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A CLASS OF SPACES WITH WEAK NORMAL STRUCTURE

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It has recently been shown that a Banach space enjoys the weak fixed point property if it is ε_0–inquadrate for some $\varepsilon_0 < 2$ and has WORTH; that is, if $x_n \xrightarrow{w} 0$ then, $\|x_n - x\| - \|x_n + x\| \longrightarrow 0$, for all x. We establish the stronger conclusion of weak normal structure under the substantially weaker assumption that the space has WORTH and is 'ε_0–inquadrate in every direction' for some $\varepsilon_0 < 2$.

A Banach space X is said to have the *weak fixed point property* if whenever C is a nonempty weak compact convex subset of X and $T : C \longrightarrow C$ is a *nonexpansive* mapping; (that is, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$), then T has a fixed point in C.

It is well known that if X fails to have the weak fixed point property then it fails to have *weak normal structure*; that is, X contains a weak compact convex subset C with more than one point which is *diametral* in the sense that, for all $x \in C$

$$\sup\{\|y - x\| : y \in C\} = \operatorname{diam} C := \sup\{\|y - z\| : y, z \in C\}.$$

Further, if X fails to have weak normal structure then there exists a sequence, (x_n), satisfying;

$$\text{(S1)} \qquad x_n \xrightarrow{w} 0$$

and for $C := \overline{\text{co}}\{x_n : n \in \mathbb{N}\}$

$$\text{(S2)} \qquad \lim_n \|x - x_n\| = \operatorname{diam} C = 1, \qquad \text{for all } x \in C.$$

That is, (x_n) is a non-constant weak null sequence which is 'diameterising' for its closed convex hull. In particular, since $0 \in C$, we have $\|x_n\| \to 1$.

Details of these and related results may be found in the monograph by Goebel and Kirk [7] for example.

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For the Banach space X we define $\delta : [0, 2] \times X\backslash\{0\} \longrightarrow |\mathbb{R}|$ by

$$\delta(\varepsilon, x) := \inf\left\{1 - \left\|y + \frac{\varepsilon}{2\|x\|}x\right\| : \|y\| \leqslant 1 \text{ and } \left\|y + \frac{\varepsilon}{\|x\|}x\right\| \leqslant 1\right\}.$$

We refer to $\delta(\varepsilon, x)$ as the *modulus of convexity in the direction* x. X is *uniformly convex in every direction* (UCED) if $\delta(\varepsilon, x) > 0$, for all $x \neq 0$ and all $\varepsilon > 0$ (Day, James and Swaminathan [**2**]).

The *modulus of convexity* of X is given by

$$\delta(\varepsilon) := \inf_{x \neq 0} \delta(\varepsilon, x),$$

and X is *uniformly convex* if $\delta(\varepsilon) > 0$, for all $\varepsilon > 0$.

Following Day, given $\varepsilon_0 \in (0, 2]$ we say X is ε_0*–inquadrate* if $\delta(\varepsilon_0) > 0$.

By analogue with this last definition, for $\varepsilon_0 \in (0, 2]$ we shall say X is ε_0*–inquadrate in every direction* if $\delta(\varepsilon_0, x) > 0$, for all $x \neq 0$.

It is readily verified that X is ε_0–inquadrate in every direction if and only if whenever $\limsup_n \|x_n\| \leqslant 1$, $\limsup_n \|x_n + \lambda_n x\| \leqslant 1$, and $\|x_n + (\lambda_n/2)x\| \longrightarrow 1$ we have $\limsup_n |\lambda_n|\ \|x\| \leqslant \varepsilon_0$.

Note there are also the weaker notions, of X being ε_0–inquadrate in some subset of directions, and for each $x \neq 0$ there being an $\varepsilon_x \in [0, 2)$ with $\delta(\varepsilon_x, x) > 0$, however; these will not concern us.

Garkavi [**5**] showed that spaces which are UCED have weak normal structure and hence enjoy the weak fixed point property. An essentially similar argument establishes the result for spaces which are ε_0–inquadrate in every direction for some $\varepsilon_0 \in (0, 1)$. To see this, suppose that X fails weak normal structure and so contains a sequence (x_n) satisfying (S1) and (S2). Choose m so that $\|x_m\| > \varepsilon_0$, then putting $x = x_m$ we have $\|x_n\| \longrightarrow 1$, $\|x_n - x\| \longrightarrow 1$ and, since $0 \in C$, so $x/2 \in C$, $\|x_n - x/2\| \longrightarrow 1$ contradicting the assumption that X is ε_0–inquadrate in every direction.

In general the situation when $1 \leqslant \varepsilon_0 < 2$ remains unresolved, even in the ε_0–inquadrate case.

Two other 'classical' conditions known to be sufficient for weak normal structure are:

(1) *The condition of Opial,* whenever $x_n \xrightarrow{w} 0$ and $x \neq 0$ we have

$$\limsup_n \|x_n\| < \limsup_n \|x_n - x\|.$$

The conditition was introduced by Opial [**10**], and shown to imply weak normal structure by Gossez and Lami Dozo [**8**]. The condition is unchanged if both lim sups are replaced by lim infs. We say X satisfies the *non-strict Opial condition* if the condition holds with strict inequality replaced by '$\leqslant$'.

(2) ε_0*–Uniform Radon–Reisz* (ε_0–URR), for some $\varepsilon_0 \in (0,1)$; there exist $\delta > 0$ so that whenever $x_n \xrightarrow{w} x$, with $\|x_n\| \leqslant 1$ and $sep(x_n) := \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon_0$ we have $\|x\| < 1 - \delta$. When the condition holds for all $\varepsilon_0 > 0$ we say X is URR. The condition is essentially due to Huff [**9**], and was shown to imply weak normal structure by van Dulst and Sims [4].

Gossez and Lami Dozo [**8**] showed that Opial's condition follows from the non-strict version in the presence of uniform convexity, however a careful reading of their argument establishes the following.

Proposition 1. *If X is UCED and satisfies the non-strict Opial condition then X satisfies the Opial condition.*

Proof: Suppose X fails the Opial condition, then there exists a sequence $x_n \xrightarrow{w} 0$ and $x \neq 0$ with

$$\liminf_n \|x_n\| \not< \liminf_n \|x_n - x\|.$$

By the non-strict Opial condition we must have equality, and without loss of generality we may assume that $\|x_n\| \longrightarrow 1$. Let $y_n := x_n - x$, then $\|x_n\|, \|y_n\| \longrightarrow 1$ and $x_n - y_n = x$. Thus, by UCED we must have that

$$\liminf_n \|x_n - x/2\| = \liminf_n \|(x_n + y_n)/2\| < 1 = \liminf_n \|x_n\|,$$

contradicting the non-strict Opial condition. ∎

In Sims [**12**], the notion of *weak orthogonality* (WORTH); if $x_n \xrightarrow{w} 0$ then for all $x \in X$ we have

$$\|x_n - x\| - \|x_n + x\| \longrightarrow 0,$$

was introduced (also see Rosenthal [**11**]), and it was asked whether spaces with WORTH have the weak fixed point property. WORTH generalises the lattice theoretic notion of 'weak orthogonality' introduced by Borwein and Sims [1] and shown to be sufficient for the weak fixed point property in Sims [**12**].

Proposition 2. *The non-strict Opial condition is entailed by WORTH.*

Proof: If $x_n \xrightarrow{w} 0$ then for any $x \in X$ we have

$$\begin{aligned}\limsup_n \|x_n\| &\leqslant \frac{1}{2}\big(\limsup_n \|x_n - x\| + \limsup_n \|x_n + x\|\big)\\ &= \limsup_n \|x_n - x\|, \quad \text{as } \lim_n \|x_n - x\| - \|x_n + x\| = 0, \text{ by WORTH.}\end{aligned}$$

∎

Combining this with proposition 1 we obtain the following.

Corollary 3. *A Banach space which has UCED and WORTH satisfies the Opial condition.*

Recently García Falset [**6**] working through the intermediate notion of the ACM–property has shown that spaces which are ε_0–inquadrate for some $\varepsilon_0 < 2$ and have WORTH have the weak fixed point property.

We give a direct and elementary proof that the stronger conclusion of weak normal structure follows from the substantially weaker premises of WORTH and ε_0–inquadrate in every direction for some $\varepsilon_0 < 2$. That ε_0–inquadrate in every direction is genuinely weaker than ε_0–inquadrate follows since spaces which are ε_0–inquadrate, for an $\varepsilon_0 < 2$ are necessarily superreflexive (van Dulst [**3**]) while every separable Banach space has an equivalent norm which is UCED [**2**], and hence ε_0–inquadrate in every direction for $0 < \varepsilon_0 < 2$.

Definition: We say a Banach space X has *property (k)* if there exists $k \in [0, 1)$ such that whenever $x_n \xrightarrow{w} 0$, $\|x_n\| \longrightarrow 1$ and $\liminf_n \|x_n - x\| \leqslant 1$ we have $\|x\| \leqslant k$. Note: By considering subsequences we see that the property remains unaltered if in the definition we replace $\liminf$ by $\limsup$.

Property (k) is an interesting condition which clearly exposes the uniformity in Opial's condition. Indeed Opial's condition corresponds to propery (k) with $k = 0$.

Proposition 4. *If X has property (k) then X has weak normal structure.*

Proof: Suppose X fails weak normal structure, then there is a sequence (x_n) satisfying (S1) and (S2). Choosing m sufficiently large so that $\|x_m\| > k$ (see the remark following S2) and taking $x := x_m$ we have that property (k) is contradicted by the sequence (x_n). □

We now turn to conditions sufficient for property (k), and hence also for weak normal structure.

Proposition 5. *If X is ε_0–URR, for some $\varepsilon_0 \in (0,1)$ then X has property (k).*

Proof: Suppose $x_n \xrightarrow{w} 0$, $\|x_n\| \longrightarrow 1$ and $\limsup_n \|x_n - x\| \leqslant 1$. Choose m so that $\|x_m\| > \varepsilon_0$, then, since $\liminf_n \|x_n - x_m\| \geqslant \|x_m\|$, we may extract a subsequence, which we continue to denote by (x_n), with $\|x_n - x_m\| > \varepsilon_0$ for all n. Continuing in this way we obtain a subsequence, still denoted by (x_n), with $\operatorname{sep}(x_n) > \varepsilon_0$. But, then $x_n - x \xrightarrow{w} x$ is a sequence in the unit ball with a separation constant greater than ε_0 and so $\|x\| \leqslant 1 - \delta$, where δ is given by the definition of ε_0–URR. Thus X has property (k) with $k = 1 - \delta$. □

Proposition 6. *If X is ε_0–inquadrate in every direction for some $\varepsilon_0 \in (0,1)$ and satisfies the non-strict Opial condition then X has property (k).*

PROOF: Suppose $x_n \xrightarrow{w} 0$, $\|x_n\| \longrightarrow 1$ and $\limsup_n \|x_n - x\| \leqslant 1$. If $x = 0$ there is nothing to prove, so we assume that $x \neq 0$. Then x_n and $y_n := x_n - x$ are two sequences in the unit ball with $x_n - y_n = x$ a fixed direction, so

$$\begin{aligned}\left\|\frac{x_n + y_n}{2}\right\| &\leqslant 1 - \delta(\|x\|, x) \\ &< 1, \quad \text{if } \delta(\|x\|, x) > 0.\end{aligned}$$

But, then $\limsup_n \|x_n - x/2\| < 1 = \limsup_n \|x_n\|$, contradicting the non-strict Opial condition. Thus we must have $\delta(\|x\|, x) = 0$ and this requires $\|x\| \leqslant \varepsilon_0$. So X has property (k) with $k = \varepsilon_0$. ∎

The case when X is ε_0–inquadrate in every direction for an $\varepsilon_0 \in [1, 2)$ is handled by the following proposition which in conjunction with proposition 3 yields our main result.

PROPOSITION 7. *If X is ε_0–inquadrate in every direction for some $\varepsilon_0 \in (0, 2)$ and has WORTH then X has property (k).*

PROOF: Suppose $x_n \xrightarrow{w} 0$, $\|x_n\| \longrightarrow 1$ and $\limsup_n \|x_n - x\| \leqslant 1$. Let $a_n : = x_n - x$ and $b_n : = x_n + x$. Then by WORTH $\|a_n\| - \|b_n\| \longrightarrow 0$, so $\limsup_n \|b_n\| = \limsup_n \|a_n\| = \limsup_n \|x_n - x\| \leqslant 1$, and $b_n - a_n = 2x$. Therefore we have

$$\limsup_n \|x_n\| = \limsup_n \|(a_n + b_n)/2\| \leqslant 1 - \delta(2\|x\|, x) < 1,$$

a contradiction, unless $2\|x\| \leqslant \varepsilon_0$. Thus X has property (k) with $k = \varepsilon_0/2$. ∎

If X is ε_0–inquadrate then the calculations of the previous proof allow some flexibility. If we measure the 'degree of WORTHwhileness' of a Banach space X by

$$w := \sup\{\lambda : \lambda \liminf_n \|x_n + x\| \leqslant \liminf_n \|x_n - x\|, \text{ whenever } x_n \xrightarrow{w} 0 \text{ and } x \in X\},$$

(so X has WORTH if and only if $w = 1$) then we can adapt the above calculations to verify the following.

PROPOSITION 8. *X has property (k) if*

$$w > \max\{\varepsilon_0/2, 1 - \delta(\varepsilon_0)\}$$

for some positive ε_0.

We close by noting that many spaces have WORTH, including all Banach lattices which are 'weakly orthogonal' in the sense introduced by Borwein and Sims [**1**]. In particular for $0 < \alpha < 1$ the space ℓ_2 with the equivalent norm $\|x\| := \max\{\alpha\|x\|_2, \|x\|_\infty\}$

has WORTH and hence satisfies the non-strict Opial condition, but fails to satisfy the Opial condition for any α.

However, many important spaces do not enjoy WORTH, for example with the exception of $p = 2$ all the spaces $\mathcal{L}_p[0,1]$ fail to satisfy the non-strict Opial condition (see the details of the example given in Opial [**10**]) and hence also fail to have WORTH. They do however enjoy property (k); for example, when $p > 2$ it follows from Clarkson's inequality that we may take $k = (1 - 2^{-p})^{1/p}$.

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