BANACH SPACE GEOMETRY AND THE FIXED POINT PROPERTY

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ABSTRACT. This is the text of four lectures given at the special international *workshop on metric fixed point theory* held at the University of Seville from September 25–29, 1995.

1. FOREPLAY

Throughout $X \equiv (X, \|\cdot\|)$ will denote a real Banach space (although our results remain valid if X is over the complex field). The closed unit ball of X is $B_X := \{x \in X : \|x\| \leq 1\}$, and the unit sphere of X is $S_X := \{x \in X : \|x\| = 1\}$. The dual space of X is the space X^* of all continuous linear functionals $f : X \to \mathbb{R}$ equipped with the norm $\|f\| := \sup f(B_X)$. For each $x \in X$ we define the evaluation functional $\hat{x} : X^* \to \mathbb{R} : f \mapsto \hat{x}(f) := f(x)$. The mapping $J : x \mapsto \hat{x}$ is an isometric isomorphism of X into X^{**} and we refer to $\hat{X} := J(X)$ as the natural embedding of X in X^{**}. The space X is reflexive if $\hat{X} = X^{**}$. By the duality map on X we understand $D : X \to 2^{X^*} : x \mapsto \{f \in X^* : f(x) = \|f\|\|x\| = \|x\|^2\}$. $D(x)$ is nonempty for each $x \in X$ courtesy of the Hahn-Banach theorem, and it is a fundamental result of R. C. James that X is reflexive if and only if D is onto.

Besides the norm topology we will be concerned with the weak topology on X, $w := \sigma(X, X^*)$, and the weak* topology on X^*, $w^* := \sigma(X^*, X)$. For what follows it will be enough to recall the following.

In the case of the weak topology:

(1) For a sequence (x_n) we have $x_n \xrightarrow{w} x$ if and only if $f(x_n) \to f(x)$, for all $f \in X^*$.
(2) A subset A of X is w-compact if and only if it is w-sequentially compact (the Eberline-Smullian theorem).
(3) The norm function on X is w-lower semi-continuous.
(4) The unit ball B_X is w-compact if and only if X is reflexive.
(5) Of 'historical' importance for metric fixed point theory is Mazur's theorem: for $A \subseteq X$ we have $\overline{\text{co}}\, A = \overline{\text{co}}^w A$.

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In the case of the weak* topology:

(1) For a net (f_α) we have $f_\alpha \xrightarrow{w^*} f$ if and only if $f_\alpha(x) \to f(x)$, for all $x \in X$.
(2) w^*-compact sets are w^*-sequentially compact if for example X is separable (in which case the relative w^*-topology on B_{X^*} is a metric topology), or more generally if X smoothable (that is, for some equivalent norm on X the duality map D is single valued).
(3) An equivalent norm on X^* is the dual norm of an equivalent norm on X if and only if it is w^*-lower semi-continuous.
(4) B_X is w^*-dense in $B_{X^{**}}$ (Goldstein's theorem).

We will concentrate on the w and w^* topologies. While many of the results extend to other locally convex linear topologies τ for which the norm is τ-lower semi-continuous, the results and arguments for the w or w^* cases are typical.

In all that follows C will denote a nonempty closed bounded convex subset, and $T : C \longrightarrow C$ a nonexpansive self mapping of C: that is, $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

Examples of nonexpansive maps on appropriate domains include the following,

(1) Isometries,
(2) Shifts,
(3) Multiplication operators,
(4) Resolvents of accretive operators,
(5) Contractions (that is, $x \neq y \Longrightarrow \|Tx - Ty\| < \|x - y\|$). It is interesting to note that while being a contraction is clearly more restrictive than being a nonexpansive map, apart from the observations that contractions can have at most one fixed point, almost nothing more seems to be known for contractions than for nonexpansive maps.
(6) Strict contractions (that is, there exists $k \in [0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$). These are the subject of the celebrated Banach contraction mapping principle, which ensures the existence of a (necessarily unique) fixed point to which any orbit $x, Tx, T^2x, T^3x, \cdots$ is norm convergent at a geometric rate.

We say X has the fixed point property, fpp, if on each nonempty closed bounded convex subset C every nonexpansive self mapping T has a fixed point.

We say X has the weak fixed point property, w-fpp , if on each nonempty w-compact convex subset C every nonexpansive self mapping T has a fixed point.

We say the dual of X, X^*, has the weak* fixed point property, w^*-fpp , if on each nonempty w^*-compact convex subset C every nonexpansive self mapping T has a fixed point.

We will be concerned with sufficient conditions for each of these properties. Of course all three properties coincide when X is reflexive.

Not all spaces enjoy these properties. For example:

(1) c_0 fails the fpp. $T : (x_1, x_2, x_3, \cdots) \mapsto (1, x_1, x_2, \cdots)$ is a fixed point free isometry on $B^+_{c_0} := \{(x_n) : x_n \to 0,\ 0 \le x_n \le 1\}$.

(2) $\mathcal{L}_1[0, 1]$ fails the w-fpp . The baker transform followed by the 'reflection' $f(t) \mapsto 1-f(t)$ is a fixed point free isometry on the w-compact order interval $0 \le f(t) \le 1$. This is a modification by R. Sine [Si] of the celebrated example of D. Alspach [A] which provided the first instance of a Banach space failing the w-fpp and so settled a question of more than 20 years standing. The example is canonical in abstract L-spaces [B-S], and remains effectively the only explicitly known example for the failure of the w-fpp (for an alternative construction see Schechtmann [Sc]).

(3a) $\ell_1 = c_0^*$ equipped with the equivalent dual norm $\|(x_n)\| := \|(x_n)^+\|_1 \vee \|(x_n)^-\|_1$ fails the w^*-fpp for $T(x_n) := (1 - \sum x_n, x_1, x_2, \cdots)$ acting on the w^*-compact convex set $C := B^+_{\ell_1}$ [Lim].

(3b) ℓ_1 as c^* fails the w^*-fpp for an affine contraction [D-L, Sm]. To see this, it is convenient to consider ℓ_1 acting on c via the action $(x_n)(c_n) := x_1c_1 + x_2 \lim c_n + x_3c_2 + x_4c_3 + \cdots$, for $(x_n) \in \ell_1$ and all $(c_n) \in c$. Then, for any $\delta \in (0, 1)$ and sequence $(\epsilon_n) \subset (0, 1)$ with $\sum \epsilon_n < \infty$ take

$$T(x_n) :=$$

$$\left(\delta(1-x_1) + \sum_{k=1}^{\infty}(1-\epsilon_k)x_{k+1},\ \delta(1-x_1),\ (1-\epsilon_1)x_2,\ (1-\epsilon_2)x_3,\ \cdots\right)$$

acting on the w^*-compact convex subset

$$C := \left\{(x_n) : 0 \le x_n,\ x_1 = \sum_{k=2}^{\infty} x_k \ \le\ 1\right\}.$$

It is unknown whether or not every Banach space has the w-fpp for contractions.

Henceforth we concentrate on sufficient conditions for the w-fpp and w^*-fpp , and so for the fpp in the case of reflexive spaces. It is a significant open question as to whether or not the fpp implies reflexivity, although there is considerable evidence in support of the conjecture that it does. The converse question is perhaps the fundamental open question in metric fixed point theory.

It is an interesting, but seldom used, result that *the w-fpp is separably determined*. To see this, suppose that C is a nonempty w-compact convex subset and that T is a fixed point free nonexpansive mapping of C into C. Choose $c \in C$ and define $K_1 := \{c\}$ and inductively $K_{n+1} := \overline{\text{co}}\,\{T(K_n) \cup K_n\}$. Then

$$K_\infty := \overline{\bigcup_n K_n}$$

is a separable, convex w-compact (by Mazur's theorem) T-invariant subset, on which $T|_{K_\infty}$ is a fixed point free nonexpansive self mapping.

The above argument actually establishes the formally stronger result that *any minimal w-compact convex T-invariant set is separable.*

A similar result in the w^*-case is potentially important, as then w^*-compactness could be assumed sequential. Unfortunately no such general result seems known. The situation in $C(\Omega)^*$ has been investigated by M. Smyth [Sm]. If Ω is an infinite compact Hausdorff space $C(\Omega)^*$ failure of the w^*–fpp is separably determined (either it contains an isometric copy of $\mathcal{L}_1$, and hence of the Alspach example, or it is an $\ell_1(\Gamma)$ and failure of the w^*–fpp can be reduced to a separable example similar to the ℓ_1 example considered above). If, however, Ω is not dispersed then $C(\Omega)^*$ contains a nonseparable w^*-compact convex subset C which admits a nonexpansive mapping $T : C \to C$ for which C is a minimal invariant subset. That is, $C(\Omega)^*$ contains inseparable minimal invariant subsets.

2. Opening moves

Suppose X (X^*) fails the w (w^*)-fpp, then there exists a nonempty w (w^*)-compact convex subset C and a fixed point free nonexpansive mapping $T : C \to C$. By the compactness and Zorn's lemma we may assume that C is minimal in the sense that no proper nonempty w (w^*)-compact convex subset of C is invariant under T. We will refer to such a set as a *minimal invariant set* for T (that it is nonempty w (w^*)-compact and convex being understood). We note in passing that for those of an intuitionist persuasion more constructive approaches to the existence of such sets are possible [see G-K, for example].

In particular we readily see that such a minimal invariant set C must contain more than one point, an observation best expressed by $\operatorname{diam}(C) := \sup\{\|x - y\| : x, y \in C\} > 0$, and that

(1) $C = \overline{\operatorname{co}}\, T(C)$, in the w-case, or

(2) $C = \overline{\operatorname{co}}^{w^*} T(C)$, in the w^*-case.

Our *meta-strategy* is to find further properties of such a minimal invariant set which are known to be ruled out by (geometric) properties of the space (or of the domain of T).

Lemma 2.1. *Let T be a fixed point free nonexpansive map and C a minimal invariant set for T. If $\psi : C \to \mathbb{R}$ satisfies*

(i) *ψ is a w (w^*)-lower semi-continuous convex function, and*

(ii) *$\psi(Tx) \leq \psi(x)$, for all $x \in C$*

then ψ is constant on C.

Proof. $D := \{x \in C : \psi(x) = \inf \psi(C)\}$ is a nonempty w (w^*)-closed convex T-invariant subset of C, and so by minimality $D = C$. □

From this we have the following basic observation of Brodskii and Mil'man [B-M].

Theorem 2.2. *: If C is a w (w^*)-compact minimal invariant set for the fixed point free nonexpansive mapping T then C is diametral in the sense that for all $x \in C$*

$$\begin{aligned} diam(C) &= rad(x, C), \quad \text{— the radius of } C \text{ about } x \\ &:= \sup_{y \in C} \|x - y\|. \end{aligned}$$

Proof. It suffices to verify that $\psi(x) := \text{rad}\,(x, C)$ satisfies the hypotheses of lemma 2.1, as then ψ is a constant on C with value equal to

$$\sup_{x \in C} \psi(x) = \sup_{x \in C} \sup_{y \in C} \|x - y\| = \text{diam}\,(C).$$

To complete the proof we first note that ψ is the supremum of convex w (w^*)-lower semi-continuous functions and so is itself convex and w (w^*)-lower semi-continuous. We next show that

$$\psi(x) = \sup_{y \in \text{co}\, T(C)} \|x - y\|.$$

In the w-case this follows immediately since $C = \overline{\text{co}}\, T(C)$. In the w^*-case $C = \overline{\text{co}}^{w^*} T(C)$, so given $\epsilon > 0$ there exists a $y_\epsilon \in C$ with $\psi(x) - \epsilon \leq \|x - y_\epsilon\|$ and a net $y_\alpha \xrightarrow{w^*} y_\epsilon$ with $y_\alpha \in \text{co}\, T(C)$. Thus,

$$\psi(x) - \epsilon \leq \|x - y_\epsilon\| \leq \liminf_\alpha \|x - y_\alpha\|$$

and so there exists a $y \in \text{co}\, T(D)$ with $\psi(x) - 2\epsilon \leq \|x - y\|$.

It now follows by standard convexity arguments that

$$\psi(x) = \sup_{y \in T(C)} \|x - y\|,$$

and so $\psi(Tx) \leq \psi(x)$, completing the proof. □

This has led to the concept of normal structure. We say that X (X^*) has w (w^*)-*normal structure* if it does not contain any nontrivial w (w^*)-compact convex diametral subsets.

For example. *If (e_n) denotes the usual basis in c_0, then $C := \overline{co}\,\{e_n\}$ is a w-compact ($e_n \xrightarrow{w} 0$) convex diametral subset. So c_0 fails to have w-normal structure.*

Clearly

$$w\ (w^*)\text{-normal structure} \implies w\ (w^*)\text{-fpp}.$$

We can classify conditions on X (X^*) as 'strong' if they imply w (w^*)-normal structure, and 'weak' if they fail to imply w (w^*)-normal structure while still ensuring the w (w^*)-fpp for X (X^*). In the next section we will survey some of the

known strong conditions. Weak conditions usually involve 'asymptotic' structure in X (X^*) and are best dealt with after we have considered Banach space ultraproducts in section 4. First, however, in preparation for later we consider the remaining important structure known for a minimal invariant set C for T.

Alongside the diametrality of C, another important observation has been the existence in C of an *approximate fixed point sequence* for T. That is, a sequence $(a_n) \subset C$ with $\|a_n - Ta_n\| \to 0$ (to obtain such a sequence, choose $x_0 \in C$ and take a_n to be the unique fixed point of the strict contraction $(1-1/n)T + 1/n\, x_0$ guaranteed courtesy of the Banach contraction mapping principle).

Now, let (a_n) be a given approximate fixed point sequence for T in the minimal invariant w (w^*)-compact convex set C. For each subsequence (x_k) of (a_n) consider

$$\psi(x) := \limsup_k \|x - x_k\|.$$

If ψ is w (w^*)-lower semi-continuous, then lemma 2.1 applies to give ψ has constant value c on C. Further, if $x_{k_\alpha} \overset{w\,(w^*)}{\longrightarrow} x_0$ we have

$$\begin{aligned} \operatorname{diam}(C) \geq c = \psi(x) &\geq \limsup_\alpha \|x - x_{k_\alpha}\| \\ &\geq \liminf_\alpha \|x - x_{k_\alpha}\| \\ &\geq \|x - x_0\|, \quad \text{for all } x \in C. \end{aligned}$$

So, since C is diametral, we have $c = \operatorname{diam}(C)$, and we have arrived at the fundamental lemma of Goebel [G] and Karlovitz [Ka]:

$$\lim_n \|x - a_n\| = \operatorname{diam}(C).$$

When this holds we refer to (a_n) as a *diameterizing sequence* for C.

In the w-case this is always the case (this was the result of Goebel and Karlovitz), as w-lower semi-continuity is ensured for the functions ψ defined above.

In the w^*-case this may not be the case.

Example. *In* $\ell_\infty = \ell_1^*$ *for*

$$a_n := (\underbrace{0, 0, \cdots, 0}_{n\ 0's}, -1, -1, \cdots)$$

and $\psi(x) := \limsup_n \|x - a_n\|$ *we have for*

$$x_n := (1, \cdots, 1, 0, 0, \cdots) \overset{w^*}{\longrightarrow} x_\infty := (1, 1, 1, 1, \cdots)$$

that $\psi(x_n) = 1$ while $\psi(x_\infty) = 2$. So ψ is not w^*-lower semi-continuous. This example along with an example in a separable dual are given in [S92b].

The question of when such ψ are w^*-lower semi-continuous, and so a w^*-Goebel and Karlovitz lemma pertains, has been taken up by M. A. Khamsi [Kh] and [S92b], who established the result for stable duals (and duals with shrinking strongly monotone bases) and w^*-orthogonal dual lattices: that is, Banach lattices for which

$$f_n \xrightarrow{w^*} 0 \implies \| |f_n| \wedge |f| \| \longrightarrow 0, \quad \text{for all } f \in X^*.$$

The extension of the Goebel/Karlovitz lemma to a wider class of dual spaces is an open problem.

The Goebel/Karlovitz lemma appears to endow minimal invariant sets with a richer structure than mere diametrality: namely, the existence of a diametrizing sequence. Unfortunately the existence of any nontrivial diameteral set implies the existence of a diametral set containing a diameterizing sequence. To see this, suppose D is a closed convex diametral set, with $d := \operatorname{diam}(D) > 0$. Starting with any $x_1 \in D$ we may inductively choose a sequence (x_n) so that $\|x_{n+1} - (1/n)\sum_1^n x_k\| > d - 1/n^2$. Then an easy calculation shows that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n) \longrightarrow d$. Thus (x_n) is diameterizing for $C := \overline{\operatorname{co}}\,\{x_k\}_{k=1}^\infty$.

If D is w-compact then so too is C. In general, we can not reach such a conclusion in the w^*-case. If, however, D is norm separable, as would be the case in a separable dual, we may use the following device of C. Lennard [Le]. Let (y_n) be a dense sequence in D, and modify the construction of the sequence (x_n) by choosing it so that

$$\|x_{n+1} - \frac{1}{2n}\sum_{k=1}^{n}(x_k + y_k)\| > d - \frac{1}{n^2},$$

then (x_n) is diameterizing for all points of D. In particular it is diameterizing for $C := \overline{\operatorname{co}}^{w^*}\,\{x_k\}_{k=1}^\infty$.

The following characterization of normal structure due to T. Landes [La84] will be of subsequent importance.

Lemma 2.3. *X contains a sequence which is diameterizing for its closed convex hull if and only if X contains a sequence (x_n) for which there is $c > 0$ with $\lim_n \|x - x_n\| = c$ for all $x \in \overline{\operatorname{co}}\,\{x_n\}_{n=1}^\infty$.*

Proof. Starting with $x_{n_1} := x_1$ extract a subsequence (x_{n_k}) so that $\|x_{n_{k+1}} - x_{n_j}\| \leq (1 + 1/k)c$, for $j \leq k$. Put

$$z_k := \frac{k}{k+1}x_{n_{k+1}} + \frac{1}{k+1}x_1$$

and let

$$C := \overline{\operatorname{co}}\,\{z_k\}_{k=1}^\infty.$$

A calculation shows that $\|z_m - z_k\| \leq c$ (take $m > k$), so $\operatorname{diam}(C) = c$ and (z_k) is diameterizing for C.

The converse is immediate. □

Note: If in addition (x_n) is weak convergent then C is w-compact.

Normalization. Given a nontrivial w (w^*)-compact convex diametral set C (minimal invariant for the fixed point free nonexpansive mapping T) and (x_n) a diameterizing sequence for C (an approximate fixed point sequence for T) it is convenient in the w-case, or in the w^*-case when the w^*-compactness is sequential, to pass to a subsequence $x_{n_k} \overset{w\,(w^*)}{\longrightarrow} x_0$ and replace C by $(1/\operatorname{diam}(C))(C - x_0)$, (x_n) by $((1/\operatorname{diam}(C))(x_{n_k} - x_0)$ (and T by $(1/\operatorname{diam}(C))[T(\operatorname{diam}(C)x + x_0) - x_0]$). In this way we may always assume without loss of generality that $\operatorname{diam}(C) = 1$, $x_n \overset{w\,(w^*)}{\longrightarrow} 0$, and so in particular , since $0 \in C$, $\|x_n\| \to 1$.

The arguments preceding lemma 2.3 make it clear that to establish the w (w^*)-fpp from weak conditions, where w (w^*)-normal structure fails, will require delicate arguments; arguments which make explicit use of the fact that the diameterizing sequence is an approximate fixed point sequence for a fixed point free nonexpansive mapping. Such arguments are considered in section 5. In the next section we consider some of the known strong conditions.

3. Strong end games

In this section we will be concerned with the following conditions on a Banach space X, which are sufficient for w-normal structure.

(UC) *Uniform convexity*: For every $\epsilon > 0$ there is a $\delta > 0$ such that, x, $y \in B_X$ and $\|(1/2)(x-y)\| > 1-\delta$ implies $\|x-y\| < \epsilon$. This may be rephrased as; if C is a closed convex subset of B_X with $\operatorname{diam}(C) \geq \epsilon$ then $C \cap (\delta B_X) \neq \emptyset$. It is well known (Mil'man/Pettis theorem) that (UC) $\Longrightarrow$ reflexivity. Indeed it implies superreflexivity, and every superreflexive space admits an equivalent (UC) norm.

(ϵ_0-InQ) *ϵ_0-inquadrate*: The same as (UC), but only for ϵ greater than or equal to some given $\epsilon_0 \in [0, 2)$. (ϵ_0-InQ) is sufficient for superreflexivity.

(ϵ_0-UKK) *ϵ_0-uniformly Kadec-Klee*: For every $\epsilon > 0$ there is a $\delta > 0$ such that, $x_n \overset{w}{\to} x_0$, $\|x_n\| \to 1$, and $\operatorname{sep}(x_n) := \inf\{\|x_m - x_n\| : m \neq n\} \geq \epsilon$, implies $\|x_0\| \leq 1-\delta$. This may be rephrased as; if C is a w-compact convex subset of B_X with a (sequential) measure of noncompactness $\gamma(C) := \sup\{\operatorname{sep}(x_n) : (x_n) \subset C\} \geq \epsilon$ then $C \cap (\delta B_X) \neq \emptyset$. This generalization of R. Huff's [H] notion of uniform Kadec-Klee was introduced by van Dulst and Sims

[vanD-S], where it was shown to imply that the Chebycheff centre $\{x \in C : \operatorname{rad}(x, C) = \inf_{y \in C} \operatorname{rad}(y, C)\}$ is norm compact. Whether or not the asymptotic centre is norm compact in the same circumstances remains an open question.

(GGLD) *Generalized Gossez-Lami Dozo property*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| < \limsup_m \limsup_n \|x_m - x_n\|,$$

(GGLD) was introduced by A. Jiménez-Melado [JM].

(ass-P) *Asymptotic–P*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| < \text{ass-diam}\,\{x_n\},$$

where $\text{ass-diam}\,\{x_n\} := \lim_n \operatorname{diam}\{x_k\}_{k=n}^{\infty}$ is the *asymptotic diameter* of the sequence (x_n). Asymptotic–P was introduced in [S-S].

(P) *Property–P*: Whenever (x_n) is a nonconstant w-null sequence we have

$$\liminf_n \|x_n\| < \operatorname{diam}\{x_n\}.$$

Property–P was first considered by K. K.Tan and H. K. Xu [T-X].

(O) *Opial's condition*: Whenever $x_n \xrightarrow{w} 0$ and $x \neq 0$ we have

$$\limsup_n \|x_n\| \ < \ \limsup_n \|x_n + x\|.$$

This condition was introduced by Z. Opial [O] and shown to imply w-normal structure by J. P. Gossez and E. Lami Dozo [G-L].

(WO) *Weak Opial*: Whenever (x_n) is a w-null sequence which is not norm convergent we have

$$\liminf_n \|x_n\| \ < \ \sup_m \limsup_n \|x_m - x_n\|,$$

this is equivalent to requiring that there exists $x \in \overline{\text{co}}\,\{x_n\}$ with $\limsup_n \|x_n\| < \limsup_n \|x - x_n\|$. This weakening of the Opial condition was introduced by Tingley [T].

Most of the following implications are clear from the respective definitions.

$$\begin{array}{ccccccc}
\text{(UC)} & \Rightarrow & (\epsilon_0\text{-Iq}),\ 0 \le \epsilon_0 < 1 & \Rightarrow & (\epsilon_0\text{-UKK}),\ 0 \le \epsilon_0 < 1 & & \\
 & & & & \Downarrow & & \\
 & & & & \text{(GGLD)} & & \\
 & & & & \Downarrow & & \\
 & & & & \text{(ass-P)} & & \\
 & & & & \Downarrow & & \\
\text{(O)} & \Rightarrow & \text{(WO)} & \Rightarrow & \text{(P)} & & \\
 & & & & \Downarrow & & \\
 & & & & w\text{-n. str.} & &
\end{array}$$

It is also clear that (GGLD) $\Longrightarrow$ (WO).

To see that, for $0 \leq \epsilon_0 < 1$, (ϵ_0-UKK) implies (ass-P), suppose there were an $x_n \xrightarrow{w} 0$ with $\liminf_n \|x_n\| = \text{ass-diam}\{x_n\} = 1$. Then since we can choose m so that $\|x_m\|$ is arbitrarily close to 1 and $w - \lim_n x_m - x_n = x_m$, we can extract a subsequence (x_{n_k}) with $\text{sep}(x_{n_k}) > \epsilon_0$ and choose an m_0 so that $1 \geq \|x_{n_k} - x_{m_0}\| \geq \|x_{m_0}\| > 1 - \delta$. But then $y_k := x_{m_0} - x_{n_k}$ defines a sequence which contradicts (ϵ_0-UKK). Whether or not (ϵ_0-UKK), for $\epsilon_0 \geq 1$, implies w-normal structure remains unanswered. The result does, however, follow if X satisfies an additional property (WORTH) [S94].

We will show that the properties (GGLD) and (ass-P) coincide, as do (WO) and (P). Apart from these equivalences all the other implications given above are strict. In most cases this follows from standard examples. For instance, all the spaces $\mathcal{L}_p$, with $p \neq 2$ fail to have (O), but for $p \geq 3$ satisfy $\limsup_n \|x_n\|_p^p \leq (1/2)\limsup_m \limsup_n \|x_m - x_n\|_p^p$ whenever $x_n \xrightarrow{w} 0$, and so these spaces are 'WO-full'.

The space c_0 equivalently renormed by

$$\|(\xi_n)\| := \|(\xi_n)\|_\infty + \sum_n \frac{|\xi_n|}{2^n},$$

was considered by A. Jiménez-Melado [JM]. It enjoys (O) [van D], but lacks (ass-P). Thus, while Opial's condition implies property-P it fails to imply any of the stronger conditions listed.

To separate w-normal structure and (P) is more delicate. We first note the following two propositions.

Proposition 3.1. *A Banach space has (WO) if and only if it has (P).*

Proof. We need only prove (P) $\Rightarrow$ (WO). Suppose X fails to have (WO), then we can find $x_n \xrightarrow{w} 0$ with $\|x_n\| \to 1$, but $\limsup_n \|x_m - x_n\| \leq 1$ for all m. Construct from (x_n) the sequence (z_k) as in the proof of Landes' result lemma 2.3. Then $z_k \xrightarrow{w} 0$, $\|z_k\| \to 1$, and $\text{diam}\{z_k\} \leq 1$, so X fails to have (P). $\square$

Proposition 3.2. *If X has w-normal structure and satisfies the nonstrict Opial condition:*

$$\limsup_n \|x_n\| \leq \limsup_n \|x + x_n\|, \quad \textit{for all } x \in X,$$

(whenever $x_n \xrightarrow{w} 0$,) then X has (P).

Proof. Suppose X fails (P), and hence (WO), then there exists a sequence $x_n \xrightarrow{w} 0$, with $\|x_n\| \to 1$, and $\limsup_n \|x - x_n\| \leq 1$, for all $x \in C := \overline{\text{co}}\{x_n\}$. But, by

the nonstrict Opial condition $\limsup_n \|x - x_n\| \geq \lim_n \|x_n\| = 1$. It follows that $\operatorname{diam}(C) = 1$ and (x_n) is a diameterizing sequence for C. Thus X fails w-normal structure. □

To separate (P) from w-normal structure we make use of results by T. Landes. Landes [La86] introduced the *weak sum property* as follows.

Say a sequence (x_n) is *limit affine* if $\psi(x) := \lim_n \|x - x_n\|$ exists and is affine on $\overline{\operatorname{co}}\{x_n\}$ (we can in fact drop the existence assumption and replace lim by limsup), and *nondecreasing* if $\psi(x_n)$ is nondecreasing. A space X has the *weak sum property* (WSP) if the only w-convergent nondecreasing limit affine sequences are the constant sequences. The name derives from Landes' result that the property is preserved under the taking of finite sums, and is the weakest Banach space property so preserved which implies w-normal structure.

It is readily seen that (P) implies (WSP): If $x_n \xrightarrow{w} 0$ is a nonconstant nondecreasing and limit affine sequence, then the function ψ it defines is affine norm continuous, and hence also w-continuous, on $\overline{\operatorname{co}}\{x_n\}$. Further $l := \lim \psi(x_n)$ exists. It follows, since ψ is nondecreasing, that $\lim_n \|x_n\| = \psi(0) = l \geq \lim_n \|x_m - x_n\|$, contradicting (WO) and hence (P).

Landes considers two renormings of c_0.

$$X_1 := (c_0, \|\cdot\|') \quad \text{and } X_2 := (c_0, \|\cdot\|'')$$

where

$$\|(x_n)\|' := \sup_{i<j} \left| \sum_{\substack{k=1 \\ k \neq i}}^{j-1} \frac{2}{3^k} x_k \; + \; (1 - 1/3^i) x_j \right|$$

and

$$\|(x_n)\|'' := \|(x_n)\|' \vee \sup_j \left| \sum_k f_k x_k \right|,$$

where

$$f_k = \begin{cases} 1/3^k, & \text{for } k < j, \\ (1 - 1/3^j) & \text{for } k = j, \\ 0 & \text{for } k > j. \end{cases}$$

X_1 has (WSP), but not (P): $e_n \xrightarrow{w} 0$, $\|e_n\|' = 1$, and $\operatorname{diam}(e_n) = 1$, while, X_2 has w-normal structure, but not (WSP).

Thus, in general, w-normal structure $\not\Rightarrow$ (WSP) $\not\Rightarrow$ (P).

Our final result is the equivalence of (GGLD) and (ass-P).

Proposition 3.3. *A Banach space has (GGLD) if and only if it has (ass-P).*

Proof. It is sufficient to prove the implication (ass-P) $\Rightarrow$ $\liminf_n \|x_n\| < \operatorname{sep}(x_n)$, whenever (x_n) is a w-null sequence which does not converge in norm. As then (ass-P) $\Rightarrow$ (GGLD) follows, and the reverse implication is clear.

Our proof is based on a technique explored by T. Benavides. To obtain a contradiction suppose there exists a sequence $x_n \xrightarrow{w} 0$, with $\|x_n\| \to 1$, and $\operatorname{sep}(x_n) \leq 1$. Given $\epsilon > 0$ define

$$A := \{\ \{x_n, x_m\}\ : \|x_m - x_n\| \geq 1 + \epsilon\}, \quad \text{and}$$
$$B := \{\ \{x_n, x_m\}\ : \|x_m - x_n\| < 1 + \epsilon \text{ and } x_m \neq x_n\}.$$

Then $A \cup B$ is the set of all two element subsets of $\{x_n\}_{n=1}^{\infty}$ and Ramsey's theorem ensures the existence of a subsequence (x_{n_k}) with $\{\ \{x_{n_k}, x_{n_j}\}\ : x_{n_k} \neq x_{n_j}\}$ either contained entirely in A, or entirely in B. Now $\operatorname{sep}(x_n) \leq 1$, so the above set lies entirely in B. That is, $\|x_{n_k} - x_{n_j}\| < 1+\epsilon$, for all k and j, and so $\operatorname{diam}(x_{n_k}) \leq 1+\epsilon$.

Repeated application of this, with $\epsilon = 1/m$, and a diagonalization argument, yields a subsequence (z_n) with $\operatorname{ass-diam}(z_n) \leq 1$, but $\|z_n\| \to 1$, so X fails to enjoy (ass-P). $\square$

Remark. w^*-analogues for the above properties exist and similar results apply, in particular they all yield w^*-normal structure, at least in dual spaces where w^*-compactness is sequential. This last requirement can often be relaxed if the properties are appropriately defined in terms of nets.

4. Ultra-techniques

In this section we develop the Banach space ultrapower and initiate its use as a tool for studying the w-fpp . Throughout I will denote an index set, for our purposes usually $\mathbb{N}$. For a more extensive and detailed development than is possible here the reader is referred to [G-K], [A-K], or [S82a].

A *filter* on I is a nonempty family of subsets $\mathcal{F} \subseteq 2^I$ satisfying

(i) $\mathcal{F}$ is closed under taking supersets. That is, $A \in \mathcal{F}$ and $A \subseteq B \subseteq I \implies B \in \mathcal{F}$.

(ii) $\mathcal{F}$ is closed under finite intersections. $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$.

We say $\mathcal{F}$ is *countably complete* if it is closed under countable intersections.

Examples.

(1) 2^I

(2) The Fréchet filter $\{A \subseteq I : I \backslash A \text{ is finite }\}$

(3) For $i_0 \in I$, $\mathcal{F}_{i_0} := \{A \subseteq I : i_0 \in A\}$. Filters of the form $\mathcal{F}_{i_0}$ for some $i_0 \in I$ are termed *trivial*, or *non-free* filters.

A filter $\mathcal{F}$ is *proper* if it does not equal the power set of I, 2^I. Equivalent conditions are: $\emptyset \notin \mathcal{F}$, or $\mathcal{F}$ has the finite intersection property.

Henceforth by filter we will mean proper filter.

An *ultrafilter* $\mathcal{U}$ on I is a filter on I which is maximal with respect to ordering of filters on I by inclusion: that is, if $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{F}$ is a filter on I, then $\mathcal{F} = \mathcal{U}$. Zorn's lemma ensures that every filter has an extension to an ultrafilter.

Lemma 4.1. *A [proper] filter $\mathcal{U} \subset 2^I$ is an ultrafilter on I if and only if for every $A \subseteq I$ precisely one of A or $I \backslash A$ is in $\mathcal{U}$.*

As a consequence of this lemma: For an ultrafilter $\mathcal{U}$ on I if $I = A_1 \cup A_2 \cup \cdots \cup A_n$ then at least one of the sets $A_1, A_2, \cdots, A_n$ is in $\mathcal{U}$, and an ultrafilter is nontrivial (free) if and only if it contains no finite subsets.

It will henceforth be a standing assumption that all the filters and ultrafilters with which we deal are nontrivial.

An ultrafilter $\mathcal{U}$ is countably incomplete if and only if there exist elements $A_0, A_1, \cdots, A_n, \cdots$ in $\mathcal{U}$ with

$$I = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \quad \text{and} \quad \cap_{n=0}^{\infty} A_n = \emptyset.$$

Countably incomplete ultrafilters are particularly convenient, as we shall see inductive and diagonal type arguments are readily extended to then. Every ultrafilter $\mathcal{U}$ over $\mathbb{N}$ is necessarily countably incomplete ($A_n := \{n, n+1, n+2, \cdots\} \in \mathcal{U}$).

For a Hausdorff topological space $(\Omega, \mathcal{T})$, ultrafilter $\mathcal{U}$ on I, and $(x_i)_{i \in I}$ we say

$$\mathcal{U} - \lim x_i \equiv \mathcal{T} - \mathcal{U} - \lim x_i = x_0$$

if for every neighbourhood N of x_0 we have $\{i \in I : x_i \in N\} \in \mathcal{U}$. Limits along $\mathcal{U}$ are unique and satisfy all the usual limit theorems when Ω is a linear topological space. If Ω is compact then $\mathcal{U} - \lim x_i$ exists for all $(x_i)_{i \in I}$. If $\mathcal{U}$ is on $\mathbb{N}$ and (x_n) is a bounded sequence in $\mathbb{R}$ then

$$\liminf_n x_n \leq \mathcal{U} - \lim x_n \leq \limsup_n x_n.$$

For a Banach space X and ultrafilter $\mathcal{U}$ on I we can form the substitution space

$$\ell_\infty(X) := \{(x_i)_{i \in I} : \|(x_i)\| = \sup_{i \in I} \|x_i\| < \infty\|.$$

$$N_{\mathcal{U}}(X) := \{(x_i)_{i\in I} : \mathcal{U} - \lim \|x_i\| = 0\}$$

is a closed linear subspace of $\ell_\infty(X)$.

The *Banach space ultrapower* of X over $\mathcal{U}$ is defined to be the quotient space

$$(X)_{\mathcal{U}} := \frac{\ell_\infty(X)}{N_{\mathcal{U}}(X)}$$

with elements denoted $[x_i]_{\mathcal{U}}$ and the quotient norm canonically given by

$$\|[x_i]_{\mathcal{U}}\| = \mathcal{U} - \lim \|x_i\|.$$

The mapping

$$\mathcal{J} : X \longrightarrow (X)_{\mathcal{U}} : x \longmapsto [x] := [x_i]_{\mathcal{U}}, \quad \text{where } x_i = x, \text{ for all } i \in I$$

is an isometric embedding of X into $(X)_{\mathcal{U}}$.

For a closed bounded convex subset $C \subset X$ and nonexpansive mapping $T : C \longrightarrow C$ we define in $(X)_{\mathcal{U}}$

$$\tilde{C} := \{[c_i]_{\mathcal{U}} : c_i \in C, \text{ for all } i \in I\}.$$

$\tilde{C}$ is convex subset, with $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C)$, containing the isometric copy $\mathcal{J}(C)$ of C, and on which

$$\tilde{T} : \tilde{C} \longrightarrow \tilde{C} : [c_i]_{\mathcal{U}} \longmapsto [Tc_i]_{\mathcal{U}},$$

where the representatives c_i are chosen from C, is a well defined nonexpansive mapping which leaves $\mathcal{J}(C)$ invariant.

Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Then for $\tilde{C}$ and $\tilde{T}$ constructed as above we have the following results.

Proposition 4.2. *If (a_n) is an approximate fixed point sequence for T, then $[a_n]_{\mathcal{U}}$ is a fixed point of $\tilde{T}$. So $\tilde{T}$ always has fixed points in $\tilde{C}$.*

Conversely, from a fixed point (indeed an approximate fixed point sequence) for $\tilde{T}$ in $\tilde{C}$ we can readily extract an approximate fixed point sequence for T. Although we will not make use of it, it is worth noting the following significant observation of B. Maurey [M] (see [El], or [A-K], for more details).

Lemma 4.3. *'Between' any two fixed points $\tilde{a}$ and $\tilde{b}$ of $\tilde{T}$ in $\tilde{C}$ there is a fixed point $\tilde{c}$ with*

$$\|\tilde{a} - \tilde{c}\| = \|\tilde{c} - \tilde{b}\| = \frac{1}{2}\|\tilde{a} - \tilde{b}\|.$$

That is, the fixed point set of $\tilde{T}$ is metrically convex.

The following generalization of the Goebel/Karlovitz lemma, due to P. K. Lin [Lin] has proved basic for establishing the w–fpp in the presence of weak conditions. The proof of this and the next result illustrate 'diagonal' arguments over an incomplete ultrafilter.

Lemma 4.4. *Suppose C is a w-compact minimal invariant set for T. If $(\tilde{a}_n)$ is an approximate fixed point sequence for $\tilde{T}$ in $\tilde{C}$ then*

$$\lim_n \|\tilde{a}_n - \mathcal{J}x\| = diam(C), \quad \text{for all } x \in C.$$

Proof. Suppose this were not the case. Without loss of generality we may take $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C) = 1$, and by passing to a subsequence if necessary assume that $\|\tilde{a}_n - \tilde{T}\tilde{a}_n\| < 1/n$.

Then there are $\epsilon_0 > 0$, $x_0 \in C$, and $n_0 \in \mathbb{N}$ with

$$\|\tilde{a}_n - \mathcal{J}x_0\| < 1 - \epsilon_0, \quad \text{for all } n > n_0.$$

Let $\tilde{a}_n = [a_m^n]_{\mathcal{U}}$, with $a_m^n \in C$, and define

$$A_n := \{m : \|a_m^n - x_0\| < 1 - \epsilon_0/2\},$$

and

$$B_n := \{m : \|a_m^n - Ta_m^n\| < 2/n\}.$$

Then A_n and B_n are in $\mathcal{U}$.

Put $m_0 = 0$ and for $n \in \mathbb{N}$ inductively choose $m_n \in A_n \cap B_n \cap \{m_{n-1}+1, m_{n-1}+2, \cdots\} \in \mathcal{U}$. Then the sequence $(a_{m_n}^n)$ is such that

$$\|a_{m_n}^n - Ta_{m_n}^n\| < 2/n.$$

That is, $(a_{m_n}^n)$ is an approximate fixed point sequence for T in C. But,

$$\|a_{m_n}^n - x\| < 1 - \epsilon_0/2,$$

an observation which is difficult to reconcile with the fact that $(a_{m_n}^n)$ is, by the Goebel/Karlovitz lemma, diameterizing for C. □

Proposition 4.5. *The set $\tilde{C}$ in $(X)_{\mathcal{U}}$ is closed. Hence when X is a superreflexive space $\tilde{C}$ is w-compact.*

Proof. While the proof is true for any ultrafilter $\mathcal{U}$, we will only prove it in the case when $\mathcal{U}$ is countably incomplete with $I = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, each $A_n \in \mathcal{U}$, and $\bigcap A_n = \emptyset$.

Suppose $[t_i^1]_{\mathcal{U}}, [t_i^2]_{\mathcal{U}}, \cdots$ is a sequence of points in $\tilde{C}$, with each $t_i^j \in C$, which converges to $[x_i]_{\mathcal{U}} \in (X)_{\mathcal{U}}$. By passing to a subsequence if necessary we may without loss of generality assume that

$$\|[t_i^m] - [x_i]\| = \mathcal{U} - \lim_i \|t_i^m - x_i\| < \frac{1}{m}.$$

For each $m \in \mathbb{N}$ let

$$B_m := \{i \in I : \|t_i^m - x_i\| < 2/m\} \cap A_m \in \mathcal{U}.$$

and put $B_0 := I$ and $t_i^0 := 0$, then

$$I = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m \supset \cdots, \quad \text{and } \cap_{m=0}^{\infty} B_m = \emptyset.$$

From this it follows that for each $i \in I$ there is a unique m such that $i \in B_m \backslash B_{m+1}$. Define $y_i := t_i^m$, for this m, in particular then $y_i \in C$.

Now, given any $m \in \mathbb{N}$, for each $i \in B_m$ there is a unique $p \geq m$ with $i \in B_p \backslash B_{p+1}$. thus,

$$\|y_i - x_i\| = \|t_i^p - x_i\| < 2/p \leq 2/m,$$

and so

$$\{i \in I : \|y_i - x_i\| < 2/m\} \supseteq B_m \ni \mathcal{U}.$$

We therefore have that $\mathcal{U} - \lim \|y_i - x_i\| = 0$, which yields the desired conclusion that $[x_i]_{\mathcal{U}} \in \tilde{C}$. $\square$

Remark. For the above results, and in many applications, a Banach space ultrapower $(X)_{\mathcal{U}}$ over $\mathbb{N}$ can be replaced by the space

$$\frac{\ell_\infty(X)}{c_0(X)}$$

where the quotient norm is canonically given by $\|[x_n]\| = \limsup_n \|x_n\|$. Disadvantages and advantages are largely cosmetic and it is up to individual readers to choose which setting is most to their taste.

5. Weak ends

The results developed in the last section have been used to establish the w–fpp from a variety of conditions on the space X. For example, B. Maurey [M] established the w–fpp for c_0, and by deeper arguments for the reflexive subspaces of $\mathcal{L}_1[0, 1]$ and the Hardy space H_1 (see also, [E] and [A-K] for more details). J. Borwein and Sims [B-S] generalized Maurey's c_0 result to obtain the w–fpp for Banach lattices with a Riesz angle

$$\alpha(X) := \sup\{\| |x| \vee |y| \| : x, y \in B_X\} < 2$$

and for which

$$\liminf_m \liminf_n \| |x_n| \wedge |x_m| \| = 0, \quad \text{whenever } x_n \xrightarrow{w} 0.$$

Lattices with this last property were referred to as *weak orthogonal* Banach lattices. A formally stronger version of weak orthogonality:

$$\liminf_n \| |x_n| \wedge |x| \| = 0, \quad \text{whenever } x_n \xrightarrow{w} 0 \text{ and } x \in X,$$

was shown to imply the w-fpp by Sims [S88], [S92a]. This was obtained using arguments similar to those developed by P. K. Lin [Lin] to prove the w-fpp for Banach spaces with a 1-unconditional basis.

Several more 'geometric' variants of these conditions have been introduced. For instance A. Jiménez-Melado and E. Lloréns-Fuster [JM-LF] considered the property of orthogonal convexity, gave examples of orthogonally convex spaces, and showed that it entails the w-fpp . A Banach space X is *orthogonally convex* if for every w-null sequence (x_n) with

$$D(x_n) := \limsup_m \limsup_n \|x_m - x_n\| > 0$$

there exists $\beta > 1/2$ such that

$$\limsup_m \limsup_n \sup\{\|z\| : \|z - x_m\|, \|z - x_n\| \leq \beta \|x_m - x_n\|\} < D(x_n)$$

The significance of a Riesz angle was transported to general Banach spaces by J. Garcia-Falset: For $\mathcal{U}$ an ultra filter over $\mathbb{N}$ define

$$R(X) := \sup\{\mathcal{U} - \lim_n \|x + x_n\| : x, x_n \in B_X \text{ and } x_n \xrightarrow{w} 0\}.$$

For example, $R(c_0) = 1$, while $R(\ell_1) = 2$, in general $1 \leq R(X) \leq 2$.

Garcia-Falset [GF94] showed that a Banach space X has the w-fpp if $R(X) < 2$ and it satisfies the nonstrict Opial condition:

$$\mathcal{U} - \lim \|x_n\| \leq \mathcal{U} - \lim \|x_n + x\|, \quad \text{whenever } x \neq 0 \text{ and } x_n \xrightarrow{w} 0.$$

Recently he has proved the result without requiring the nonstrict Opial condition [GF95].

We illustrate how the machinery of section 4. may be used by proving the special case when $R(X) = 1$.

Proposition 5.1. *If X is a Banach space with $R(X) = 1$ then X has the w-fpp .*

Proof. We first observe that $R(X) = 1$ implies the nonstrict Opial condition. If this were not the case we could find a sequence $x_n \xrightarrow{w} 0$, with $\mathcal{U} - \lim \|x_n\| = 1$ and $\mathcal{U} - \lim \|x + x_n\| < 1$ for some $x \neq 0$. Since $x + x_n \xrightarrow{w} x$ we have $\|x\| \leq \mathcal{U} - \lim \|x +$

$x_n\| < 1$. From the definition of $R(X)$ we therefore have $\mathcal{U}-\lim\|x-x_n\| \leq 1$. But, then $\mathcal{U}-\lim\|x_n\| \leq \mathcal{U}-\lim(\|x+x_n\|+\|x-x_n\|)/2$, a contradiction.

Now, suppose X fails the w-fpp . Let C be a w-compact convex set with $\operatorname{diam}(C)=1$ which is a minimal invariant set for the nonexpansive mapping T, and let (a_n) be a w-null approximate fixed point sequence for T. For an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ let $\tilde{C}$ and $\tilde{T}$ be defined as in section 4. Define

$$W := \{\tilde{w}\in\tilde{C} : \|\tilde{w}-[a_n]\| \leq 1/2, \text{ and } \exists\, x\in C \text{ with } \|\tilde{w}-\mathcal{J}x\| \leq 1/2\}$$

Then, W is a $\tilde{T}$-invariant, closed convex nonempty (as $(1/2)[a_n]\in W$) set, which, by the standard construction using Banach contraction mapping principle, contains an approximate fixed point sequence for $\tilde{T}$. So by the generalized Goebel/Karlovitz lemma (lemma 4.4) $\sup\{\|\tilde{w}\| : \tilde{w}\in W\}=1$.

On the other hand, let $\tilde{w}=[w_n]\in W$, where without loss of generality $w_n\in C$, for all $n\in\mathbb{N}$, and let $v=\text{w}\text{-}\mathcal{U}-\lim w_n$. Then, since $a_n \xrightarrow{w} 0$, the w-lower semi-continuity of the norm gives

$$(1/2) \geq \|\tilde{w}-[a_n]\| := \mathcal{U}-\lim\|w_n-a_n\| \geq \|v\|,$$

and the nonstrict Opial condition established above gives

$$\|\tilde{w}-\mathcal{J}v\| := \mathcal{U}-\lim\|w_n-v\| \leq \mathcal{U}-\lim\|w_n-x_0\| =: \|\tilde{w}-\mathcal{J}x_0\| \leq 1/2.$$

for some $x_0\in C$, which exists by the definition of W. Thus

$$\|\tilde{w}\| \leq \|\tilde{w}-\mathcal{J}v+\mathcal{J}v\| \leq R(X)\|\tilde{w}-\mathcal{J}v\| \vee \|v\| \leq 1/2.$$

A contradiction establishing the result. □

The above proof is typical of those for many of the results mentioned above. A numeric contradiction is arrived at, here $1\neq 1/2$, and by carefully analyzing the proof the gap, here between $1/2$ and 1, can be exploited to establish the w-fpp for spaces whose Banach-Mazur distance from a space satisfying the conditions is not too great. We will not pursue such stability issues. The interested reader is directed to the papers cited above.

As mentioned above, proposition 5.1 has been generalized to only requiring $R(X)<2$. Since in a Banach lattice $R(X)\leq\alpha(X)$, this substantially extends the result of Borwein and Sims, by showing that in the presence of a Riesz angle condition the assumption of weak orthogonality is unnecessary. A generalization in another direction is encompassed by the following.

For the rest of this section $\mathcal{U}$ will be an ultrafilter on $\mathbb{N}$.

A *weakly null type* on a Banach space X is a function of the form

$$\psi_{(x_n)}(x) = \mathcal{U}-\lim_n\|x-x_n\|,$$

where (x_n) is a w-null sequence. We say $\psi_{(x_n)}$ is nontrivial if $\psi_{(x_n)}(0) \neq 0$: that is, if $\|x_n\| \not\to 0$. If X is a separable space we may replace (x_n) by a subsequence so that $\psi_{(x_n)}(x) = \lim_n \|x - x_n\|$, for all $x \in X$.

As we have seen, weakly null types play an important rôle in metric fixed point theory. Note, for example, our proof of the Brodskii-Mil'man theorem and the Goebel/Karlovitz lemma in section 2, many of the conditions considered in section 3 (for instance, Opial's condition is essentially a statement concerning the behaviour of weakly null types), the definition of $R(X)$, and the proof for proposition 5.1 given above.

Recently N. Kalton [K] introduced *property (M)*: Weakly null types are constant on spheres about the origin. That is, for $x_n \xrightarrow{w} 0$ the weakly null type $\psi_{(x_n)}(x) = \mathcal{U} - \lim_n \|x - x_n\|$ is a function of $\|x\|$ only.

Examples of spaces with property (M) include c_0, ℓ_p for $1 \leq p < \infty$, indeed all the Orlicz sequence spaces with the Δ_2-condition. Subspaces of $\mathcal{L}_1[0, 1]$ with property (M) have the uniform Opial property of Prus [P], and so have w-normal structure (see Kalton and Werner [K-W], for details).

Property (M) was one of the two essential ingredients in Kalton's characterization of those separable Banach spaces X for which the compact operators $\mathcal{K}(X)$ form an M-ideal in the algebra of all bounded linear operators, $\mathcal{L}(X)$. That is,

$$\mathcal{L}(X)^* = \left(\mathcal{K}(X)^\perp \oplus V\right)_1, \quad \text{for some closed subspace V.}$$

Åsvald Lima [Li] effectively proved that the dual of such a space is weak*-uniformly Kadec-Klee (UKK*) and hence has weak* normal structure (see, [S82b], for more details). This, combined with the rôle of weakly null types noted above, makes it natural to inquire into connections between property (M), weak normal structure, and the w–fpp .

Lemma 5.2. *Each of the following properties implies the one below it.*

(i) *X has property (M).*
(ii) *If $x_n \xrightarrow{w} 0$ and $\|x\| \leq \|y\|$ then $\psi_{(x_n)}(x) \leq \psi_{(x_n)}(y)$.*
(iii) *X satisfies the non-strict Opial condition.*

Proof. Only (i) $\Longrightarrow$ (ii) requires proof, and from the definition of property (M) it is enough to show that $\psi_{(x_n)}(tx)$ is an increasing function of t on $[0, \infty)$. To see this, note that for $0 < t_1 < t_2$ there exists $\beta \in (0, 1)$ such that $t_1 x = \beta(-t_2)x + (1-\beta)t_2 x$ and so, since $\psi_{(x_n)}$ is convex and by property (M) $\psi_{(x_n)}(-t_2 x) = \psi_{(x_n)}(t_2 x)$, we have $\psi_{(x_n)}(t_1 x) \leq \beta\psi_{(x_n)}(-t_2 x) + (1-\beta)\psi_{(x_n)}(t_2 x) = \psi_{(x_n)}(t_2 x)$. □

Proposition 5.3. *Let X be a Banach space with property (M). Then X has w-normal structure if and only if there is no nontrivial weakly null type which is identically equal to 1 on B_X.*

Proof. ($\Longleftarrow$) Suppose X fails to have w-normal structure then B_X contains $x_n \xrightarrow{w} 0$ with $\lim_n \|x - x_n\| = 1$, for all $x \in \overline{\text{co}}\,\{x_n\}_{n=1}^{\infty}$. In particular, since $0 \in \overline{\text{co}}\,\{x_k\}_{k=1}^{\infty}$, we have $\|x_n\| \longrightarrow 1$.

Thus, $\psi_{(x_n)}(0) = 1$ and $\psi_{(x_n)}(x_m) = 1$, for all m. Further, since $\|x_m\| \to 1$, it follows from lemma 5.2 that $\psi_{(x_n)}$ equals 1 on the open unit ball, and hence by continuity on B_X.

($\Longrightarrow$) This follows immediately from lemma 2.3. □

As a consequence of this lemma we have the following.

Theorem 5.4. *A Banach space X with property (M) has weak normal structure if there exists a point $x_0 \in S_X$ at which the relative weak and norm topologies agree.*

Proof. Suppose X fails to have weak normal structure. Let $\psi_{(x_n)}$ be the weakly null type of proposition 5.3, and let $y_n := x_0 - x_n$. Then, $y_n \xrightarrow{w} x_0$ and $\liminf_n \|y_n\| \leq \psi_{(x_n)}(x_0) = 1$, so there is a subsequence with $\|y_{n_k}\| \to 1$. But, then (y_{n_k}), and hence (x_{n_k}), is norm convergent, a difficult thing for a diametrizing sequence to achieve. □

Corollary 5.5. *X has property (M) and satisfies any of the following then X has weak normal structure.*

(i) *X has the Kadec-Klee property (the relative weak and norm topologies agree on S_X).*
(ii) *X is reflexive.*
(iii) *X has the Radon-Nikodym property.*
(iv) *X has the point of continuity property: for every weakly closed bounded subset A, the identity map (A, weak) to (A, norm) has at least one point of continuity, see [E-W] for details.*

A dual property to (M), property (M*), is defined in X^* by requiring that

$$\psi_{(f_n)} : X^* \longrightarrow \mathbb{R}^+ : f \longmapsto \mathcal{U} - \lim_n \|f - f_n\|$$

be a function of $\|f\|$ only, whenever $f_n \xrightarrow{w^*} 0$. Since (M*) implies X^* has the Radon-Nikodym we conclude that if X^* has property (M*) then X^* has w^*-normal structure and hence the w^*–fpp .

It follows from the work in Kalton [K] Lemma 3.6 and the discussion preceding it that if X has property (M) and $c_0 \not\hookrightarrow X$ then X has w-normal structure. Thus, for spaces with property (M) the presence of c_0 is the only impediment to weak normal structure. It is not however an impediment for the w–fpp .

Directly after these lectures were given in Seville, J. Garcia-Falset and myself succeeded in proving the following, using an adaptation of the methods illustrated in proposition 5.1.

Theorem 5.6. *Let X be a Banach space with property (M) then X has the w-fpp* (see, [GF-S], for details).

It can also be seen that if X is a separable, nonatomic σ-order complete Banach lattice which admits an equivalent norm with property (M), then X has the w-fpp .

We close this section with an observation by T. Dalby which justifies our earlier claim that this work represents a generalization of proposition 5.1.

proposition 5.7. *For a Banach space X, if $R(X) = 1$ then X has property (M).*

Proof. We have seen (proof of proposition 5.1) that $R(X) = 1$ implies the nonstrict Opial condition. Now, let $x_n \xrightarrow{w} 0$ and suppose x, y are such that $\|x\| = \|y\| \neq 0$, then

$$\|y\| = \|x\| \leq \liminf_n \|x + x_n\| \leq \psi_{(x_n)}(x).$$

Also, by the nonstrict Opial condition

$$\mathcal{U} - \lim_n \|x_n\| \leq \mathcal{U} - \lim_n \|x + x_n\| = \psi_{(x_n)}(x).$$

Let $d := \psi_{(x_n)}(x)$, then from the definition of R(X) we have

$$\mathcal{U} - \lim_n \|(1/d)y + (1/d)x_n\| \leq R(X) = 1.$$

That is, $\psi_{(x_n)}(y) \leq \psi_{(x_n)}(x)$, and the result follows from symmetry in x and y. □

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