A new approach to generalized metric spaces

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ABSTRACT.

To overcome fundamental flaws in B. C. Dhage's theory of generalized metric spaces, flaws that invalidate most of the results claimed for these spaces, we introduce an alternative more robust generalization of metric spaces. Namely, that of a G-metric space, where the G-metric satisfies the axioms:

(1) $G(x, y, z) = 0$ *if* $x = y = z$,
(2) $0 < G(x, x, y)$; *whenever* $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ *whenever* $z \neq y$,
(4) G *is a symmetric function of its three variables*, *and*
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$

1. Introduction

During the sixties, 2-metric spaces were introduced by Gahler [**6**], [**7**].

DEFINITION 1. Let X be a nonempty set, and let $\mathbf{R}$ denote the real numbers. A function $d : X \times X \times X \to \mathbf{R}^+$ satisfying the following properties:

(A1) For distinct points $x, y \in X$, there is $z \in X$, such that $d(x, y, z) \neq 0$.
(A2) $d(x, y, z) = 0$ if two of the triple $x, y, z \in X$ are equal.
(A3) $d(x, y, z) = d(x, z, y) = \cdots$ (symmetry in all three variables),
(A4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$,
    for all $x, y, z, a \in X$,

is called a 2-*metric*, on X. The set X equipped with such a 2-metric is called a 2-*metric space*.

It is clear that taking $d(x, y, z)$ to be the area of the triangle with vertices at x, y and z in $\mathbf{R}^2$ provides an example of a 2 -metric.

Gahler claimed that a 2-metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. For instance Ha et al in [**8**] show that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is, further there is no easy relationship between results obtained in the two settings, in particular the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated.

These considerations led Bapure Dhage in his PhD thesis [1992] to introduce a new class of generalized metrics called D-metrics.

DEFINITION 2. A function $D : X \times X \times X \to \mathbf{R}^+$ is a D-*metric* if it satisfies axioms (A3) and (A4), but with (A1) and (A2) replaced by the single axiom

   (A0) $D(x, y, z) = 0$ if and only if $x = y = z$.

An additional property sometimes imposed by Dhage on a D–metric is,

   (A5) $D(x, y, y) \leq D(x, z, z) + D(z, y, y)$ for all $x, y, z \in X$.

The perimeter of the triangle with vertices at x, y and z in $\mathbf{R}^2$ provides the prototypical example of a D-metric. Indeed, for any metric space (X, d) Dhage gave as examples of D-metrics on X;

   (E$_s$) $D_s(d)(x, y, z) = \frac{1}{3}(d(x, y) + d(y, z) + d(x, z))$, and
   (E$_m$) $D_m(d)(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$.

However, as exploited in [**10**], for these to satisfy the axioms of a D-metric it is not necessary that d satisfy the triangle inequality, only that it be a semi-metric.

In a subsequent series of papers Dhage attempted to develop topological structures in such spaces. He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results. Subsequently, these works have been the basis for over 40 papers by Dhage and other authors.

But, in 2003 we demonstrated in [**10**] that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect (also see, [**9**]). For instance a D-metric need not be a continuous function of its variables, the axiom (A4) is rarely sharp and, despite Dhage's attempts to construct such a topology, D-convergence of a sequence (x_n) to x, in the sense that $D(x_m, x_n, x) \to 0$ as $n, m \to \infty$, need not correspond to convergence in any topology.

These considerations lead us to seek a more appropriate notion of generalized metric space.

DEFINITION 3. Let X be a nonempty set, and let $G : X \times X \times X \to \mathbf{R}^+$, be a function satisfying:

   (G1) $G(x, y, z) = 0$ if $x = y = z$
   (G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
   (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
   (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables), and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality),

then the function G is called a *generalized metric*, or, more specifically a *G-metric* on X, and the pair (X, G) is a *G-metric space*.

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and z in $\mathbf{R}^2$, further taking a in the interior of the triangle shows that (G5) is best possible.

If (X, d) is an ordinary metric space, then E_s and E_m above define G-metrics on X, however, for this to be so it is now necessary that d satisfy the triangle inequality.

DEFINITION 4. Following Dhage's terminology, a G-metric space (X, G) is *symmetric* if

(G6) $G(x, y, y) = G(x, x, y)$, for all $x, y \in X$,

Clearly, any G-metric space where G derives from an underlying metric via E_s or E_m is symmetric.

The following example presents the simplest instance of a nonsymmetric G-metric and so also one which does not arise from any metric in the above ways.

EXAMPLE 1. Let $X = \{a, b\}$, let,

$G(a, a, a) = G(b, b, b) = 0$
$G(a, a, b) = 1,\ G(a, b, b) = 2$

and extend G to all of $X \times X \times X$ by symmetry in the variables. Then it is easily verified that G is a G-metric, but $G(a, b, b) \neq G(a, a, b)$.

2. Properties of G–Metric Spaces

The following useful properties of a G-metric are readily derived from the axioms.

PROPOSITION 1. *Let (X, G) be a G-metric space, then for any x, y, z and $a \in X$ it follows that:*

(1)   *if $G(x, y, z) = 0$, then $x = y = z$,*
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$,
(7) $|G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(z, a, a)\}$,
(8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z)$,
(9) $|G(x, y, z) - G(y, z, z)| \leq \max\{G(x, z, z), G(z, x, x)\}$,
(10) $|G(x, y, y) - G(y, x, x)| \leq \max\{G(y, x, x), G(x, y, y)\}$.

Easy calculations establish the following.

PROPOSITION 2. *Let (X, G) be a G-metric space and let $k > 0$, then G_1 and G_2 are also G-metrics on X, where,*

$$(1) \quad G_1(x, y, z) = \min\{k, G(x, y, z)\}, \; and$$

$$(2) \quad G_2(x, y, z) = \frac{G(x, y, z)}{k + G(x, y, z)}.$$

Further, if $X = \bigcup_{i=1}^{n} A_i$ is any partition of X then,

$$(3) \quad G_3(x, y, z) = \begin{cases} G(x, y, z), & \text{if for some } i \text{ we have } x, y, z \in A_i, \\ k + G(x, y, z), & \text{otherwise,} \end{cases}$$

is also a G-metric.

PROPOSITION 3. *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) *(X, G) is symmetric.*
(2) *$G(x, y, y) \le G(x, y, a)$, for all $x, y, a \in X$.*
(3) *$G(x, y, z) \le G(x, y, a) + G(z, y, b)$, for all $x, y, z, a, b \in X$.*

PROOF. That (1) implies (2) follows from (G3) whenever $a \ne x$, and from (X, G) being symmetric when $a = x$. Combining (2) of proposition 1 and (2) above we have

$$G(x, y, z) \le G(x, y, y) + G(z, y, y) \le G(x, y, a) + G(z, y, b)$$

, so (2) implies (3). Finally, that (3) implies (1) follows by taking $a = x$, and $b = y$ in (3). $\qquad\square$

3. The G-metric topology

For any nonempty set X, we have seen that from any metric on X we can construct a G-metric (by (E_s) or (E_m)), conversely, for any G-metric G on X,

$$(E_d) \quad d_G(x, y) = G(x, y, y) + G(x, x, y),$$

is readily seen to define a metric on X, *the metric associated with G*, which satisfies,

$$G(x, y, z) \le G_s(d_G)(x, y, z) \le 2G(x, y, z).$$

Similarly,

$$\frac{1}{2}G(x, y, z) \le G_m(d_G)(x, y, z) \le 2G(x, y, z).$$

Further, starting from a metric d on X we have,

$$d_{G_s(d)}(x, y) = \frac{4}{3}d(x, y), \quad \text{and} \quad d_{G_m(d)}(x, y) = 2d(x, y).$$

DEFINITION 5. Let (X, G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}$$

PROPOSITION 4. *Let (X, G) be a G-metric space, then for any $x_0 \in X$ and $r > 0$, we have,*

    (1)   *if $G(x_0, x, y) < r$ then $x, y \in B_G(x_0, r)$,*
    (2)   *if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that*
             $B_G(y, \delta) \subseteq B(x_0, r)$.

    PROOF. (1) follow directly from (G3), while, (2) follows from (G5) with $\delta = r - G(x_0, y, y)$. $\square$

It follows from (2) of the above proposition that the family of all G-balls, $\mathcal{B} = \{B_G(x, r) : x \in X, r > 0\}$, is the base of a topology $\tau(G)$ on X, the *G-metric topology*.

PROPOSITION 5. *Let (X, G) be G-metric space, then for all $x_0 \in X$, and $r > 0$ we have*

$$B_G(x_0, \frac{1}{3}r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).$$

Consequently, the G-metric topology $\tau(G)$ coincides with the metric topology arising from d_G. Thus, while 'isometrically' distinct, every G-metric space is topologically equivalent to a metrics space. This allows us to readily transport many concepts and results from metric spaces into the G-metric space setting.

3.1. Convergence and Continuity in G-metric spaces.

DEFINITION 6. Let (X, G) be a G-metric space. The sequence $(x_n) \subseteq X$ is *G-convergent* to x if it converges to x in the G-metric topology, $\tau(G)$.

PROPOSITION 6. *Let (X, G) be G-metric space, then for a sequence $(x_n) \subseteq X$ and point $x \in X$ the following are equivalent.*

    (1)   *(x_n) is G-convergent to x.*
    (2)   *$d_G(x_n, x) \to 0$, as $n \to \infty$ (that is, (x_n) converges to x relative to the metric d_G).*
    (3)   *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
    (4)   *$G(x_n, x, x) \to 0$, as $n \to \infty$.*
    (5)   *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

    PROOF. The equivalence of (1) and (2) follows from proposition 5. That (2) implies (3) (and (4)) follows from (E_d), the definition of d_G. (3) implies (4) is a consequence of (3) of proposition 1, while (4) entails (5) follows from (2) of proposition 1. Finally, that (5) implies (2) follows from (E_d) and (3) of proposition 1. $\square$

DEFINITION 7. Let (X, G), (X', G') be G-metric spaces, a function $f : X \longrightarrow X'$ is *G-continuous at a point* $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0), r)) \in \tau(G)$, for all $r > 0$. We say f is *G-continuous* if it is G-continuous at all points of X; that is, continuous as a function from X with the $\tau(G)$-topology to X' with the $\tau(G')$-topology.

Since G-metric topologies are metric topologies we have:

PROPOSITION 7. *Let (X, G), (X', G') be G-metric spaces, then a function $f : X \longrightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x we have $(f(x_n))$ is G-convergent to $f(x)$.*

PROPOSITION 8. *let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

PROOF. Suppose (x_k), (y_m) and (z_n) are G-convergent to x, y and z respectively. Then, by (G5) we have,

$$G(x, y, z) \leq G(y, y_m, y_m) + G(y_m, x, z)$$
$$G(z, x, y_m) \leq G(x, x_k, x_k) + G(x_k, y_m, z)$$

and

$$G(z, x_k, y_m) \leq G(z, z_n, z_n) + G(z_n, y_m, x_k),$$

so,

$$G(x, y, z) - G(x_k, y_m, z_n) \leq G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n).$$

Similarly,

$$G(x_k, y_m, z_n) - G(x, y, z) \leq G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z).$$

But then, combining these using (3) of proposition 1 we have,

$$|\, G(x_k, y_m, z_n) - G(x, y, z)\,| \leq 2(G(x, x_k, x_k) + G(y, y_m, y_m) + G(z, z_n, z_n)),$$

so $G(x_k, y_m, z_n) \to G(x, y, z)$, as $k, m, n \to \infty$ and the result follows by proposition 7. □

3.2. Completeness of G-metric spaces.

DEFINITION 8. Let (X, G) be a G-metric space, then a sequence $(x_n) \subseteq X$ is said to be *G-Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

The next proposition follows directly from the definitions.

PROPOSITION 9. *In a G-metric space, (X, G), the following are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*
(3) *(x_n) is a Cauchy sequence in the metric space (X, d_G).*

COROLLARY 1. *Every G-convergent sequence in a G-metric space is G-Cauchy.*

COROLLARY 2. *If a G-Cauchy sequence in a G-metric space (X, G) contains a G-convergent subsequence, then the sequence itself is G-convergent.*

DEFINITION 9. A G-metric space (X, G) is said to be *G-complete* if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

PROPOSITION 10. *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

COROLLARY 3. *if Y is a non-empty subset of a G-complete metric space (X, G), then $(Y, G|_Y)$ is complete if and only if Y is G-closed in (X, G).*

COROLLARY 4. *Let (X, G) be a G-metric space and let $\{F_n\}$ be a descending sequence $(F_1 \supseteq F_2 \supseteq F_3, ...)$ of non-empty G-closed subsets of X such that $\sup\{G(x, y, z) : x, y, z \in F_n\} \longrightarrow 0$ as $n \longrightarrow \infty$, then (X, G) is G-complete if and only if $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.*

Thus one can readily develop and exploit a Baire Category theorem in G-metric spaces; every complete G-metric space is non-meager in itself.

3.3. Compactness in G-metric spaces.

DEFINITION 10. Let (X, G) be a G-metric space, and let $\epsilon > 0$ be given, then a set $A \subseteq X$ is called an *ϵ-net* of (X, G) if given any x in X there is at least one point a in A such that $x \in B_G(a, \epsilon)$, if the set A is finite then A is called a *finite ϵ-net* of (X, G). Note that if A is an ϵ-net then $X = \cup_{a \in A} B_G(a, \epsilon)$.

DEFINITION 11. A G-metric space (X, G) is called *G-totally bounded* if for every $\epsilon > 0$ there exists a finite ϵ-net.

DEFINITION 12. A G-metric space (X, G) is said to be a *compact G-metric space* if it is G-complete and G-totally bounded.

PROPOSITION 11. *For a G-metric space, (X, G), the following are equivalent.*

(1) *(X, G) is a compact G- metric space.*
(2) *$(X, \tau(G))$ is a compact topological space.*
(3) *(X, d_G) is a compact metric space.*
(4) *(X, G) is G-sequentially compact; that is, if the sequence $(x_n) \subseteq X$ is such that $\sup\{G(x_n, x_m, x_l) : n, m, l \in \mathbf{N}\} < \infty$, then (x_n) has a G-convergent subsequence.*

4. Products of $G-$ Metric Spaces

In this section we discuss G-metric spaces arising as the product of G-metric spaces.

For $i = 1, 2, \cdots, n$ let (X_i, G_i) be G-metric spaces and let $X = \prod_{i=1}^{n} X_i$, then natural definitions for G-metrics on the product space X would be

$$G_m(x, y, z) = \max_{1 \leq i \leq n} \{G_i(x_i, y_i, z_i)\} \quad \text{and} \quad G_s(x, y, z) = \sum_{i=1}^{n} G_i(x_i, y_i, z_i).$$

However, unless all the (X_i, G_i) are symmetric, G_m and G_s may fail to be G-metrics.

EXAMPLE 2. Let X_1 denote the G-metric space defined in Example 1 and let $X_2 = \{1, 2\}$ with $G_2(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$. Then $G_m(x, y, z) = \max\{G_1(x_1, y_1, z_1), G_2(x_2, y_2, z_2)\}$ is not a G-metric on $X = X_1 \times X_2$. It satisfies all the axioms except (G3). For instance, if $x = (a, 1)$, $y = (b, 1)$ and $z = (a, 2)$ then $G_m(x, y, y) = 2$, but $G_m(x, y, z) = 1$.

THEOREM 4.1. *For $i = 1, \cdots, n$ let (X_i, G_i) be G-metric spaces, let $X = \prod_{i=1}^{n} X_i$, then for G defined by either*

$$G(x, y, z) = \max_{1 \leq i \leq n} \{G_i(x_i, y_i, z_i)\} \quad or \ \ G(x, y, z) = \sum_{i=1}^{n} G_i(x_i, y_i, z_i),$$

(X, G) is a symmetric G-metric space, if and only if each (X_i, G_i) is symmetric.

PROOF. That (X, G) is symmetric when all the (X_i, G_i) are is easily checked, with (G3) following from (2) of Proposition 3.

Conversely, arbitrarily choose elements $p_i \in X_i$ for $i = 1, 2, \cdots, n$. Given $j \in \{1, 2, \cdots, n\}$ and $x_j, y_j \in X_j$ let

$$x = (p_1, p_2, \cdots, p_{j-1}, x_j, p_{j+1}, \cdots, p_n) \in X$$

and

$$y = (p_1, p_2, \cdots, p_{j-1}, y_j, p_{j+1}, \cdots, p_n) \in X$$

Then $G_j(x_j, y_j, y_j) = G(x, y, y) = G(y, x, x) = G_j(y_j, x_j, x_j)$, as required. □

This leads us to seek alternative constructions for products of (not necessarily symmetric) G-metric spaces.

THEOREM 4.2. *For $i = 1, \cdots, n$, let (X_i, G_i) be G-metric spaces, then the following define symmetric G-metrics on $X = \prod_{i=1}^{n} X_i$*

(1) $G_s^m(x, y, z) = \max_{1 \leq i \leq n}\{G_s(d_{G_i})(x_i, y_i, z_i)\}$
(2) $G_s^s(x, y, z) = \sum_{i=1}^{n} G_s(d_{G_i})(x_i, y_i, z_i)$
(3) $G_m^m(x, y, z) = \max_{1 \leq i \leq n}\{G_m(d_{G_i})(x_i, y_i, z_i)\}$
(4) $G_m^s(x, y, z) = \sum_{i=1}^{n} G_m(d_{G_i})(x_i, y_i, z_i)$.

PROOF. We only prove (1), the other cases follow by similar arguments. Further, most of the axioms are readily established, so by way of illustration, we only verify (G3) and (G5)

Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, $z = (z_1, z_2, \cdots, z_n)$ and $a = (a_1, a_2, \cdots, a_n)$ be elements of X.

(G3) By (E$_s$), for each i, we have,
$G_s(d_{G_i})(x_i, y_i, y_i) = \frac{2}{3} d_{G_i}(x_i, y_i) \leq \frac{1}{3}(d_{G_i}(x_i, y_i) + d_{G_i}(x_i, z_i) + d_{G_i}(z_i, y_i))$
$= G_s(d_{G_i})(x_i, y_i, z_i)$, and it follows that $G_s^m(x, y, y) \leq G_s^m(x, y, z)$. This also shows via (2) of Proposition 3 that G_s^m is symmetric.

(G5) Again, by (E$_s$), for each i we have,
$G_s(d_{G_i})(x_i, y_i, z_i) = \frac{1}{3}(d_{G_i}(x_i, y_i) + d_{G_i}(x_i, z_i) + d_{G_i}(z_i, y_i)) \leq \frac{1}{3}(d_{G_i}(x_i, a_i) + d_{G_i}(a_i, y_i) + d_{G_i}(y_i, z_i) + d_{G_i}(x_i, a_i) + d_{G_i}(a_i, z_i)) = G_s(d_{G_i})(x_i, a_i, a_i) + G_s(d_{G_i})(a_i, y_i, z_i)$ and so, $G_s^m(x, y, z) \leq G_s^m(x, a, a) + G_s^m(a, y, z)$.

□

THEOREM 4.3. *For $i = 1, \cdots, n$, let (X_i, G_i) be G-metric spaces and let $X = \prod_{i=1}^{n}$, then all the G-metrics (G_k^j), where $j, k \in \{s, m\}$ are equivalent and the topology they induce is the product topology of the $\tau(G_i)$.*

PROOF. Starting from the definitions easy calculations yield,

$$G_s^m(x,y,z) \leq G_m^m(x,y,z) \leq G_m^s(x,y,z) \leq 3G_s^s(x,y,z) \leq 3nG_s^m(x,y,z),$$

for all $x, y, z \in X$. □

The following theorem is also easily verified.

THEOREM 4.4. *For $i = 1, \cdots, n$, let (X_i, G_i) be G-metric spaces and let $X = \prod_{i=1}^{n} X_i$. Then for all choices of $j, k \in \{s, m\}$, the product (X, G_k^j) is a complete (compact) G-metric space if and only if each of the (X_i, G_i) is G-complete (compact).*

These results provide the basis for carrying out analysis in G-metric spaces, in particular for the development of G-metric fixed point theory for mappings satisfying a variety of contractive type conditions. This will be taken up in a subsequent paper.

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