $f(x_n) \to 1$. Thus, x_n is w-convergent to some $y \in B[X]$. Further, $f(y) = \text{limit } f(x_n) = 1$, so $\hat{y} \in \mathcal{D}(f)$. Now, the sequence $(z_n) = (x,y,x,y,x,\ldots)$ is such that $f(z_n) \to 1$, thus z_n is w-convergent and so $x = y$, but this is a contradiction, as $g(x) < k$ while $g(y) = \text{limit } g(x_n) = \text{limit } \hat{x}_n(g) \geq k$. $\square$

EXERCISE: Show that $(1,0,0,0,\ldots)$ is a smooth point of ℓ_∞.

We now consider the important question of existence of smooth points.

EXAMPLE (*a space with no smooth points*):

The sequence spaces listed on p.11 can be generalized to uncountable index sets. In particular, let Γ be any uncountable set and

define $\ell_1(\Gamma)$ to be the set of functions

$$x: \Gamma \to \mathbb{R} : \gamma \mapsto x_\gamma$$

such that

i) $x_\gamma = 0$ for all but a countable number of $\gamma \in \Gamma$

and ii) $\sum_{\gamma \in \Gamma} |x_\gamma| < \infty$.

[Note: the sum in ii) makes sense, since by i) it is really an absolutely convergent series and so is independent of any ordering on Γ.]

$\ell_1(\Gamma)$ is a non-separable Banach space with norm defined by $\|x\|_1 = \sum_{\gamma \in \Gamma} |x_\gamma|$.
The dual of $\ell_1(\Gamma)$ is $\ell_\infty(\Gamma)$ the set of bounded functions $f: \Gamma \to \mathbb{R} : \gamma \mapsto f_\gamma$, with norm defined by $\|f\|_\infty = \sup_{\gamma \in \Gamma} |f_\gamma|$.
For $x \in \ell_1(\Gamma)$ and $f \in \ell_\infty(\Gamma)$ $\quad f(x) = \sum_{\gamma \in \Gamma} f_\gamma x_\gamma$.

To see that any $x \in S(\ell_1(\Gamma))$ is not a smooth point it suffices to note that $f \in \mathcal{D}(x)$ whenever f has the form $f_\gamma = sgn\ x_\gamma$ if $x_\gamma \neq 0$ and $|f_\gamma| \leq 1$ for the remaining, uncountably many, values of γ. (Here and elsewhere, $sgn\ \lambda = \lambda/|\lambda|$ for $\lambda \neq 0$.)

In contrast to the above example we have

THEOREM 4 (MAZUR, <u>über konvexe mengen in linearem normierten Räumen</u>, Studia Math. 4 (1933) pp.70-84.): *In a separable Banach space* X *the set of smooth points is a dense* G_δ *subset of* $S(X)$.

The proof of Mazur's Theorem occupies the remainder of this section. By definition, a G_δ set is the intersection of a countable number of open sets. Further, since X is complete, Baire's Category Theorem shows that any countable intersection of dense open sets is itself dense. (This follows immediately from De Morgan's rules of set theory and the observation that the complement of a dense open subset is nowhere dense.)

Thus, to show the set of smooth points is a dense G_δ subset of $S(X)$ it suffices to show it is a countable intersection of dense open subsets of $S(X)$.

[These remarks also indicate the significance of "G_δ" in the above theorem: *The intersection of the set of smooth points with any other dense open subset is itself a* <u>*dense*</u> *G_δ subset.*]

Since X is separable there exists a dense sequence (y_n) in $S(X)$. For any m, n $\in N$, let

$$G_{m,n} = \{x \in S(X): \text{ For all } f_x, g_x \in \mathcal{D}(x),\ (f_x - g_x)(y_n) < \tfrac{1}{m}\},$$

or equivalently, by Theorem 6 and 7) of §2, p.31, 32,

$$G_{m,n} = \{x \in S(X): g^+(x,y_n) - g^-(x,y_n) < \tfrac{1}{m}\} .$$

For each m and n it is clear that $G_{m,n}$ contains the smooth points of $S(X)$.

Conversely, if x is not a smooth point, then there exist two distinct elements $f_x, g_x \in \mathcal{D}(x)$. Since X separates the points of X^* there exists $y_0 \in S(X)$ and $m \in N$ such that $(f_x - g_x)(y_0) > \frac{1}{m}$, but then, by their density in $S(X)$ there exists y_n such that $(f_x - g_x)(y_n) \geq \frac{1}{m}$, so $x \notin G_{m,n}$. Thus, the set of smooth points of $S(X)$ is precisely $\bigcap_{m,n} G_{m,n}$. To establish Mazur's Theorem it is therefore sufficient to show $G_{m,n}$ is a dense open subset of $S(X)$ for each m, n $\in N$.

We first prove $G_{m,n}$ is open in $S(X)$ by showing that $F_{m,n} = S(X)\backslash G_{m,n}$ is closed.

LEMMA 5: $F_{m,n} = \{x \in S(X)$: *there exists* $f_x, g_x \in \mathcal{D}(x)$ *with* $(f_x - g_x)(y_n) \geq \frac{1}{m}\}$ *is closed.*

Proof. Let $(x_k) \subseteq F_{m,n}$ be such that $x_k \to x$, and for each k let $f_k, g_k \in \mathcal{D}(x_k)$ be such that $(f_k - g_k)(y_n) \geq \frac{1}{m}$. Since $B[X^*]$ is w^*-compact, there exists a subnet of (f_k), (f_{k_α}) with $f_{k_\alpha} \xrightarrow{w^*} f$ for some $f \in B[X^*]$. Similarly there exists a subnet of (g_{k_α}), $(g_{k_{\alpha_\beta}})$ such that $g_{k_{\alpha_\beta}} \xrightarrow{w^*} g$ for some $g \in B[X^*]$.

Now,

$$\begin{aligned} |f(x) - 1| &\leq |(f - f_{k_\alpha})(x)| + |f_{k_\alpha}(x) - 1| \\ &= |(f - f_{k_\alpha})(x)| + |f_{k_\alpha}(x) - f_{k_\alpha}(x_{k_\alpha})| \\ &\leq |(f - f_{k_\alpha})(x)| + \|f_{k_\alpha}\| \, \|x - x_{k_\alpha}\| \\ &\to 0 \quad \text{as } f_{k_\alpha} \xrightarrow{w^*} f \text{ and } \|x - x_{k_\alpha}\| \to 0. \end{aligned}$$

So $f \in \mathcal{D}(x)$.

Similarly, $g \in \mathcal{D}(x)$.

Further, $(f - g)(y_n) = \underset{\beta}{\text{limit}}(f_{k_{\alpha_\beta}} - g_{k_{\alpha_\beta}})(y_n) \geq \frac{1}{m}$, so $x \in F_{m,n}$. □

We complete the proof of Mazur's Theorem by establishing the density of each $G_{m,n}$ in $S(X)$.

For any x, y $\in$ X let $D^+\|x + ry\|(\lambda)$ denote the "right-hand" derivative at $\lambda \in \mathbb{R}$ of the real valued function of a real variable $r \mapsto \|x + ry\|$. That is,

$$\begin{aligned} D^+ \|x + ry\|(\lambda) &= \underset{r\to\lambda^+}{\text{limit}} \frac{\|x + ry\| - \|x + \lambda y\|}{r - \lambda} \\ &= \underset{h\to 0^+}{\text{limit}} \frac{\|x + \lambda y + hy\| - \|x + \lambda y\|}{h} \end{aligned}$$

$$= \lim_{h\to 0^+} \frac{\left\| \frac{x+\lambda y}{\|x+\lambda y\|} + \frac{h}{\|x+\lambda y\|} y \right\| - 1}{\frac{h}{\|x+\lambda y\|}}$$

$$= g^+\left(\frac{x+\lambda y}{\|x+\lambda y\|}, y\right), \text{ provided } x+\lambda y \neq 0.$$

Similarly, $D^-\|x + ry\|(\lambda) = \lim_{r\to\lambda^-} \frac{\|x + ry\| - \|x + \lambda y\|}{r - \lambda}$

$$= g^-\left(\frac{x+\lambda y}{\|x+\lambda y\|}, y\right).$$

The final lemma will show that the set of positive λ for which

$$D^+\|x + ry\|(\lambda) - D^-\|x + ry\|(\lambda) \geq \frac{1}{m}$$

is finite.

In particular then, taking $y = y_n$ and putting $z_\lambda = \frac{x+\lambda y}{\|x+\lambda y_n\|}$ we have for any $x \in S(X)$ that the set of positive λ for which $g^+(z_\lambda, y) - g^-(z_\lambda, y) \geq \frac{1}{m}$, is finite.

Thus, it is possible to choose a sequence of scalars (λ_k) such that $\lambda_k \to 0^+$ and $z_{\lambda_k} \in G_{m,n}$. So since $z_{\lambda_k} = \frac{x+\lambda_k y_n}{\|x+\lambda_k y\|} \to x$, we conclude that $G_{m,n}$ is dense in $S(X)$.

Lemma 6: *The set of positive values of λ for which*

$$D^+\|x + ry\|(\lambda) - D^-\|x + ry\|(\lambda) \geq \frac{1}{m}$$

is a finite set.

Proof. Let $0 < \lambda_1 < \lambda_2$ then

$$D^+\|x + ry\|(\lambda_1) = \lim_{r\to\lambda_1^+} \frac{\|x + ry\| - \|x + \lambda_1 y\|}{r - \lambda_1}$$

$$\leq \frac{\|x + \lambda_2 y\| - \|x + \lambda_1 y\|}{\lambda_2 - \lambda_1}, \quad \text{by lemma 3 of §2, p.30.}$$

$$= \frac{\|x + \lambda_1 y\| - \|x + \lambda_2 y\|}{\lambda_1 - \lambda_2}$$

$$\leq \lim_{r\to\lambda_2^-} \frac{\|x + ry\| - \|x + \lambda_2 y\|}{r - \lambda_2}, \text{ again by lemma 3 of §2.}$$

$$= D^-\|x + ry\|(\lambda_2)$$

$$= g^-\left(\frac{x + \lambda_2 y}{\|x + \lambda_2 y\|}, y\right), \text{ by above.}$$

$$\leq g^+\left(\frac{x + \lambda_2 y}{\|x + \lambda_2 y\|}, y\right), \text{ by Theorem 6 and 7) of §2.}$$

$$= D^+\|x + ry\|(\lambda_2), \text{ again by above.}$$

$$\leq \|y\|, \text{ by Exercise i) on p.31.}$$

This shows that $D^+\|x + ry\|(\lambda)$ is an increasing function of λ which is bounded above.

Further, taking λ_2 to be a point at which

$$D^+\|x + ry\|(\lambda_2) - D^-\|x + ry\|(\lambda_2) \geq \frac{1}{m}$$

we see that, for all $\lambda < \lambda_2$

$$D^+\|x + ry\|(\lambda) \leq D^+\|x + ry\|(\lambda_2) - \frac{1}{m}.$$

So, λ_2 is a point at which the value of $D^+\|x + ry\|(\lambda)$ "jumps" by at least $\frac{1}{m}$. Clearly there can only be a finite number of such points. □

REMARK: Our proof of the last lemma is a specialization of the "classical" proof that *any convex function on an open interval of real numbers fails to be differentiable for at most a countable number of points.*

Some final REMARKS

1) With only minor modifications to the proof, Mazur's Theorem can be generalized to show: *If Φ is any continuous convex function on a separable space* X *then the set of points in* S(X) *at which Φ is Gâteaux differentiable is a dense G_δ subset.*

2) The extension of Mazur's Theorem to spaces other than separable ones is a problem of current interest. (It is known to hold in a wide class of spaces, including all reflexive spaces.) In 1968 [Acta Mathematica 121, pp.31-47] Edgar Asplund initiated the study of spaces for which the generalized Mazur Theorem (above) holds. He called such spaces "weak-differentiability spaces". Since his death they have become known as weak Asplund Spaces, and have been the subject of considerable interest (see Day §4, Ch.7 for some details), though many open questions still remain, for example: *Is every smooth space a weak Asplund space?*

§4 Rotundity

Let P, P' be two Banach space properties (for example smoothness). We say P' is a *dual property* to P if

$$X \text{ has } P \Leftrightarrow X^* \text{ has } P'$$

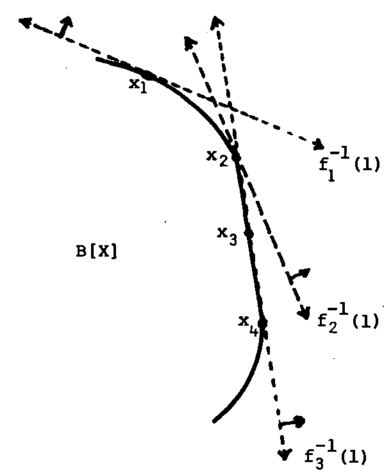
P and P' are in (complete) duality if each is a dual property of the other, that is X has P (P') $\Leftrightarrow$ X* has P' (P). When some, but not all, of the above implications hold we will speak of "partial duality"; for example, when X* has P' $\Rightarrow$ X has P.

In this section we are interested in properties which are in duality with smoothness.

We begin by strengthening the notion of support.

DEFINITION: $x \in S(X)$ is an exposed point of B[X] if there exists a support hyperplane H to B[X] such that $H \cap B[X] = \{x\}$. That is, if there exists $f \in S(X^*)$ such that $f(x) > f(y)$ for all $y \in B[X]$, $y \neq x$. In this case we say x is exposed by f. (Note: such an f is necessarily an element of $\mathcal{D}(x)$).

Some possibilities are illustrated below.



x_1 exposed by f_1.

x_2 exposed by f_2 but not by f_3.

x_3 and x_4 are not exposed points.

[You might like to construct specific (2 dimensional) spaces in which each of these possibilities occur.]

If $f \in S(X^*)$ is exposed by F where $F = \hat{x} \in \hat{X}$, so F is w*-continuous, then we say f is *w*-exposed* by $\hat{x}$ (or simply x).

Our starting point is provided by

PROPOSITION 1: *x is a smooth point of $S(X)$ if and only if there exists an $f \in S(X^*)$ which is w*-exposed by $\hat{x}$.*

Proof. $\mathcal{D}(x) = \hat{x}^{-1}(1) \cap B[X^*]$ so, if x is a smooth point, $\hat{x}$ w*-exposes the unique element of $\mathcal{D}(x)$. Conversely if $\hat{x}$ w*-exposes f then f is the unique element of $\hat{x}^{-1}(1) \cap B[X^*]$ and x is a smooth point. □

COROLLARY 2: *f is w*-exposed by $\hat{x}$ if and only if whenever $(f_n) \subset S(X^*)$ is such that $f_n(x) \to 1$, then we have $f_n \overset{w^*}{\to} f$.*

Proof. Replace "smooth" in the above proposition by the Šmul'yan equivalent given in Theorem 1 of §3, p.35, noting that f is the only possible w* limit of (f_n). □

EXERCISE 3: Let $x \in S(X)$. Show that the following are equivalent for $f_x \in \mathcal{D}(x)$.

i) f_x is a smooth point of $S(X^*)$.

ii) $\hat{x}$ is w*-exposed by $\hat{f}_x$.

iii) if (x_n) is such that $f_x(x_n) \to 1$, then $x_n \overset{w}{\to} x$.

Show that any (and hence all) of these conditions implies that x is an exposed point. (Would you consider it likely that the converse is generally true?)

LOCAL RESULTS

DEFINITION: $x \in B[X]$ is an <u>extreme point</u> of $B[X]$ if whenever $x_1, x_2 \in B[X]$ are such that $x = \frac{1}{2}(x_1 + x_2)$, then we have $x_1 = x_2 = x$. [A point which is not an extreme point is termed a "passing point".]

EXAMPLE: The points x_1, x_2 and x_4 of the previous diagram are extreme points of $B[X]$, x_3 is a passing point.

EXERCISE: i) Show that x is an extreme point of $B[X]$ if and only if x does not belong to any open line segment in $B[X]$.

ii) Show that the only possible extreme points of $B[X]$ are points of $S(X)$.

iii) Show that x is an extreme point of $B[X]$ if and only if whenever y is such that $\|x + y\| = \|x - y\| = 1$ then $y = 0$.

PROPOSITION 4: *If x is an exposed point of $B[X]$, then x is an extreme point.* (That the converse need not hold, is illustrated by the point x_4 of the previous diagram.)

Proof. Assume $x \in S(X)$ is not an extreme point of $B[X]$. That is, there exists $x_1, x_2 \in B[X]$, $x_1 \neq x_2$ with $x = \frac{1}{2}(x_1 + x_2)$. Now, let $f \in S(X^*)$ be such that $f(x) = 1$, then $\frac{1}{2}(f(x_1) + f(x_2)) = 1$ while $f(x_1), f(x_2) \leq 1$ and so $f(x_1) = f(x_2) = 1$. Thus f does not expose x, and so x is not exposed by any $f \in S(X^*)$. □

COROLLARY 5: *If $x \in S(X)$ is a smooth point of $B[X]$, then the unique element of $\mathcal{D}(x)$ is an extreme point of $B[X^*]$.*

Proof. Immediate from Proposition 4 and (the proof) of Proposition 1. □

EXERCISE 1: i) Prove that: if $f_x \in \mathcal{D}(x)$ is a smooth point of $B[X^*]$, then x is an extreme point of $B[X]$

ii) Show that $B[c_0]$ has no extreme points. Hence, deduce that no support functional in $S(\ell_1)$ is a smooth point.

2) Characterize the extreme points of $B[\ell_1]$, $B[\ell_\infty]$ and $B[C[a,b]]$.

PROBLEM (Optional): Say that $x \in S(X)$ is *fully exposed* if x is exposed by every element of $\mathcal{D}(x)$. (The point x_2 of the previous diagram is not fully exposed, though it is exposed.) Such points were implicitly considered by Ruston (1949) and explicitly by Giles (1976), who calls them "Rotund points". Show that

i) (a) x is fully exposed if every point of $\mathcal{D}(x)$ is a smooth point;

(b) if $f \in S(X^*)$ is fully exposed, then every point of $\mathcal{D}^{-1}(f) = \{x \in S(X): f(x) = 1\}$ is a smooth point. (Note: $\mathcal{D}^{-1}(f)$ may be empty).

and ii) $x \in S(X)$ is fully exposed if and only if $\|x + y\| = 2$, $y \in S(X)$ implies $y = x$.

GLOBAL RESULTS

From our work so far we have:

PROPOSITION 6: *The following are equivalent.*

i) *X is smooth ;*

ii) *every point of $\mathcal{D}(S(X))$ is a w*-exposed point ;*

iii) *every point of $\mathcal{D}(S(X))$ is an extreme point.*

Proof. i) $\Rightarrow$ ii) is immediated from Proposition 1.

ii) $\Rightarrow$ iii) is immediated from Proposition 4.

iii) $\Rightarrow$ i) assume x is not a smooth point, then there exists f_1, $f_2 \in \mathcal{D}(x)$ with $f_1 \neq f_2$. Let $f = \frac{1}{2}(f_1 + f_2)$, then $f \in \mathcal{D}(x)$ and f is not an extreme point. $\square$

If every point of $\mathcal{D}(S(X))$ is a smooth point, then, by Exercise 3, every element of $S(\hat{X})$ is w*-exposed and so every element of $S(X)$ is an exposed point. We therefore have:

If every point of $\mathcal{D}(S(X))$ is a smooth point, then every point of $S(X)$ is an extreme point.

DEFINITION: X is <u>rotund</u> if every point of $S(X)$ is an extreme point of $B[X]$.

PROPOSITION 7: *The following are equivalent:*

i) *X is rotund;*

ii) *If $x, y \in S(X)$ and $\left\|\frac{x + y}{2}\right\| = 1$, then $x = y$;*

iii) *every point of $S(X)$ is a (fully) exposed point.*

Proof. i) $\Rightarrow$ ii) For $x, y \in S(X)$ let $z = \frac{1}{2}(x + y)$, if $\|z\| = 1$, then z is an extreme point (definition of rotundity) and so $x = y = z$.

ii) $\Rightarrow$ iii) For any $x \in S(X)$ and $f \in \mathcal{D}(x)$, if $f(y) = 1$ and $y \in S(X)$ then $1 \geq \left\|\frac{x + y}{2}\right\| \geq f\left(\frac{x + y}{2}\right) = 1$ and so $y = x$. Thus f exposes x.

iii) $\Rightarrow$ i) follows from propostion 4. $\square$

Some other useful characterizations of rotundity are provided by

EXERCISE 8: Show that each of the following is equivalent to X being rotund.

i) $S(X)$ contains no non-trivial line segment.

ii) For $x, y \in S(X)$, $x \neq y$ and $\lambda \in (0,1)$, $\|\lambda x + (1-\lambda)y\| < 1$.

iii) If $x, y \in X$ are such that $\|x + y\| = \|x\| + \|y\|$ and $y \neq 0$, then $x = \lambda y$ for some λ.

iv) Every subspace of X is rotund.

v) Every 2-dimensional subspace of X is rotund.

vi) For every convex subset C of X and every $x \in X$ there is at most one best approximation from C to x.

vii) Every $f \in S(X^*)$ attains its norm on $S(X)$ at most once, that is, if $x, y \in S(X)$ and $x \neq y$, then $\mathcal{D}(x) \cap \mathcal{D}(y) = \emptyset$.

THEOREM 9 (V. Klee, 1953) i) *If X^* is smooth, then X is rotund and* ii) *If X^* is rotund, then X is smooth.*

Proof. These results follow directly from preceeding results and as an EXERCISE you should obtain them that way. We will however give direct proofs, based on the above arguments.

i) Assume X is not rotund, then there exists distinct points x, y and $z = \frac{1}{2}(x + y)$ in $S(X)$. Choose any $f \in \mathcal{D}(z)$, then $\frac{1}{2}(f(x) + f(y)) = 1$ while $f(x), f(y) \leq 1$, so $f(x) = f(y) = 1$ and $\hat{x}, \hat{y} \in \mathcal{D}(f)$, so f is not a smooth point.

ii) Assume X is not smooth, then there exists $x \in S(X)$ and $f_1, f_2 \in \mathcal{D}(x)$ with $f_1 \neq f_2$. $f = \frac{1}{2}(f_1 + f_2)$ is a point of $S(X)$ and so X^* is not rotund. □

The partial duality between rotundity and smoothness of Theorem 9 is in general best possible (see latter), however when X is reflexive the complete duality between rotundity and smoothness is an immediate corollary.

EXAMPLE: Since a Hilbert space H is smooth (Example 1) p.35) and $H^* = H$, *every Hilbert space is both rotund and smooth.*

EXERCISE: 1) From the parallelogram Law, deduce directly that any Hilbert space is rotund.

*2) i) Show that ℓ_p $(1 < p < \infty)$ is rotund.

ii) Show that c_0, ℓ_1, ℓ_∞ and $C[a,b]$ are not rotund spaces.

Equivalent Renormings to gain Rotundity

Whether or not a space is smooth (rotund) depends on the particular norm used. For example, the space of ordered pairs of real numbers is both rotund and smooth with respect to the euclidean norm $\|\cdot\|_2$, however, it is neither with respect to either of the equivalent norms $\|\cdot\|_1$, $\|\cdot\|_\infty$ A norm dependent property of this type is an *isometric* property. Properties retained by all equivalent norms are known as *isomorphic* properties (for example, 'reflexivity'.) For any isometric property P the question naturally arises of whether a given space $(X, \|\cdot\|)$ admits an equivalent norm, $\|\cdot\|'$, with respect to which X has P. If this is the case we write X is <P>. Thus, X is <rotund> if X can be equivalently renormed to be rotund.

LEMMA 10 (Klee 1953): *If there exists a continuous linear one-to-one mapping T from $(X, \|\cdot\|)$ into a rotund space Y, then X can be equivalently renormed to be rotund.*

Proof. For $x \in X$ let $\|x\|' = \|x\| + \|T x\|$, then

$$\|x\| \leq \|x\|' \leq (1 + \|T\|)\|x\|$$

so $\|\cdot\|'$ is an equivalent norm on X. Further, if $\|x + y\|' = \|x\|' + \|y\|'$, and $y \neq 0$ then $T y \neq 0$ (T one-to-one) and

$$\|x\| + \|T x\| + \|y\| + \|T y\| = \|x + y\| + \|T(x + y)\| \leq \|x\| + \|y\| + \|Tx + Ty\|.$$

So, $\|Tx + Ty\| \leq \|T x\| + \|T y\| \leq \|Tx + Ty\|$.

Now Y is rotund, so $T x = \lambda T y$ for some λ, and since T is one-to-one we have $x = \lambda y$. Thus, X is <rotund> . □

THEOREM 11: *If* $S(X^*)$ *contains a countable total subset, then* X *can be equivalently renormed to be rotund.*

Proof. Let (f_n) be a total sequence in $S(X^*)$, that is, if $f_n(x) = 0$ for all n, then $x = 0$. Define T by

$$T(x) = (f_1(x), f_2(x)/2, \ldots, f_n(x)/n, \ldots),$$

then T is linear and one-to-one (as (f_n) is total).

Further,

$$\|Tx\|_2 = \sqrt{\sum_{n=1}^{\infty} |f_n(x)/n|^2} \le \sqrt{\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)} \|x\| ,$$

so T is a continuous mapping from X into ℓ_2. Since ℓ_2 is rotund the result now follows from lemma 10. □

COROLLARY 12 (Clarkson, 1936): *Every separable Banach space can be equivalently renormed to be rotund.* In particular, then c_0, ℓ_1 and $C[a,b]$ are <rotund> .

Proof. Let (x_n) be a dense sequence in $S(X)$. For each n select an $f_n \in \mathcal{D}(x_n)$. It suffices to show (f_n) is a total subset of $S(X^*)$. Now, for any $\varepsilon > 0$ and $x \neq 0$ there exists n with $\left\|x_n - \frac{x}{\|x\|}\right\| < \varepsilon$, thus $f_n(\frac{x}{\|x\|}) > 1 - \varepsilon$ or $f_n(x) > (1-\varepsilon)\|x\|$. We therefore have $\|x\| = \sup_n f_n(x)$ and so certainly (f_n) is total. □

COROLLARY 13: *The dual of any separable Banach space can be equivalently renormed to be rotund.* In particular then ℓ_∞ is <rotund> .

Proof. Let (x_n) be a dense sequence in $S(X)$. To prove X^* is rotund it suffices to show $(\hat{x}_n)$ is total over X^*. Now if $f \neq 0$, then there exists $x \in S(X)$ with $f(x) \neq 0$ and so by their density there exists x_n such that $f(x_n) \neq 0$, so $\hat{x}_n(f) \neq 0$. □

EXERCISE: In case $X = \ell_1$ or ℓ_∞ give an explicit one-to-one continuous linear mapping from X into ℓ_2.

A word of caution about <u>equivalent dual norms</u>.

If $\|\cdot\|'$ is an equivalent norm on $(X^*, \|\cdot\|)$ the dual of $(X, \|\cdot\|)$, it does not follow that $\|\cdot\|'$ is an *equivalent <u>dual</u> norm*, that is, there exists an equivalent norm on $(X, \|\cdot\|')$, such that $\|f\|' = \sup\{f(x): \|x\|' \le 1\}$.
If such a norm on X exists it must be given by

$$\|x\|' = \|\hat{x}\|' = \sup\{f(x): \|f\|' \le 1\}.$$

In other words, we must have for each $f \in X^*$

$$\|f\|' = \sup\{f(x): \ \sup\{g(x): \|g\|' \le 1\} \le 1\}$$

or

$$\|f\|' = \sup_{x\neq 0} \frac{f(x)}{\sup\limits_{g\neq 0} \frac{g(x)}{\|g\|'}} .$$

Denoting the Right Hand Side by $\Phi(f)$ we clearly have

$$\Phi(f) \le \sup_{x\neq 0} \frac{f(x)}{\frac{f(x)}{\|f\|'}} = \|f\|' .$$

The difficulty is that for some f we may have $\Phi(f) \lneq \|f\|'$, however if this is the case, for some $\lambda > 1$

$$\lambda\ \Phi(f) = \sup_{x\neq 0} \frac{\lambda f(x)}{\sup\limits_{g\neq 0} \frac{g(x)}{\|g\|'}} < \|f\|'.$$

For each x we therefore have

$$\frac{\lambda f}{\|f\|'}(x) < \sup_{g\neq 0} \frac{g(x)}{\|g\|'} = \sup\{g(x): g \in B'[X^*]\},$$

where $B'[X^*]$ denotes the unit ball in $(X^*, \|\cdot\|')$.

It now follows, from the separation Theorem for the w*-topology (see p.7), that $\frac{\lambda f}{\|f\|'} \in \overline{B'[X^*]}^{w^*}$. Hence, $B'[X^*]$ is not w*-closed and so certainly not w*-compact.

Thus a sufficient condition for $\|\cdot\|'$ to be a dual norm on X^* is that $B'[X^*]$ be w*-compact. That this is also necessary follows immediately from the Banach-Alaoglu Theorem of p.7. We therefore have:

THEOREM 14: *An equivalent norm, $\|\cdot\|'$, on X^* is a dual norm if and only if its unit ball,* $B'[X^*] = \{g \in X^*: \|g\|' \le 1\}$ *is w*-compact.*

A useful reformulation of this result is provided by

LEMMA 15: *An equivalent norm, $\|\cdot\|'$, on X^* is a dual norm if and only if it is w*-lower semi-continuous, that is, if $f_\alpha \overset{w^*}{\to} f$ then* $\|f\|' \le \liminf_\alpha \|f_\alpha\|'$.

Proof. ($\Leftarrow$) If $B'[X^*]$ is not w*-compact, then there exists a net $(f_\alpha) \subset B'[X^*]$ such that $f_\alpha \overset{w^*}{\to} f \notin B'[x^*]$ (why?), but then $\|f\|' > 1 \ge \limsup \|f_\alpha\|' \ge \liminf \|f_\alpha\|'$.

($\Rightarrow$) Assume $f_\alpha \overset{w^*}{\to} f$ but $\|f\|' > \liminf \|f_\alpha\|'$. Let $k = \liminf \|f_\alpha\|'$ and choose a subnet (f_{α_β}) such that $\|f_{\alpha_\beta}\|' \to k^+$. Clearly $k \ne 0$ (otherwise $f_{\alpha_\beta} \xrightarrow{\|\cdot\|'} 0$ so $\|f\|' = 0 = k$). Now $\frac{f_{\alpha_\beta}}{\|f_{\alpha_\beta}\|'} \overset{w^*}{\to} \frac{f}{k}$, thus we have a net in $B'[X^*]$ w*-convergent to an element, f/k, of norm greater than one, so $B'[X^*]$ is not w*-compact. □

EXERCISES: 1) Let T be a continuous linear one-to-one mapping from the dual space X^* into a rotund dual space Y^*. Show that the equivalent rotund norm on X^* given by $\|f\|' = \|f\| + \|T f\|$ (c.f. the proof of lemma 10) is a dual norm if T is w* to w* continuous.

2) Using 1) show that

i) *ℓ_1 admits an equivalent rotund dual norm* hence conclude that c_0 *may be equivalently renormed to be smooth.*

[Hint, let $T: \ell_1 \to \ell_2$ be the identity map. Note that $\ell_2^* = \ell_2$ and that in ℓ_2 the w and w* topologies coincide.]

ii) *ℓ_∞ admits an equivalent rotund dual norm, so ℓ_1 is* <*smooth*> .

3) (optional) *If X is separable, X^* admits an equivalent rotund dual norm.*

[Hint. See the proof of Corollary 13 and theorem 11. Note, that since X is separable the w*-topology on $B[X^*]$ is a metric topology and so sequences rather than nets are sufficient.]

4) If X and Y are both smooth spaces and there exists a continuous linear mapping $T: X \to Y$ show that $\|x\|' = \|x\| + \|T x\|$ is an equivalent smooth norm on X.

[Hint. Deduce the Gâteaux differentiability of $\|\cdot\|'$ from that of the two original norms.]

5) Using 4) deduce that

i) *c_0 and ℓ_1 can be equivalently renormed so as to be both rotund and smooth.* [Hint. see exercise 2.]

ii) (Day) *Every separable space can be equivalently renormed so as to be both rotund and smooth.*

[Hint. Assume exercise 3.]

REMARKS: 1) If X* admits an equivalent dual norm which has property P we sometimes write X* is <P>*. Thus ℓ_1 and ℓ_∞ are <rotund>*.

2) Day ["Strict Convexity and Smoothness of Normed Spaces", Trans. Amer. Math. Soc., Vol. 78 (1955)] made a thorough study of renorming the spaces c_0, ℓ_1 and ℓ_∞ (his m(I)) to gain rotundity or smoothness. Among other results he proved (his, Theorem 9) that ℓ_∞ has no equivalent smooth norm. Since ℓ_1 has a rotund norm, this provides an example of a rotund space whose dual is not smooth. A smooth space whose dual is not rotund has been constructed by Troyanski.

3) Finding equivalent smooth norms for a space X is a difficult problem. The only known way appears to be the construction of an equivalent rotund dual norm on X*.

4) Renorming Theorems are extremely important to the study of Banach spaces, however the above results represent most of what was known until the mid-sixties. During the last decade considerable progress has been made by Asplund (averaging technique) and, following the fundamental work of J. Lindenstrauss and D. Amir on Weakly Compactly Generated spaces, a number of powerful renorming results have been obtained by Troyanski, John, Zizler and many others. (See Diestel Chapters 4 and 5 for details and references; also see Day Chapter VII §4.)

§5 Fréchet Differentiability

DEFINITION: The norm of X is Fréchet (or Strongly) differentiable at $x \in S(X)$ if

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists and is approached uniformly over $y \in S(X)$.

We say the norm is Fréchet differentiable if it is Fréchet differentiable at every $x \in S(X)$.

If the norm is Fréchet differentiable at x it is certainly Gâteaux differentiable at x (see p.36) and so for each $y \in S(X)$ the above limit equals $f_x(y)$ where f_x is the unique element of $\mathcal{D}(x)$.

The next proposition collects together some obvious equivalents to Fréchet differentiability at x.

PROPOSITION 1: *The following are equivalent:*

i) *The norm is Fréchet differentiable at* $x \in S(X)$.

ii) *There exists* $f_x \in \mathcal{D}(x)$ *such that*

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} - f_x(y) = 0$$

and the limit is approached uniformly over $y \in S(X)$.

iii) *There exists* $f_x \in \mathcal{D}(x)$ *such that*

$$\lim_{\lambda \to 0} \sup_{y \in S(X)} \left| \frac{\|x + \lambda y\| - \|x\|}{\lambda} - f_x(y) \right| = 0.$$

iv) *There exists* $f_x \in \mathcal{D}(x)$ *such that*

$$\lim_{z \to 0} \frac{\|x + z\| - \|x\| - f_x(z)}{\|z\|} = 0.$$

v) *There exists* $f_x \in \mathcal{D}(x)$ *such that whenever the sequence* (z_k) *converges to* 0 *we have*

$$\frac{\|x + z_k\| - \|x\| - f_x(z_k)}{\|z_k\|} \to 0.$$

vi) *There exists* $f_x \in \mathcal{D}(x)$ *such that given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\big|\|x + z\| - \|x\| - f_x(z)\big| < \varepsilon\|z\| \text{ whenever } \|z\| < \delta.$$

Further, in each of the last five equivalents f_x *is the only element of* $\mathcal{D}(x)$.

Another important characterization of Fréchet differentiability is provided by

THEOREM 2 (Šmul'yan, 1940): *The norm is Fréchet differentiable at* $x \in S(X)$ *if and only if whenever the sequence* $(f_n) \subset B[X^*]$ *is such that* $f_n(x) \to 1$, *then we have* f_n *is norm convergent.*
Further this can only happen if $\mathcal{D}(x)$ *contains a single element,* f_x, *and this element is the limit of the* f_n.

Proof. ($\Rightarrow$) If necessary, by replacing f_n by $(1 - \frac{1}{n})f_n$ we may assume $\|f_n\| < 1$ and so $1 - f_n(x) > 0$ for all n. Now, assume $f_n \not\to f_x$, then there exists $\varepsilon > 0$ and a subsequence (f_{n_k}) with $\|f_{n_k} - f_x\| > 2\varepsilon$ for every k. So there exists $y_k \in S(X)$ with $(f_{n_k} - f_x)(y_k) > 2\varepsilon$. Let $\alpha_k = \varepsilon^{-1}[1 - f_{n_k}(x)] > 0$ and put $z_k = \alpha_k y_k$, then $\|z_k\| = \alpha_k \to 0$, so by the Fréchet differentiability at x, there exists $N > 0$ such that

$$\frac{\varepsilon}{2}\alpha_k = \frac{\varepsilon}{2}\|z_k\| \geq \|x + z_k\| - \|x\| - f_x(z_k), \text{ for } k > N.$$

However

$$\begin{aligned} \|x + z_k\| - \|x\| - f_x(z_k) &\geq f_{n_k}(x + z_k) - 1 - f_x(z_k) \\ &= (f_{n_k} - f_x)(z_k) - [1 - f_{n_k}(x)] \\ &\geq \alpha_k(f_{n_k} - f_x)(y_k) - \varepsilon\alpha_k \\ &\geq 2\varepsilon\alpha_k - \varepsilon\alpha_k \\ &= \varepsilon\alpha_k . \end{aligned}$$

We therefore have $\frac{1}{2}\alpha_k \geq \alpha_k$ so $\alpha_k \leq 0$, a contradiction.

($\Leftarrow$) Let $z_k \to 0$, then putting $y = z_k/\|z_k\|$, $\lambda = \|z_k\|$ in lemma 1 on p.28 we have, for any support mapping $x \mapsto f_x$,

$$f_x\left(\frac{z_k}{\|z_k\|}\right) \leq \frac{\|x + z_k\| - \|x\|}{\|z_k\|} \leq f_{\frac{x+z_k}{\|x+z_k\|}}\left(\frac{z_k}{\|z_k\|}\right)$$

or

$$0 \leq \frac{\|x + z_k\| - \|x\| - f_x(z_k)}{\|z_k\|} \leq \left(f_{\frac{x+z_k}{\|x+z_k\|}} - f_x\right)\left(\frac{z_k}{\|z_k\|}\right).$$

It is therefore sufficient to show $f_{\frac{x+z_k}{\|x+z_k\|}} \xrightarrow{\|\cdot\|} f_x$ and so by our assumption it is sufficient to show $f_{\frac{x+z_k}{\|x+z_k\|}}(x) \to 1$. However,

$$\left|f_{\frac{x+z_k}{\|x+z_k\|}}(x) - 1\right| = \left|f_{\frac{x+z_k}{\|x+z_k\|}}\left(x - \frac{x + z_k}{\|x + z_k\|}\right)\right|$$

$$\to 0 \text{ as } z_k \to 0 . \qquad \square$$

(Remark: The last part of the proof shows that it is sufficient to assume that $(f_n) \subset \mathcal{D}(S(X))$, indeed, that (f_n) is contained in the range of some support mapping.)

COROLLARY 3: *The following are equivalent:*

i) *The norm is Fréchet differentiable at* x.

ii) *All support mappings are continuous norm to norm at* x.

iii) *There exists a support mapping which is continuous norm to norm at* x.

Proof. i) $\Rightarrow$ ii) Let $x \mapsto f_x$ be any support mapping and suppose $(x_n) \subset S(X)$ is such that $x_n \xrightarrow{\|\cdot\|} x$, then

$$|f_{x_n}(x) - 1| = |f_{x_n}(x - x_n)| \leq \|x - x_n\| \to 0 .$$

So $f_{x_n}(x) \to 1$ and by Theorem 2 $f_{x_n} \xrightarrow{\|\cdot\|} f_x$.

ii) $\Rightarrow$ iii) is immediate

iii) $\Rightarrow$ i) Let $x \mapsto f_x$ be a support mapping which is continuous norm to norm at x. Then, if $z_k \to 0$ we have $\frac{x + z_k}{\|x + z_k\|} \xrightarrow{\|\cdot\|} x$ and so $f_{\frac{x+z_k}{\|x+z_k\|}} \xrightarrow{\|\cdot\|} f_x$. The result now follows from the first five lines in in the ($\Leftarrow$) part of the proof to lemma 2. □

EXERCISE: Assuming the Bishop-Phelps-Bollobás Result (Theorem 11, §2, p.25) deduce iii) $\Rightarrow$ i) in Corollary 3 directly from the statement of Theorem 2.

The following is another sometimes useful characterization of Fréchet differentiability.

COROLLARY 4: *The norm is Fréchet differentiable at* $x \in S(X)$ *if and only if, given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\big|\|x + z\| + \|x - z\| - 2\big| < \varepsilon\|z\| \quad \textit{whenever} \quad \|z\| < \delta$$

Proof. ($\Leftarrow$) Assume the norm is not Fréchet differentiable at x, then by Theorem 2, there exists $(f_n) \subset S(X^*)$ with $f_n(x) \to 1$ but (f_n) does not converge to f_x. By passing to a subsequence if necessary we may

assume that $\|f_n - f_x\| > \varepsilon$ for all n and some $\varepsilon > 0$. Again by passing to another subsequence if necessary we may assume that $f_n(x) > 1 - \frac{\varepsilon}{2n}$.

Now, choose $y_n \in S(X)$ such that $(f_n - f_x)(y_n) > \varepsilon$, then

$$\begin{aligned} \|x + \tfrac{1}{n} y_n\| + \|x - \tfrac{1}{n} y_n\| &\geq f_n(x + \tfrac{1}{n} y_n) + f_x(x - \tfrac{1}{n} y_n) \\ &= 1 + f_n(x) + \tfrac{1}{n}[(f_n - f_x)(y_n)] \\ &\geq 2 - \frac{\varepsilon}{2n} + \frac{1}{n}\varepsilon \\ &\geq 2 + \frac{\varepsilon}{2n} . \end{aligned}$$

Thus, putting $z_n = \frac{1}{n} y_n$ we have $\|z_n\| = \frac{1}{n} \to 0$ while

$$\|x + z_n\| + \|x - z_n\| - 2 > \varepsilon\|z_n\| .$$

($\Rightarrow$) From vi) of Proposition 1 we have, there exists $\delta > 0$ such that, whenever $\|z\| < \delta$

$$-\frac{\varepsilon}{2}\|z\| \leq \|x + z\| - 1 - f_x(z) \leq \frac{\varepsilon}{2}\|z\| .$$

In particular then, replacing z by $-z$ we also have

$$-\frac{\varepsilon}{2}\|z\| \leq \|x - z\| - 1 + f_x(z) \leq \frac{\varepsilon}{2}\|z\| \text{ whenever } \|z\| < \delta.$$

Adding these two inequalities together yields the result. □

Fréchet differentiability of a dual norm at $f \in S(X^*)$ is characterized by any of the previous results, in addition we have the pre-dual characterization

THEOREM 5 (Šmul'yan): *The norm of* X^* *is Fréchet differentiable at* $f \in S(X^*)$ *if and only if there exists* $x \in S(X)$ *such that whenever the sequence* $(x_n) \subset B[X]$ *has* $f(x_n) \to 1$ *we have* $x_n \xrightarrow{\|\cdot\|} x$.

Proof. ($\Rightarrow$) If $(x_n) \subset B[X]$ and $f(x_n) \to 1$, then by Theorem 2, $\hat{x}_n$ is norm convergent to the unique element F_f of $\mathcal{D}(f)$. Since $\hat{X}$ is norm closed,

$F_f \in \hat{X}$, that is $F_f = \hat{x}$ for some $x \in S(X)$ and so $x_n \xrightarrow{\|\cdot\|} x$.

($\Leftarrow$) Our hypothesis is equivalent to: given $\epsilon > 0$ there exists $\delta > 0$ such that $\|y\| \le 1$ and $f(y) > 1 - \delta$ implies $\|y - x\| < \epsilon$ (otherwise, taking $\delta = \frac{1}{n}$ for each n we could construct a sequence violating the hypothesis). It suffices to show: given $\epsilon > 0$ there exists $\delta_1 > 0$ such that $\|g\| < \delta_1$ $(g \in X^*)$ implies $\big|\|f+g\|-1-g(x)\big| \le \epsilon\|g\|$.

If $g = 0$ there is nothing to prove so assume $g \neq 0$.

Now, $1 + g(x) = f(x) + g(x)$ (as $f(x) = 1$, why?)

$\le \|f + g\|$.

or $0 \le \|f + g\| - 1 - g(x)$, thus we need only prove the upper inequality $\|f + g\| - 1 - g(x) \le \epsilon \|g\|$.

Choose $y_n \subset S(X)$ such that $(f + g)(y_n) \to \|f + g\|$ and let $\delta_1 = \frac{\delta}{3}$.

Then, there exists N such that, for $n > N$ we have

$$\begin{aligned}(f+g)(y_n) &\ge \|f+g\| - \|g\| \ge \big|\|f\| - \|g\|\big| - \|g\| \\ &= 1 - 2\|g\| \\ &\ge 1 - 2\delta_1\end{aligned}$$

and so, for $n > N$

$$\begin{aligned}f(y_n) &\ge 1 - 2\delta_1 - g(y_n) \\ &\ge 1 - 2\delta_1 - \|g\| \ge 1 - 3\delta_1 = 1 - \delta.\end{aligned}$$

So, by hypothesis $\|y_n - x\| < \epsilon$, for $n > N$. Consequently we have

$$\begin{aligned}\|f+g\| &= \lim_{n\to\infty} (f+g)(y_n) \\ &\le 1 + \sup_{n>N} g(y_n) \\ &= 1 + \sup_{n>N} [g(y_n - x) + g(x)]\end{aligned}$$

$$\begin{aligned}&= 1 + g(x) + \sup_{n>N} g(y_n - x) \\ &\le 1 + g(x) + \|g\|\|y_n - x\| \\ &\le 1 + g(x) + \epsilon\|g\|, \text{ as required.}\end{aligned}$$ □

COROLLARY 6: *If the norm of* X^* *is Fréchet differentiable at* $f \in S(X^*)$, *then* $f \in \mathcal{D}(S(X))$.

Proof. For x as in Theorem 5, $f(x) = 1$, so $f \in \mathcal{D}(x)$. □

EXERCISE: 1) Assuming James' Theorem (see pp.17-18), show that X is reflexive if the norm of X^* is Fréchet differentiable. [We will develop an alternative proof of this result shortly.]

2) Show that the norm is Fréchet differentiable at $x \in S(X)$ if and only if the norm of X^{**} is Fréchet differentiable at $\hat{x}$.

Properties dual to Fréchet differentiability.

DEFINITION: x is a strongly exposed point of B[X] if there exists $f \in S(X^*)$ such that whenever the sequence $(x_n) \subset B[X]$ is such that $f(x_n) \to 1$, then we have $x_n \xrightarrow{\|\cdot\|} x$.

Clearly such a point is an exposed point (exposed by f) and $f \in \mathcal{D}(x)$. We say f strongly exposes x or x is strongly exposed by f.

If $f \in S(X^*)$ is strongly exposed by $\hat{x} \in S(\hat{X})$ we say f is a w*-*strongly exposed point* (f is w*-strongly exposed by x).

Theorems 2 and 5 can now be restated as

Theorem 7: i) *The norm is Fréchet differentiable at* $x \in S(X)$ *if and only if* x *w*-strongly exposes some element of* $S(X^*)$, *which must necessarily be the unique element of* $\mathcal{D}(x)$.

ii) *The norm is Fréchet differentiable at* $f \in S(X^*)$ *if and only if* f *strongly exposes some element* x *of* $S(X)$. *Further* $\hat{x}$ *must be the unique element of* $\mathcal{D}(f)$ *and* $f \in \mathcal{D}(x)$.

EXERCISE: Show that the norm of ℓ_1 is not Fréchet differentiable at any point of $S(\ell_1)$.

PROPOSITION 8: i) *The norm of* X *is Fréchet differentiable if and only if every point of* $\mathcal{D}(S(X))$ *is a w*-strongly exposed point.*

ii) *The norm of* X^* *is Fréchet differentiable at every point of* $\mathcal{D}(S(X))$ *if and only if every point of* $S(X)$ *is a strongly exposed point.*

Proof. i) ($\Rightarrow$) is immediate from i) of theorem 7.

($\Leftarrow$) By theorem 7 i) it is sufficient to show that, given any $x \in S(X)$ and $f_x \in \mathcal{D}(x)$ we have f_x is w*-strongly exposed by x, that is, if $f_n(x) \to 1$, then $f_n \xrightarrow{\|\cdot\|} f_x$. By assumption f_x is w*-strongly exposed by some $y \in S(X)$, that is, if $f_n(y) \to 1$, then $f_n \xrightarrow{\|\cdot\|} f_x$. Further, f_x is certainly w*-exposed by x. Otherwise there exists $g \in \mathcal{D}(x)$, $g \neq f_x$, but then $\frac{1}{2}(g + f_x)$ is an element of $\mathcal{D}(x)$ which is not an extreme point, so not a w*-exposed point and so certainly not a w*-strongly exposed point, contrary to the assumption that every point of $\mathcal{D}(S(X))$ is w*-strongly exposed. By corollary 2 of §4, p.46 we therefore have $f_n \xrightarrow{w^*} f_x$ whenever $f_n(x) \to 1$, but then $f_n(y) \to f_x(y) = 1$ and so $f_n \xrightarrow{\|\cdot\|} f_x$ as required.

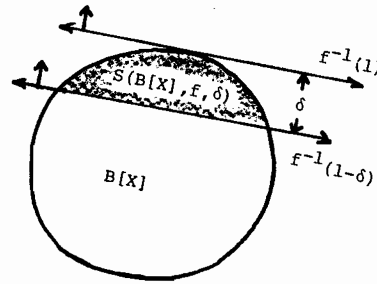
ii) The proof of ii) is left as an EXERCISE. It is similar to that of i) using theorem 7 ii) and Exercise 3 on p.46, provided we "note" that; x is strongly exposed by f implies $\hat{x}$ is w* (strongly) exposed by f. □

A valuable geometric interpretation of strongly exposed points is provided by the following.

SLICES.

DEFINITION: The <u>slice</u> of $B[X]$ determined by $f \in S(X^*)$ and $\delta \in (0,1)$ is

$$S(B[X],f,\delta) = \{x \in B[X] : f(x) > 1-\delta\}.$$

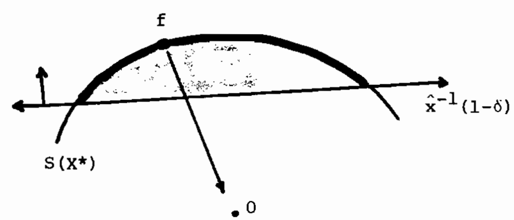


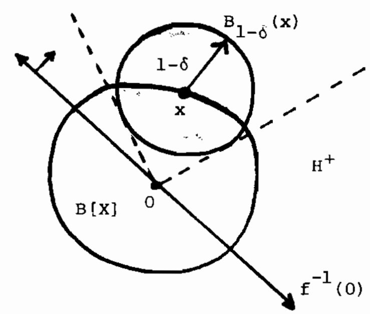
Clearly, $S(B[X],f,\delta)$ is a non-empty relatively w-open subset of $B[X]$.

REMARK: The concept of "slice" appears in Day and has been extensively used by R.R. Phelps and others since about 1970.

When X is replaced by a dual space X^*, a slice of the form $S(B[X^*],\hat{x},\delta) = \{f \in B[X^*] : f(x) > 1-\delta\}$ is referred to as a <u>w*-slice</u> of $B[X^*]$, that is, the "slicing" functional is a w*-continuous embedding element.

EXERCISE (optional) *Dual picture of a slice* (The following interpretation is due essentially to Anantharaman, Lewis and Whitefield, 1976). Show that, for $x \in S(X)$ and $\delta \in (0,1)$, $f \in S(B[X^*],\hat{x},\delta) \cap S(X^*)$ if and only if the ball $B_{1-\delta}(x)$ is contained in the positive open half space $H^+ = \{y \in X : f(y) > 0\}$.





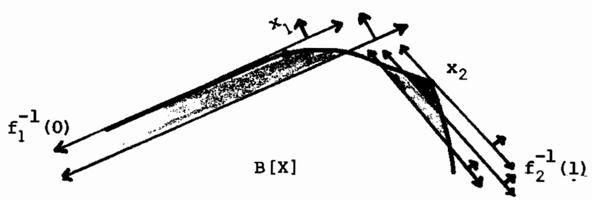
Similarly show that, for $f \in S(X^*)$ and $\delta \in (0,1)$,

$$x \in S(B[X],f,\delta) \cap S(X^*) \text{ if and only if } g(x) > 0 \text{ for all } g \in B_{1-\delta}(f).$$

Strongly exposed points may be viewed in terms of slices as follows.

PROPOSITION 9; i) $x \in S(X)$ *is strongly exposed by* $f \in S(X^*)$ *if and only if* f *determines slices of* $B[X]$ *of arbitrarily small diameter which contain* x. (That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in S(B[X],f,\delta)$ and diam $S(B[X],f,\delta) < \varepsilon$. By the diameter of a set A we mean, as usual, diam $A = \mathrm{Sup}\{\|x - y\|: x, y \in A\}$.)

ii) $f \in S(X^*)$ *is* w*-*strongly exposed by* $x \in S(X)$ *if and only if* $\hat{x}$ *determines* w*-*slices of* $B[X^*]$ *of arbitrarily small diameter which contain* f.



x_1 is not strongly exposed by f_1, x_2 is strongly exposed by f_2.

Proof. i) ($\Leftarrow$) Let $(x_n) \subset B[X]$ be such that $f(x_n) \to 1$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in S(B[X],f,\delta)$ and diam $S(B[X],f,\delta) < \varepsilon$. Further, there exists N such that $f(x_n) > 1-\delta$ for $n > N$. Thus $x_n \in S(B[X],f,\delta)$ for $n > N$ and so for $n > N$ $\|x_n - x\| \le$ diam $S(B[X],f,\delta) < \varepsilon$. That is, $x_n \xrightarrow{\|\cdot\|} x$.

($\Rightarrow$) For each n choose $x_n \in S(B[X],f,\frac{1}{n})$ such that $\|x_n - x\| > \mathrm{Sup}\,\{\|y - x\|: y \in S(B[X],f,\frac{1}{n})\} - \frac{1}{n}$. Then, $f(x_n) > 1-\frac{1}{n}$, so $f(x_n) \to 1$ and hence $\|x_n - x\| \to 0$. Further,

$$\text{diam } S(B[X],f,\tfrac{1}{n}) \;\le\; 2\|x_n - x\| + \tfrac{2}{n} \to 0.$$

ii) The proof is similar to i) and is left as an EXERCISE. □

COROLLARY 10: i) *If* x *is a strongly exposed point of* $B[X]$, *then the relative* w *and norm topologies on* $B[X]$ *agree at* x. That is, if (x_α) is a net in $B[X]$ with $x_\alpha \xrightarrow{w} x$, then $x_\alpha \xrightarrow{\|\cdot\|} x$.)

ii) *If* f *is a* w*-*strongly exposed point of* $B[X^*]$, *then the relative* w* *and norm topologies on* $B[X^*]$ *agree at* f.

Proof. i) Let x be strongly exposed by f, then given any $\varepsilon > 0$ there exists, by Proposition 9 i), a $\delta > 0$ such that $x \in S(B[X],f,\delta)$ and diam $S(B[X],f,\delta) < \varepsilon$. Now $S(B[X],f,\delta)$ is a w-open neighbourhood of x and so, since $x_\alpha \xrightarrow{w} x$ we have: there exists α_0 such that for $\alpha > \alpha_0$, $x_\alpha \in S(B[X],f,\delta)$, but then, for $\alpha > \alpha_0$ we have $\|x_\alpha - x\| \le$ diam $S(B[X],f,\delta) < \varepsilon$, so $x_\alpha \xrightarrow{\|\cdot\|} x$.

ii) The proof of ii) is similar to that of i) and is left as an EXERCISE. □

EXERCISE. (Optional) Gregory, 1977. Show that, f is a w*-strongly exposed point of $B[X^*]$ if and only if f is a w*-exposed point at which the relative w* and norm topologies on $B[X^*]$ agree.

Using Proposition 9 and our earlier characterization of Fréchet differentiability (Theorem 7) we have:

THEOREM 11: i) *The norm is Fréchet differentiable at* $x \in S(X)$ *if and only if* x *determines* w*-*slices of* $B[X^*]$ *of arbitrarily small diameter.*

ii) *The norm is Fréchet differentiable at* $f \in S(X^*)$ *if and only if* f *determines slices of* $B[X]$ *of arbitrarily small diameter.*

Proof. i) ($\Rightarrow$) If the norm is Fréchet differentiable at x, then the unique element f_x of $\mathcal{D}(x)$ is w*-strongly exposed by x (Theorem 7) and so, by Proposition 9, x determines w*-slices of $B[X^*]$ of arbitrarily small diameter.

($\Leftarrow$) Choose any $f_x \in \mathcal{D}(x)$ then $f_x \in S(B[X],\hat{x},\delta)$ for any $\delta > 0$ (and so f_x is necessarily unique). Thus f_x is contained in w*-slices of $B[X^*]$ which are determined by x and are of arbitrarily small diameter. Hence f_x is w*-strongly exposed by x and the result follows from Theorem 7 i).

ii) ($\Rightarrow$) The proof is similar to that of i) ($\Rightarrow$) and so is left as an EXERCISE.

($\Leftarrow$) Since f determines slices of B[X] of arbitrarily small diameter, there exists a sequence.

$\delta_1, \delta_2, \ldots, \delta_n, \ldots$ of strictly positive real numbers such that as $n \to \infty$, diam $S(B[X],f,\delta_n) \to 0$ monotonically. We therefore have that $\overline{S(B[X],f,\delta_n)}$ is a nested sequence of closed sets in the complete metric space $(B[X],d(x,y) = \|x-y\|)$ whose diameters tend to 0. Thus Cantor's intersection theorem (see footnote p.21) applies to show $\bigcap_n \overline{S(B[X],f,\delta_n)}$ contains exactly one point, x, necessarily a member of S(X). Thus, by Proposition 9 i) f strongly exposes x, and so by Theorem 7 ii) the norm is Fréchet differentiable at f. □

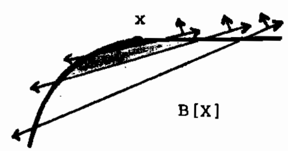
EXERCISE: If $f \in S(X^*)$ is not a support functional, show that there exists $r > 0$ such that

$$\operatorname{diam} S(B[X],f,\delta) > r \quad \text{for all } \delta > 0.$$

[Hint: Examine the proof of ii) ($\Leftarrow$) in the above Theorem.]

PROBLEMS: A] *Denting Points*

DEFINITION: $x \in S(X)$ is a denting point of B[X] if x is contained in slices of $B[X^*]$ of arbitrarily small diameter. That is, given $\varepsilon > 0$ there exists $f \in S(X^*)$ and $\delta \in (0,1)$ such that $x \in S(B[X],f,\delta)$ and diam $S(B[X],f,\delta) < \varepsilon$. The distinction between a denting point and a strongly exposed point is that in the case of a denting point different slicing functionals may be needed to obtain smaller diameter slices.



x is a denting point, however, x is not strongly exposed.

Similarly, if $f \in S(X^*)$ is contained in w*-slices of $B[X^*]$ of arbitrarily small diameter, then we say f is a w*-denting point.

REMARK: Denting points (introduced by Rieffel in 1967) and w*-denting points have played an important role in the recent geometric characterizations of Banach spaces with the "Radon-Nikodým Property". (See, for example, Diestel, Diestel and Uhl's survey paper: "The Radon-Nikodým Theorem for Banach Space Valued Measures", Rocky Mountain J. of Maths. Vol.6 No.1, 1976; or their monograph "Vector Measures" No. 15 in Math. Surveys, Amer. Math. Soc., 1977.)

1) Use the separation Theorems (see pp.4, 5 and 7) to show that

i) x *is a denting point of* $B[X]$ *if and only if the norm* (equals weak) *closed convex hull of* $B[X]\backslash B_\varepsilon(x)$ *does not contain* x *for any* $\varepsilon > 0$

and ii) f *is a* w*-*denting point of* $B[X^*]$ *if and only if the* w*-*closed convex hull of* $B[X^*]\backslash B_\varepsilon(f)$ *does not contain* f *for any* $\varepsilon > 0$.

2) Show that, *if* x (f) *is a* (w*-) *denting point of* $B[X]$ $(B[X^*])$, *then*

i) x (f) *is an extreme point.*

and ii) *the relative* w (w*) *and norm topologies on* $B[X]$ $(B[X^*])$ *agree at* x (f).

(Remark: In the case of w*-denting points the converse is also true - Sims, 1978.)

Deduce that $f \in S(X^*)$ *is a* w*-*strongly exposed point if and only if it is a* w*-*exposed,* w*-*denting point.* Hence conclude that *every point of* $\mathcal{D}(S(X))$ *is a* w*-*strongly exposed point (and so the norm of* X *is Fréchet differentiable) if every point of* $S(X^*)$ *is a* w*-*denting point.*

(Remark: No similar results are known to hold for denting points of $B[X]$ in a non-reflexive space. However it is known (Larman/Sims, 1977) that *if every point of* $S(X)$ *is a denting point then the strongly exposed points of* $B[X]$ *are dense in* $S(X)$; see 5 below.)

3) (Larman) If x is a denting point of $B[X]$ and $f_x \in \mathcal{D}(x)$ show that it is possible to obtain slices of $B[X]$ containing x and of arbitrarily small diameter by using functionals which are arbitrarily near to f_x. That is, given $\varepsilon_1, \varepsilon_2 > 0$ there exists $f \in S(X^*)$ with $\|f - f_x\| < \varepsilon_1$ and $\delta > 0$ such that

$$x \in S(B[X],f,\delta) \quad \text{and diam } S(B[X],f,\delta) < \varepsilon_2.$$

[Hint: Consider f's in the line segment joining f_x and f_{ε_2} where f_{ε_2} is any functional in $S(X^*)$ known to determine a slice of diameter less than ε_2 which contains x.]

4) (Optional) Bishop 1967 (see Phelps "Dentability and extreme points in Banach spaces", Lemma 7; J. of Functional Anal. Vol.17, 1974 pp.78-90).

Show that, *if for every* $f \in S(X^*)$, $\delta > 0$ *and* $\varepsilon > 0$ *there exists* $g \in S(X^*)$ *and* $\delta_0 > 0$ *such that* $\|f - g\| < \varepsilon$, $S(B[X],g,\delta_0) \subseteq S(B[X],f,\delta)$ *and* diam $S(B[X],g,\delta_0) < \varepsilon$ (that is, for each f and δ $S(B[X],f,\delta)$ contains slices of arbitrarily small diameters determined by functionals arbitrarily near to f), *then* $S(B[X],f,\delta)$ *contains a strongly exposed point of* $B[X]$.

[Hint: Use the assumptions to inductively construct sequences $(g_k) \subset S(X^*)$ and (δ_k) such that $g_0 = f$,

$$\|g_{k+1} - g_k\| < 2^{-k}\,\delta_k\ , \quad \delta_{k+1} < \tfrac{1}{2}\,\delta_k\ ,$$

$$\text{diam } S(B[X],g_{k+1},\delta_{k+1}) < 2^{-1}\,\delta_k \quad \text{and}$$

$$S(B[X],g_{k+1},\delta_{k+1}) \subseteq S(B[X],g_k,\delta_k)\ .$$

Deduce that (g_k) is a Cauchy sequence and so converges to some $g \in S(X^*)$.

Also deduce that $\bigcap_k \overline{S(B[X],g_k,\delta_k)}$ contains exactly one point x, say.

Now show that g strongly exposes x.]

5) i) From 3) deduce that, if every point of S(X) is a denting point, then for each $\varepsilon > 0$

$$\{f \in S(X^*) : f \text{ determines a slice of } B[X] \text{ of diameter less than } \varepsilon\}$$

is dense in $S(X^*)$.

[Hint: Use the subreflexivity of X.]

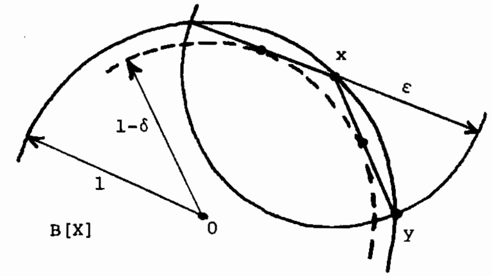
ii) Assuming 4) deduce from 5 i) that every slice of B[X] contains a strongly exposed point of B[X]. Hence conclude that the result of Larman/Sims referred to in 2) is correct.

B] *Local Uniform Rotundity*

The following idea was introduced and extensively studied by Lovaglia "Locally uniformly convex Banach Spaces", Trans. Amer. Maths. Soc. 78 (1955) pp.225-238.

DEFINITION: B[X] is <u>locally uniformly rotund</u> (lur) at $x \in S(X)$ if, given $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in S(X)$ and $\left\|\frac{y+x}{2}\right\| > 1-\delta$ implies $\|x - y\| < \varepsilon$.

We say X is locally uniformly rotund if B[X] is lur at every point x of S(X). (Note: For any given ε the δ in general will depend on x.)



1) Show that B[X] *is lur at* $x \in S(X)$ *if and only if whenever the sequence* $(x_n) \subset B[X]$ *is such that* $\left\|\frac{x_n + x}{2}\right\| \to 1$, *then* $x_n \xrightarrow{\|\cdot\|} x$.

[Hint for ($\Leftarrow$): If not lur at x observe that for some $\varepsilon > 0$ and each $n \in \mathbb{N}$ there exists $x_n \in B[X]$ with $\left\|\frac{x_n + x}{2}\right\| > 1 - \frac{1}{n}$ but $\|x_n - x\| \geq \varepsilon$.]

2) Show that, *If* B[X] *is lur at* $x \in S(X)$, *then* x *is a strongly exposed point of* B[X].

[Hint: Show that x is strongly exposed by <u>any</u> $f \in \mathcal{D}(x)$, note that $\left\|\frac{x_n + x}{2}\right\| \geq \frac{1}{2}(f(x_n) + 1)$.]

Hence deduce that:

i) *If* B[X] *is lur at* $x \in S(X)$, *then the norm of* X^* *is Fréchet differentiable at each point in* $\mathcal{D}(x)$.

ii) *If* X *is lur, then the norm of* X^* *is Fréchet differentiable at every point of* $\mathcal{D}(S(X))$.

iii) *If* X *is lur and reflexive, then the norm of* X^* *is Fréchet differentiable.*

3) Show that, *if for* $x \in S(X)$, $B[X^*]$ *is lur at some* $f \in \mathcal{D}(x)$, *then* f *is* w*-*strongly exposed by* x.

Hence deduce that:

i) *If for* $x \in S(X)$, $B[X^*]$ *is lur at some* $f \in \mathcal{D}(x)$, *then the norm is Fréchet differentiable at* x.

ii) *If X^* is lur, then the norm of X is Fréchet differentiable.*

4) Show that:

i) *If X^* is lur, then the relative w^* and norm topologies on $S(X^*)$ agree* (this is the property (**) introduced by Namioka in 1974).

ii) The converse of i) is not generally valid. ℓ_1 (indeed $\ell_1(\Gamma)$) *has* (**). [Hint. Let $(f_\alpha) \subset S(\ell_1)$ be such that $f_\alpha \xrightarrow{w^*} f \in S(\ell_1)$. Let $f = (f_1, f_2, \ldots, f_n, \ldots)$ and $f_\alpha = (f_1^\alpha, \ldots, f_n^\alpha, \ldots)$. Given any $\varepsilon > 0$, show that there exists N with $\sum_{n=1}^{N} |f_n| > 1-\varepsilon$. Since $f_\alpha \xrightarrow{w^*} f$ and in finite dimensional spaces the w^* and norm topologies coincide deduce that for some α_0 we have

$$\sum_{n=1}^{N} |f_n^\alpha| \;>\; 1-2\varepsilon \quad \text{for all } \alpha > \alpha_0.$$

Hence conclude that

$$\|f_\alpha - f\|_1 \le \sum_{n=1}^{N} |f_n^\alpha - f_n| + \sum_{n=N}^{\infty} |f_n^\alpha| + \sum_{n=N}^{\infty} |f_n| \;\to\; 0.]$$

5) *The converse of 3 ii) is not in general valid.* Consider ℓ_1 with the equivalent rotund dual norm $\|f\|' = \frac{1}{2}(\|f\|_1 + \|f\|_2)$ (see Exercise 2 on p.55). Show that for this norm the relative w^* and norm topologies on the unit sphere agree. Hence deduce that every support functional is a w^*-strongly exposed point and so the corresponding equivalent norm on c_0 is Fréchet differentiable. Now show that $(\ell_1, \|\cdot\|')$ is not lur at $(1,0,0,\ldots)$. [Hint. Consider (f_n) where $f_n = g_n/\|g_n\|'$ and

$$g_n = (0, \underbrace{\tfrac{2}{n}, \tfrac{2}{n}, \ldots, \tfrac{2}{n}}_{n \text{ terms}}, 0, 0, \ldots).]$$

6) (Sullivan) *Let X be lur.* Show that $A \subseteq S(X)$ *is a dense subset of* $S(X)$ *if* $\mathcal{D}(A)$ *is dense in* $S(X^*)$.

[Hint: For $x \in S(X)$ choose $(a_n) \subset A$ such that $f_{a_n} \xrightarrow{\|\cdot\|} f_x$, observe that $f_{a_n}(a_n + x) \to 2.]$

(* Show that the assumption X lur may be weakened to: the denting points of $B[X]$ are dense in $S(X)$.)

REMARK: *Renorming to gain LUR.*

Kadec (see Diestel) has shown that any separable space X is $\langle$ LUR $\rangle$. The proof begins along similar lines to those used on pp.51 and 52 to show $\langle$ rotund $\rangle$.

(outline): Let $T: X \to \ell_2$ be a continuous linear one-to-one mapping (see proof of Corollary 12 and Theorem 11 on p.52) and take $|||x||| = (\|x\|^2 + \|Tx\|_2^2)^{\frac{1}{2}}$. Then $|||\cdot|||$ is equivalent to $\|\cdot\|$ (show this). Further, if $|||x||| = |||x_m||| = 1$ and $|||x + x_m||| \to 2$ then

$$\begin{aligned}
|||x+x_m|||^2 + \|T(x-x_m)\|_2^2 &= \|x+x_m\|^2 + \|T(x+x_m)\|_2^2 + \|T(x-x_m)\|_2^2 \\
&= \|x+x_m\|^2 + 2\|Tx\|_2^2 + 2\|Tx_m\|_2^2, \text{ by the parallelogram law} \\
&\le 2\|x\|^2 + 2\|x_m\|^2 + 2\|Tx\|_2^2 + 2\|Tx_m\|_2^2, \text{ by the triangle inequality} \\
&= 2|||x||| + 2|||x_m||| = 4
\end{aligned}$$

and so, since $|||x_m + x|||^2 \to 4$ we have $\|T(x - x_m)\|_2^2 \to 0$, or $Tx_m \xrightarrow{\|\cdot\|_2} Tx$. For any $f \in \ell_2^*$ $(= \ell_2)$ we therefore have $f(Tx_m) \to f(Tx)$, or $T^*(f)(x_m) \to T^*(f)(x)$ for all f. Since T is one-to-one, $\Gamma = T^*(\ell_2)$ is a weak*-dense subspace of X^* and we therefore have

$$x_m \xrightarrow{\Gamma} x,$$

where Γ is a <u>norming</u> subspace of X^*.

The proof is completed by establishing that, for any norming subspace Γ of X^*, X could have first been renormed to have the Kadec property H_Γ:

$$\textit{if } \|x_m\| \to \|x\| \textit{ and } x_m \xrightarrow{\Gamma} x, \textit{ then } \|x_m - x\| \to 0.$$

Appendix to §5

APPLICATIONS TO REFLEXIVITY

When establishing Reflexivity we will work through the following ancillary concept and results.

DEFINITION: A subspace M of X^* is <u>strictly norming</u> for X (Dixmier's "*Of Characteristic 1*", 1948) if for every $x \in X$ we have
$\operatorname{Sup}\{f(x) : f \in S(M)\} = \|x\|$.

EXAMPLE: *$\hat{X}$ is strictly norming for X^*.*

LEMMA 1: *X is reflexive if and only if X^{**} contains no proper closed strictly norming subspaces.*

Proof. ($\Leftarrow$) If X^{**} contains no proper closed strictly norming sub-subspace, then, from the above example, $\hat{X}$ must equal X^{**} and so X is reflexive.

($\Rightarrow$) Let M be any proper closed subspace of X^{**}, then by 3) on p.4, there exists $F \in S(X^{***})$ such that $F|_M = 0$. However X is reflexive, so $X^{***} = X^*$ and $F = \hat{f}$ for some $f \in S(X^*)$. We therefore have, $\operatorname{Sup}\{m(f) : m \in S(M)\} = 0$ while $\|f\| = 1$. Thus M is not strictly norming. $\square$

REMARK: The absence of closed proper strictly norming subspaces not only unifies our approach to reflexivity problems but it also serves as a useful intermediary for other Banach space properties. (For example, the Radon-Nikodým property - see p.69.)

EXERCISE: If X is separable and X^* contains no proper closed strictly norming subspaces, conclude that X^* is separable.
[Hint: Consider the closed linear span of

$$\{f_{x_n} : (x_n) \text{ is a dense sequence in } S(X)\},$$

where $x \mapsto f_x$ is any support mapping.]

LEMMA 2: *If M is a strictly norming subspace of X^*, then $B(M)$ is w^*-dense in $B[X^*]$.*

Proof. The proof is identical to that for Goldstine's Theorem (p.7) which is a special case and is therefore left as an EXERCISE. $\square$

EXERCISE: Prove the converse of lemma 2.

COROLLARY 3: *If $B[X^*]$ is the norm closed convex hull of points of $S(X^*)$ at which the relative w^* and norm topologies on $B[X^*]$ agree, then X^* contains no proper closed strictly norming subspaces.*

Proof. If M is a proper closed strictly norming subspace of X^*, then $B[M]$ is a proper norm closed subset of $B[X^*]$, so there exists a point $f \in S(X^*)$ at which the relative w^* and norm topologies on $B[X^*]$ agree which is not in $B[M]$. By lemma 2, there exists $(f_\alpha) \subset B[M]$ with $f_\alpha \xrightarrow{w^*} f$, but then by assumption $f_\alpha \xrightarrow{\|\cdot\|} f$, so $f \in S[M]$, a contradiction. $\square$

THEOREM 4: *If the norm of X is Fréchet differentiable, then $B[X^*]$ is the norm closed convex hull of points of $S(X^*)$ at which the relative w^* and norm topologies on $B[X^*]$ agree.*

Proof. By proposition 8 of §V every point of $\mathcal{D}(S(X))$ is a w^*-strongly exposed point and hence, by Corollary 10 ii) of the same section, a point at which the relative w^* and norm topologies on $B[X^*]$ agree. The result now follows from the subreflexivity of X (Theorem 11 p.25) which states $\overline{\mathcal{D}(S(X))} = S(X^*)$, so $\overline{co}\,\mathcal{D}(S(X)) = B[X^*]$. $\square$

COROLLARY 5: *If the norm of X^* is Fréchet differentiable, then X is reflexive.*

Proof. The proof is immediate from theorem 4, Corollary 3 and lemma 1. $\square$

REMARK: It is a corollary to a renorming theorem of Troyanski (1971) that any reflexive space may be equivalently renormed to be Fréchet differentiable. We therefore have: *a Banach space is reflexive if and only if it admits an equivalent Fréchet differentiable norm.*

s dual (dual)

EXERCISE: i) Show that X is reflexive if $B[X^{**}]$ is the norm closed convex hull of its w*-denting points (see Problem A] of §V).

ii) If X^{**} is lur (see Problem B] of §V), show that X is reflexive.

PROBLEM A] *"Weak" Results*

The motivation for what follows is provided by:

i) Using Mazur's Theorem on p.6, show that the assumption in Corollary 3 above may be weakened to yield:

If $B[X^*]$ *is the norm (equals w) closed convex hull of points of* $S(X^*)$ *at which the relative* w* *and* w *topologies on* $B[X^*]$ *agree, then* X^* *contains no proper closed strictly norming subspaces.*

Motivated by earlier work of Leonard and Sundaresan, Diestel and Faires ["On Vector Measures", trans. Amer. Maths. Soc. 198 (1974), pp.253-271] introduced the following notion.

DEFINITION: X is <u>very smooth</u> if there exists a support mapping from $S(X)$ into $S(X^*)$ which is continuous norm to w. Clearly, X Fréchet differentiable $\Rightarrow$ X is very smooth $\Rightarrow$ X is smooth (Gâteaux differentiable), so a very smooth space has a unique support mapping. Very smoothness does not correspond to any known differentiability property of the norm.

2) Show that *the following are equivalent:*

i) X *is very smooth*

ii) *For each* $x \in S(X)$ *whenever* $(f_n) \subset B[X^*]$ *is such that* $f_n(x) \to 1$, *then* f_n *is w-convergent* (necessarily to the unique element of $\mathcal{D}(x)$.)

iii) *Every point of* $\mathcal{D}(S(X))$ *is an extreme point at which the relative* w* *and* w *topologies on* $B[X^*]$ *agree.*

[Hints: i) $\Rightarrow$ ii) If $f_n(x) \to 1$, use theorem 11 p.25 to show there exists sequences $(x_n) \subset S(X)$ and (f_{x_n}) with $\|x_n - x\| < \frac{1}{n}$, $\|f_{x_n} - f_n\| < \frac{1}{n}$ and $f_{x_n} \in \mathcal{D}(x_n)$. By i) we have $f_{x_n} \xrightarrow{w} f_x$ as $\|x_n - x\| \to 0$ and so, since $\|f_{x_n} - f_n\| \to 0$, we can conclude that $f_n \xrightarrow{w} f_x$.

ii) $\Rightarrow$ iii) Use ii) to show every point of $\mathcal{D}(S(X))$ is a w*-exposed point and hence an extreme point. Now argue as follows: Assume $f_\alpha \not\xrightarrow{w} f_x$, then by passing to a subnet if necessary, we may assume there exists a w-neighbourhood N of f_x with $f_\alpha \notin N$ for any α. If however $f_\alpha \xrightarrow{w^*} f_x$, then $f_\alpha(x) \to 1$ and so for each n, there exists α_n such that $f_{\alpha_n}(x) > 1 - \frac{1}{n}$, by ii) $f_{\alpha_n} \xrightarrow{w} f_x$ as $n \to \infty$, and so f_{α_n} is eventually in N, a contradiction.

iii) $\Rightarrow$ i) By proposition 6 p.48, X is smooth and so by Theorem 2 p.36, if $x_n \xrightarrow{\|\cdot\|} x$ we have $f_{x_n} \xrightarrow{w^*} f_x$ but then by iii) $f_{x_n} \xrightarrow{w} f_x$.]

3) Using 1) above and 2 iii), deduce that X *is reflexive if* X^* *is very smooth.*

4) Show that, $\hat{x} \in S(\hat{X})$ *is a smooth point of* X^{**} *if and only if*

all support mappings from $S(X)$ *into* $S(X^*)$ *are norm to weak continuous at* x.

[Hints: ($\Rightarrow$) Let $\phi: x \to f_x$ be a support mapping from $S(X)$ into $S(X^*)$. Let $\Phi: F \to F_F$ be any support mapping from $S(X^{**})$ into $S(X^{***})$ such that $F_{\hat{x}} = \hat{f}_x$. Deduce that the norm to w* continuity of Φ at $\hat{x}$ implies the norm to w continuity of ϕ at x.

($\Leftarrow$) Let $(f_n) \subset S(X^*)$ be such that $f_n(x) \to 1$ by theorem 3 p.38 to show $\hat{x}$ is a smooth point it suffices to show f_n is w-convergent. To show this use an argument similar to that suggested for i) $\Rightarrow$ ii) in 2) above.]

As a corollary of 4) we have

5) Giles, 1975. X is very smooth if and only if every point of $S(\hat{X})$ is a smooth point of X^{**}.

REMARK: The condition: all support mappings are continuous norm to w at $x \in S(X)$ seems an appropriate localized definition of <u>very smooth at x</u>.

Unlike the cases of norm to w* or norm to norm continuity, it does <u>not</u> seem to be known whether or not the existence of one support mapping which is norm to w continuous at $x \in S(X)$ ensures that all support mappings are norm to w continuous at x.

6) Note that from 5) we have

X^{***} rotund $\Rightarrow X^{**}$ smooth $\Rightarrow \hat{X}$ smooth in $X^{**} \Leftrightarrow X$ very smooth.

Using 3) deduce the following "higher dual" conditions for reflexivity.

i) Dixmier (1948): X *is reflexive if* X^{****} *is rotund.*

ii) Giles/Rainwater (1973): X *is reflexive if* X^{***} *is smooth*

Thus, for a non-reflexive space the properties of smoothness and

rotundity must break down in sufficiently high duals.

§6 Uniform Conditions

One of the earliest studied geometric conditions for a Banach space was that of "uniform convexity", introduced by J.A. Clarkson in 1936 to determine a class of Banach spaces for which a Radon-Nikodým type thoerem held.

DEFINITION: X is uniformly rotund (or uniformly convex) if given $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x, y \in S(X)$ have $\left\|\frac{x+y}{2}\right\| > 1 - \delta$, then $\|x - y\| < \varepsilon$.

[This definition should be compared with that for local uniform rotundity - see p..72-75. They are the same except that here δ does not depend on the particular point x and so may be chosen uniformly on S(X).]

A sometimes useful concept is the *modulus of rotundity of* X, that is the function $\delta\colon (0,2] \to \mathbb{R}^+$ defined by

$$\delta(\varepsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S(X) \text{ and } \|x - y\| \geq \varepsilon\}.$$

Clearly X *is uniformly rotund if and only if* $\delta(\varepsilon) > 0$ *for all* $\varepsilon \in (0,2]$.

EXERCISES: 1) From the parallelogram law, determine the modulus of rotundity for a Hilbert space. Hence deduce that *any Hilbert space is uniformly rotund.*

[Ans: $\delta(\varepsilon) = 1 - \sqrt{1 - (\varepsilon/2)^2}$]

*2) Determine the modulus of convexity $\delta_p(\varepsilon)$ for each of the two dimensional spaces ℓ_p^2. Deduce that ℓ_p^2 is uniformly rotund for $1 < p < \infty$.

[REMARK: *The spaces* $\mathcal{L}_p(\Omega,\mu)$ *and in particular* ℓ_p *are all uniformly rotund for* $1 < p < \infty$. Clearly for $p = 1$ or ∞ they are not. The

modulus of rotundity in ℓ_p was the subject of a thorough study by Hanner (1956). Among other results he shows that

$$\delta(\varepsilon) \sim \begin{cases} \frac{p-1}{2}\left(\frac{\varepsilon}{2}\right)^2 & \text{for } 1 < p < 2 \\ \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p & \text{for } p \geq 2. \end{cases}]$$

3) If X is finite dimensional it follows from the Heine-Borel theorem that S(X) is compact. Show that

$$\delta(\varepsilon) = \min_{x\in S(X)} \min_{y\in S(X)} \{1 - \left\|\frac{x+y}{2}\right\| : \|x - y\| \geq \varepsilon\}.$$

Hence deduce that a finite dimensional space is uniformly rotund if and only if there does not exist $x, y \in S(X)$ with $x \neq y$ and $\left\|\frac{x+y}{2}\right\| = 1$. That is, if and only if X is rotund.

REMARK: Many other moduli of rotundity (including local ones) may be defined. These have been the subject of extensive studies - see for example V.D. Mil'man "Geometric Theory of Banach spaces II", Russian Maths. Surveys 26 (1971) pp.72-163.

Uniform rotundity may be characterized "sequentially" as follows.

PROPOSITION 1 (Clarkson): X *is uniformly rotund if and only if whenever the sequences* (x_n), $(y_n) \subset B[X]$ *are such that* $\left\|\frac{x_n + y_n}{2}\right\| \to 1$ *we have that* $\|x_n - y_n\| \to 0$.

Proof. ($\Rightarrow$) Since $\|x_n\|, \|y_n\| \leq 1$ and $2 \geq \|x_n\| + \|y_n\| \geq \|x_n + y_n\| \to 2$ we have that $\|x_n\|, \|y_n\| \to 1$. Let $x_n' = x_n/\|x_n\|$ and $y_n' = y_n/\|y_n\|$, then $(x_n'), (y_n') \subset S(X)$ and $\|x_n' - x_n\|, \|y_n' - y_n\| \to 0$. Thus $\left\|\frac{x_n' + y_n'}{2}\right\| \to 1$ and it follows immediately from the definition of uniform convexity that $\|x_n' - y_n'\| \to 0$, but then $\|x_n - y_n\| \to 0$.

($\Leftarrow$) Assume X is not uniformly rotund, then for some $\varepsilon > 0$ and every n there exists a pair $x_n, y_n \in S(X)$ with $\left\|\frac{x_n + y_n}{2}\right\| > 1 - \frac{1}{n}$ but $\|x_n - y_n\| \geq \varepsilon$. The sequences (x_n), (y_n) violate the assumption. □

EXERCISES: 1) Show that X *is uniformly rotund if and only if every separable subspace of* X *is uniformly rotund.*
[Hint for ($\Leftarrow$): If X is not uniformly rotund choose (x_n) and (y_n) as in the previous proof and consider $M = \mathrm{span}(\{x_n\} \cup \{y_n\})$.]

2) If X *is uniformly rotund,* show that *whenever* $(x_n) \subset B[X]$ *is such that* $\left\|\frac{x_n + x_m}{2}\right\| \to 1$ *as* n, m $\to \infty$ *we have* (x_n) *is a Cauchy* (and hence by the completeness of X, convergent) *sequence.*

3) If X is uniformly rotund show that every element of $S(X^*)$ is a support functional. Assuming Jame's Theorem (pp.17-18) deduce the Mil'man-Pettis Theorem: Every uniformly rotund Banach space is reflexive.
[REMARK: We will develop an alternative proof shortly. The Mil'man-Pettis Theorem is one of the "most proved" results in the Geometric Theory of Banach spaces. It was first proved by D.P. Mil'man (1938) and independently by B.J. Pettis (1939). Since, then a number of shorter proofs have been found, in particular alternative proofs have been given by Kakutani, Ruston (1949) and Ringrose.]

4) Show that *every non-empty closed convex subset of a uniformly rotund space is a Tchebycheff set.* That is, every such set contains a unique closest point to any point in the space.

5) (optional) Show that X is uniformly convex if and only if whenever (x_n), $(y_n) \subset S(X)$ are such that $\|f_{x_n} + f_{y_n}\| \to 2$ for some support map $x \mapsto f_x$ we have $\|x_n - y_n\| \to 0$. (Sims/Yorke 1978).

Inquadrate Spaces

DEFINITION: X is <u>inquadrate</u> if $\delta(\varepsilon) > 0$ for some $\varepsilon \in (0,2]$. That is, there exists $\varepsilon \in (0,2]$ and $\delta > 0$ such that $\|x - y\| < \varepsilon$ whenever $x, y \in S(X)$ and $\left\|\frac{x + y}{2}\right\| > 1-\delta$. It will be convenient to refer to such a space as being "ε-inquadrate". Clearly, X *is uniformly rotund if and only if* X *is ε-inquadrate for every* $\varepsilon \in (0,2]$.

The following is an adaptation of Ringrose's proof that uniformly rotund Banach spaces are reflexive.

LEMMA 2: *Let* M *be a closed proper subspace of* X^*. *If* M *is ε-inquadrate for some* $\varepsilon \in (0,1)$, *then* M *is not strictly norming for* X.

Proof. By Reisz' lemma (p.4), there exists $f \in S(X^*)$ with $\mathrm{dist}(f,M) > \varepsilon$. Let $\delta > 0$ be such that, for $x,y \in S(M)$ with $\left\|\frac{x + y}{2}\right\| > 1-\delta$ we have $\|x - y\| < \varepsilon$ (δ exists since M is ε-inquadrate). Choose $x \in S(X)$ such that $f(x) > 1-\delta$ and let N be the relative w*-open neighbourhood of f in $B[X^*]$ given by

$$N = \{g \in B[X^*]: g(x) > 1-\delta\}.$$

Now, assume M is strictly norming for X, then by lemma 2 of §V, p.77, there exists $m_1 \in M \cap N$. Moreover, if m_2 is any other element of $M \cap N$ we have $\left\|\frac{m_1 + m_2}{2}\right\| \geq \left(\frac{m_1 + m_2}{2}\right)(x) > 1-\delta$, so $\|m_1 - m_2\| < \varepsilon$. Thus, $N \cap M \subseteq B_\varepsilon[m_1]$.

We therefore have

$$f \in \overline{N}^{w^*} = \overline{M \cap N}^{w^*}, \quad \text{again by lemma 2 of §V.}$$
$$\subseteq B_\varepsilon[m_1], \quad \text{since } B_\varepsilon[m_1] \text{ is w*-compact and hence w*-closed.}$$

So $\|f - m_1\| \leq \varepsilon$, contradicting the fact that $\mathrm{dist}(f,M) > \varepsilon$. □

As a corollary we have:

THEOREM 3: *If* X *is* ε*-inquadrate for some* $\varepsilon \in (0,1)$, *then* X *is reflexive.*

Proof. If X is ε-inquadrate, then $\hat{X}$ is an ε-inquadrate closed subspace of X^{**} which strictly norms X^*, so by lemma 2 $\hat{X}$ cannot be a proper subspace. That is, $\hat{X} = X^{**}$, or X is reflexive. □

COROLLARY 4 (Mil'man-Pettis Theorem): *If* X *is uniformly rotund, then* X *is reflexive.*

REMARK: The above definition of inquadrate is due to R.C. James (1964) who termed such spaces "uniformly non-square". The term inquadrate is due to Day. The concept was motivated by earlier work of Anatole Beck In 1963, Beck characterized separable Banach spaces X in which sequences of "identically distributed X-valued random variables satisfied a law of large numbers". He introduced the notion of (k,ε)-convexity.

DEFINITION: X is <u>(k,ε)-convex</u>, where $k \in N$ and $\varepsilon > 0$ if for each set of k elements, $x_1, x_2, \ldots, x_k \in B[X]$, there is at least one choice of + and - signs such that $\left\| \frac{x_1 \pm x_2 \pm \ldots \pm x_k}{k} \right\| < (1-\varepsilon)$.

EXERCISE. Show that X *is* $(2,\varepsilon)$*-convex for some* $\varepsilon \in (0,1)$ *if and only if* X *is inquadrate.* Hence conclude that X is reflexive if X is $(2,\varepsilon)$-convex for some $\varepsilon \in (0,1)$.

Geometric aspects of (k,ε)-convexity were considered in detail by Giesy in 1964 [Trans. Amer. Math. Soc., 125, pp.114-146]. James [Israel J. of Maths, 18 (1974) pp.145-155] gave an example of a non-reflexive space which is $(3,\varepsilon)$-convex, thus showing that the conclusion of the above exercise is in general false for $k > 2$.

The notion of inquadrate has played a part in the recently developed theory of "super-reflexive" spaces.

DEFINITION (James 1972). For Banach spaces X and Y, Y is <u>finitely represented</u> in X (Day's X mimics Y) if given $\varepsilon > 0$, for each finite dimensional subspace M of Y there exists a one-to-one linear mapping $T: M \to X$ with $|\|Tx\| - \|x\|| < \varepsilon$ for all $x \in S(M)$. That is, every finite dimensional subspace of Y is arbitrarily nearly isometric with a subspace of X.

EXERCISES: 1) Show that the condition "For each $\varepsilon > 0$ there exists $T: M \to X$ with $|\|T x\| - \|x\|| < \varepsilon$ for all $x \in S(X)$" is equivalent to "For each $\varepsilon > 0$ there exists $T : M \to X$ with $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$, where T^{-1} denotes the inverse of T as a mapping from M to T(M).

2) Show that, if Y is finitely represented in X and X is uniformly rotund (inquadrate) then Y is uniformly rotund (inquadrate).

3) Show that ℓ_2^2 is finitely represented in c_0. (Indeed, it may be seen that every Banach space is finitely represented in c_0). [Hint: approximate the unit circle by a polygon made up of lines of the form $f_i^{-1}(1)$, with $\|f_i\| = 1$ $i = 1, 2, \ldots, n$, and consider the mapping

$$T: \underset{\sim}{x} \to (f_1(\underset{\sim}{x}), f_2(\underset{\sim}{x}), \ldots, f_n(\underset{\sim}{x}), 0, 0, 0, \ldots).]$$

If P is any Banach space property, we say X is <u>super-P</u> if whenever Y is finitely represented in X, then Y has P.

EXERCISE: Show that i) Every finite dimensional Banach space is super-reflexive.

ii) Every uniformly convex Banach space is super-reflexive.

James and Enflo have shown that the following are equivalent.

i) X is super-reflexive.

ii) X is $<$ uniformly rotund $>$.

iii) X is $<$ inquadrate $>$.

Many other super-properties have been investigated (for example, super-Radon-Nikodým property) and have proved to be equivalent to super-reflexivity.

Properties in duality with uniform rotundity.

DEFINITION: X is uniformly Fréchet differentiable if

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \quad \text{exists, and is approached uniformly over}$$

$x, y \in S(X)$.

As in Proposition 1 of §V, X *is uniformly Fréchet differentiable if and only if given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for each* $x \in S(X)$ *and* $y \in X$ *with* $\|y\| < \delta$ *there exists some* $f \in S(X^*)$ (necessarily the unique element of $\mathcal{D}(x)$) *with* $\left| \|x + z\| - \|x\| - f(z) \right| < \varepsilon \|z\|$.

DEFINITION: X is uniformly smooth if given $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $x \in S(X)$ and $y \in X$ with $\|y\| < \delta$ we have $\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$.

REMARK: $\rho(t) = \operatorname{Sup}\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1 \text{ and } \|y\| \le t\}$ is known as the modulus of smoothness. X is uniformly smooth if and only if $\rho(t)/t \to 0$ as $t \to 0^+$.

Not surprisingly, as shown in the next proposition, uniform smoothness and uniform Fréchet differentiability are equivalent.

THEOREM 5: *The following are equivalent.*

i) X^* *is uniformly rotund.*

ii) *there exists a support mapping* $x \mapsto f_x$ *from* $S(X)$ *into* $S(X^*)$ *which is uniformly continuous norm to norm.* In which case $x \mapsto f_x$ is necessarily the unique support mapping on X.

iii) X *is uniformly Fréchet differentiable.*

iv) X *is uniformly smooth.*

Proof. i) $\Rightarrow$ ii). Given $\varepsilon > 0$, let $\delta > 0$ be such that $\|f\| = \|g\| = 1$ and $\left\|\frac{f + g}{2}\right\| > 1-\delta$ implies $\|f - g\| < \varepsilon$. Then, if $\|x\| = \|y\| = 1$ and $\|x - y\| < 2\delta$ we have

$$\begin{aligned} \|f_x + f_y\| &\ge (f_x + f_y)(x) \\ &= 1 + f_y(x) \\ &= 1 + f_y(y) + f_y(x - y) \\ &= 2 + f_y(x - y) \\ &\ge 2 - \|x - y\| \quad , \text{ as } |f_y(x-y)| \le \|x-y\| \\ &\ge 2 - 2\delta . \end{aligned}$$

Thus, $\left\|\frac{f_x + f_y}{2}\right\| > 1-\delta$ and so $\|f_x - f_y\| < \varepsilon$, establishing the uniform continuity of $x \mapsto f_x$.

ii) $\Rightarrow$ iii). Given $\varepsilon > 0$ let $\delta \in (0,\frac{1}{2})$ be such that, for $x, y \in S(X)$ with $\|x - y\| < 4\delta$ we have $\|f_x - f_y\| < \varepsilon$. Then, for any $x \in S(X)$ and $z \neq 0$ with $\|z\| < \delta$ we have

$$\begin{aligned} \left\|\frac{x + z}{\|x + z\|} - x\right\| &\le \frac{1}{\|x + z\|}(\|z\| + |1 - \|x + z\||\|x\|) \\ &\le \frac{2\delta}{1 - \delta} \\ &\le 4\delta . \end{aligned}$$

So $\left\| f_{\frac{x+z}{\|x+z\|}} - f_x \right\| < \varepsilon.$

The result now follows from lemma 1 of §2 on p.28 which yields

$$0 \le \frac{\|x + z\| - \|x\| - f_x(z)}{\|z\|} \le \left(f_{\frac{x+z}{\|x+z\|}} - f_x\right)\left(\frac{z}{\|z\|}\right)$$

$$\le \left\| f_{\frac{x+z}{\|x+z\|}} - f_x \right\| < \varepsilon .$$

iii) $\Rightarrow$ iv). Given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x \in S(X)$ and $\|z\| < \delta$ we have

$$\|x + z\| - \|x\| - f_x(z) \le \frac{\varepsilon}{2} \|z\| .$$

Now, $\|-z\| = \|z\| < \delta$ so

$$\|x - z\| - \|x\| + f_x(z) \le \frac{\varepsilon}{2} \|z\| .$$

Adding these inequalities yields,

$$\|x + z\| + \|x - z\| < 2 + \varepsilon\|z\| \quad \text{for all } z \text{ with } \|z\| < \delta .$$

iv) $\Rightarrow$ i). Given $\varepsilon > 0$ let $\delta > 0$ be such that for $x \in S(X)$ and y with $\|y\| < \delta$ we have $\|x + y\| + \|x - y\| < 2 + \frac{\varepsilon}{4} \|y\|$.
Let $f, g \in S(X^*)$ have $\|f - g\| \ge \varepsilon$, then for some $y \in X$ with $\|y\| < \delta$ we have $(f - g)(y) \ge \varepsilon \frac{\delta}{2}$. Then

$$\begin{aligned}\|f + g\| &= \operatorname{Sup}\{(f + g)(x): x \in S(X)\} \\ &= \operatorname{Sup}\{f(x + y) + g(x - y) - (f - g)(y): x \in S(X)\} \\ &< \operatorname{Sup}\{\|x + y\| + \|x - y\| - (f - g)(y): x \in S(X)\} \\ &< 2 + \varepsilon\|y\|/4 - \varepsilon\delta/2 \\ &< 2 - \varepsilon\delta/4 ,\end{aligned}$$

or $\left\|\frac{f + g}{2}\right\| < 1 - \varepsilon\delta/8$. Thus X^* is uniformly rotund. □

Since uniform rotundity implies reflexivity (Corollary 4) we see that *uniform smoothness* (or uniform Fréchet differentiability) *is in complete duality with uniform rotundity*, and that any of these conditions on X ensures reflexivity.

EXERCISE: Show that X^* *is uniformly smooth if and only if given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $x \in S(X)$ *and* $f_x \in \mathcal{D}(x)$ *if* $y \in S(X)$ *has* $f_x(y) > 1-\delta$ *then* $\|x - y\| < \varepsilon$. That is the support functionals of $S(X)$ uniformly determine slices of $B[X]$ of arbitrarily small diameter.

[Hint: Replace X^* uniformly smooth by X uniformly rotund. For ($\Leftarrow$): if $x, y \in S(X)$ have $\left\|\frac{x + y}{2}\right\| > 1 - \delta/2$, let $z' = \frac{x + y}{2}$ and $z = z'/\|z'\|$. Choose $f \in \mathcal{D}(z)$ and note that $f(x), f(y) > 1-\delta$.]