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A GEOMETRICALLY ABERRANT BANACH SPACE WITH NORMAL STRUCTURE

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An example is given of a Banach space with normal structure which does not satisfy the geometrical conditions commonly expected to be related to normal structure.

A Banach space is said to have *normal structure* if for each non-trivial bounded convex subset K there exists a point $p \in K$ such that

$$\sup\{\|p-x\| : x \in K\} < \operatorname{diam} K .$$

A Banach space is said to have *uniformly normal structure* if there exists a $0 < k < 1$ such that for each bounded convex subset K there exists a point $p \in K$ such that

$$\sup\{\|p-x\| : x \in K\} \leq k \operatorname{diam} K .$$

Normal structure was introduced by Brodskiĭ and Milman [2] and has been significant in the development of fixed point theory. A recent survey of results on normal structure has been given by Swaminathan [11].

Considerable research has been directed into finding geometrical conditions which imply normal structure.

A Banach space is said to be *uniformly rotund* if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for x, y, $\|x\| = \|y\| = 1$, $\|x-y\| < \varepsilon$

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when $\|x+y\| > 2 - \delta$; such a space has uniformly normal structure, [5].

A Banach space is said to be *uniformly rotund in every direction* if for any given $z \neq 0$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, z) > 0$ such that $|\lambda| < \varepsilon$ for x, y , $\|x\| = \|y\| = 1$ and $x - y = \lambda z$ when $\|x+y\| > 2 - \delta$; such a space has normal structure, [4], but not necessarily uniformly normal structure, [8].

A Banach space is said to be *weakly uniformly Kadec-Klee* if there exists an $\varepsilon < 1$ and a $\delta > 0$ such that for every sequence $\{x_n\}$, $\|x_n\| \leq 1$ which converges weakly to x and $\inf\{\|x_n - x_m\| : m \neq n\} \geq \varepsilon$ we have $\|x\| \leq 1 - \delta$. Van Dulst and Sims have recently shown that such a space has *weak normal structure*, that is, the normal structure property holds for weakly compact convex sets, [12].

A Banach space is said to be *locally uniformly rotund* if for any given x , $\|x\| = 1$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, x) > 0$ such that $\|x-y\| < \varepsilon$ for $\|y\| = 1$, when $\|x+y\| > 2 - \delta$. Smith and Turett have recently provided an example of a reflexive locally uniformly rotund space which does not have normal structure, [10].

In this paper we give an example of a reflexive Banach space which lacks all of these geometrical properties but which does have normal structure.

In order to produce an example of a discontinuous metric projection Brown devised a geometrically interesting equivalent renorming of Hilbert sequence space l^2 , [3]. Given natural basis $\{e_n\}$ and writing

$$M \equiv \left\{\{\lambda_n\} \in l^2 : \lambda_1 = 0\right\}$$

and

$$M_k \equiv \operatorname{sp}\{e_1, e_k\} \quad \text{for} \quad k \geq 3 ,$$

l^2 can be given an equivalent rotund norm $\|\cdot\|$ such that its restriction to M remains the original l^2-norm $\|\cdot\|_2$ and its restriction to M_k is an $l^{p(k)}$-norm where $p(k) \to \infty$ as $k \to \infty$.

Brown's space is not uniformly rotund in every direction. For $x_k \equiv (e_1+e_k)/\|e_1+e_k\|$ and $y_k = (-e_1+e_k)/\|-e_1+e_k\|$,

$$x_k - y_k = \frac{2}{2^{1/p(k)}} \cdot e_1 \quad \text{and} \quad \|x_k-y_k\| \to 2 \quad \text{as} \quad k \to \infty$$

but

$$x_k + y_k = \frac{2}{2^{1/p(k)}} \cdot e_k \quad \text{and} \quad \|x_k+y_k\| \to 2 \quad \text{as} \quad k \to \infty .$$

Brown's space is not weakly uniformly Kadec-Klee. For $x_k \equiv (e_1+e_k)/\|e_1+e_k\|$ and any $y \equiv \sum \alpha_n e_n$,

$$\begin{aligned}(x_k, y) &= \frac{1}{2^{1/p(k)}} (\alpha_1+\alpha_k) \\ &\to \alpha_1 \quad \text{as} \quad k \to \infty \\ &= (e_1, y) .\end{aligned}$$

So the sequence $\{x_k\}$ converges weakly to e_1. But

$$\begin{aligned}\|x_k-x_l\|_2^2 &= \left\| e_1\left(\frac{1}{2^{1/p(k)}} - \frac{1}{2^{1/p(l)}}\right) + \frac{e_k}{2^{1/p(k)}} - \frac{e_l}{2^{1/p(l)}}\right\|_2^2 \\ &= \left(\frac{1}{2^{1/p(k)}} - \frac{1}{2^{1/p(l)}}\right)^2 + \frac{1}{2^{2/p(k)}} + \frac{1}{2^{2/p(l)}} \\ &\to 2 \quad \text{as} \quad k, l \to \infty .\end{aligned}$$

However, $(1/\sqrt{2})\|x\|_2 \le \|x\| \le \|x\|_2$ for all $x \in l_2$ so

$$\liminf_{k,l\to\infty} \|x_k-x_l\| \ge 1 .$$

Therefore, for every $0 < \varepsilon < 1$,

$$\liminf\{\|x_k-x_l\| : k \ne l\} \ge \varepsilon ;$$

but $\|e_1\| = 1$.

Brown's space is not locally uniformly rotund. For $x_k \equiv (e_1+e_k)/\|e_1+e_k\|$,

$$\|e_1+x_k\| = \left\|\left(1 + \frac{1}{2^{1/p(k)}}\right)e_1 + \frac{1}{2^{1/p(k)}}\, e_k\right\|$$

$$= \left\{\left(1 + \frac{1}{2^{1/p(k)}}\right)^{p(k)} + \tfrac{1}{2}\right\}^{1/p(k)}$$

$$\to 2 \quad \text{as} \quad k \to \infty .$$

But

$$\|e_1-x_k\| = \left\|\left(1 - \frac{1}{2^{1/p(k)}}\right)e_1 - \frac{1}{2^{1/p(k)}}\, e_k\right\|$$

$$= \left\{\left(1 - \frac{1}{2^{1/p(k)}}\right)^{p(k)} + \tfrac{1}{2}\right\}^{1/p(k)}$$

$$\to 1 \quad \text{as} \quad k \to \infty .$$

Nevertheless, as a reflexive Banach space containing a Hilbert subspace of codimension one, Brown's space does have normal structure as is evident from the following general result.

LEMMA. *If a Banach space X contains a closed subspace M of finite codimension with uniformly normal structure then X has normal structure.*

Proof. Since M has uniformly normal structure it is reflexive, [9], and since X contains a reflexive subspace of finite codimension it too is reflexive. Suppose that X does not have normal structure. Then by the characterisation theorem of Brodskii and Milman [2], there exists a weakly compact convex subset K containing a sequence $\{x_n\}$ such that

$$d\left(x_{n+1},\ \mathrm{co}\left\{x_1,\ \ldots,\ x_n\right\}\right) \to \operatorname{diam} K \quad \text{as} \quad n \to \infty .$$

Subsequences of $\{x_n\}$ satisfy this property so we may, by weak compactness assume that $\{x_n\}$ converges weakly; by translation we may assume that $\{x_n\}$ is weakly convergent to 0; by scaling we may assume that $\operatorname{diam} K = 1$. Consider a linear projection P from X onto M. Since $\{x_n\}$ converges weakly to 0 so $\{x_n-Px_n\}$ is convergent to 0 in the finite dimensional complement of M.

Given $0 < k < 1$ the constant associated with the uniformly normal structure of M, choose $0 < \varepsilon < (1-k)/4(1+k)$. Then there exists a ν such that

$$\|x_n - Px_n\| < \varepsilon \quad \text{for all} \quad n > \nu \ .$$

Consider $K' \equiv \text{co}\{x_n : n > \nu\}$. Now K' has the *diametral* property that, for any $x \in K'$,

$$\sup\{\|x-y\| : y \in K'\} = \text{diam } K' = 1 \ .$$

Since $\|x-Px\| < \varepsilon$ for all $x \in K'$, it follows that $\text{diam } P(K') \le 1 + 2\varepsilon$. From the uniformly normal structure of M there exists a $p \in K'$ such that

$$\|Pp-Px\| \le k(1+2\varepsilon) \quad \text{for all} \quad x \in K' \ .$$

But then, for all $x \in K'$,

$$\begin{aligned} \|p-x\| &\le \|p-Pp\| + \|Pp-Px\| + \|Px-x\| \\ &\le 2\varepsilon + k(1+2\varepsilon) \\ &< \tfrac{1}{2}(1+k) < 1 \end{aligned}$$

and this contradicts the diametral property of K' .

Bernal and Sullivan have recently provided a condition under which an equivalent renorming of Hilbert space has normal structure, [1]. Given a Hilbert space $(X, \|\cdot\|_2)$ and a norm $\|\cdot\|$ on X such that

$$\frac{1}{\beta} \|x\|_2 \le \|x\| \le \|x\|_2 \quad \text{for all} \quad x \in X$$

where $1 \le \beta < \sqrt{2}$, then the Banach space $(X, \|\cdot\|)$ has normal structure. However, Brown's renorming of Hilbert space has $\beta = \sqrt{2}$ and is therefore an example which shows that, for equivalent renormings of Hilbert space the Bernal-Sullivan condition is not necessary for normal structure.

As a reflexive Banach space containing a closed subspace with discontinuous metric projection it can be deduced indirectly from Fan and Glicksberg [6] that Brown's space lacks a variety of geometrical properties. Brown's space has also been used by Giles [7] to demonstrate the relationship between geometrical properties used by Vlasov in the convexity of the Chebychev set problem.

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