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DENSENESS OF OPERATORS WHICH ATTAIN THEIR NUMERICAL RADIUS

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Abstract

We show that a bounded linear operator on a uniformly convex space may be perturbed by a compact operator of arbitrarily small norm to yield an operator which attains its numerical radius.

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The second author, in his dissertation [Sims, 1972], raised the question of the norm-denseness of those operators which attained their numerical radii, paralleling the norm-attaining operator investigations of Lindenstrauss [1963] and the related results of Bishop and Phelps [1961]. He observed there that self-adjoint operators on a Hilbert space could be approximated by self-adjoint operators which attained their numerical radii.

In this note we show that any bounded operator on a Hilbert space can be approximated in norm by operators which attain their numerical radii and, incidentally, differ from the original operator by a compact operator. Indeed, our argument establishes this result for any uniformly convex Banach space.

It is not clear how far the argument or result extend. The numerical radius is a natural object of contemplation for Hilbert space but is less accessible for general Banach spaces. At present we do not know a counterexample to the theorem on any Banach space.

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We let X be a Banach space, X^* its dual space and $B(X)$ the algebra of bounded linear operators on X.

Let

$$\Pi := \{(x, f) \in X \times X^* : f(x) = \|f\| = \|x\| = 1\}.$$

Recall: The *numerical radius* of $T \in B(X)$ is

$$\nu(T) := \operatorname{Sup}\{|f(Tx)| : (x, f) \in \Pi\}$$

and we say T *attains* its numerical radius if there exists $(x_0, f_0) \in \Pi$ with

$$|f_0(Tx_0)| = \nu(T).$$

(TECHNICAL) LEMMA. *For X a normed linear space and $T \in B(X)$, given constants a_1, b, $a_2 > 0$ and $(x_1, f_1) \in \Pi$ with $|f_1(Tx_1)| > \nu(T) - a_1$, there exists $T' \in B(X)$ with*

(1) $T' - T$ *a rank-one operator of norm b,*

and there exists $(x_2, f_2) \in \Pi$ with

(2) $$|f_2(T'x_2)| > \nu(T') - a_2$$

and

(3) $$f_1(x_2) > 1 - \frac{a_1 + a_2}{b}.$$

Moreover,

(4) $$\left\|\frac{x_1 + x_2}{2}\right\| > 1 - \frac{a_1 + a_2}{2b}.$$

PROOF. (1) is satisfied by taking

$$T'(x) = T(x) + be^{i\theta} f_1(x)x_1$$

f. all $x \in X$, where $\theta = \arg f_1(Tx_1)$.

Now choose $(x_2, f_2) \in \Pi$ such that (2) is satisfied and simultaneously $f_1(x_2) \geqslant 0$. This is possible since $(x, f) \in \Pi$ implies $(\lambda x, \bar{\lambda} f) \in \Pi$ for any λ with $|\lambda| = 1$. To establish (3) note that

$$\begin{aligned}\nu(T') - a_2 &< |f_2(T'x_2)| = |f_2(Tx_2) + be^{i\theta} f_1(x_2) f_2(x_1)| \\ &\leqslant |f_2(Tx_2)| + b|f_1(x_2)||f_2(x_1)| \\ &\leqslant \nu(T) + bf_1(x_2) \quad \text{as } |f_1(x_2)| = f_1(x_2) \text{ and } |f_2(x_1)| \leqslant 1.\end{aligned}$$

Further,

$$\begin{aligned}\nu(T') &\geqslant |f_1(T'x_1)| = |f_1(Tx_1) + be^{i\theta}| \\ &= |f_1(Tx_1)| + b \quad (\text{by choice of } \theta) \\ &\geqslant \nu(T) - a_1 + b.\end{aligned}$$

Combining these yields

$$\nu(T) - a_1 + b - a_2 < \nu(T) + bf_1(x_2)$$

or

$$f_1(x_2) > 1 - \frac{a_1 + a_2}{b} \quad \text{as required.}$$

We now establish (4) by noting

$$\left\|\frac{x_1 + x_2}{2}\right\| \geq f_1\left(\frac{x_1 + x_2}{2}\right) \geq \frac{1}{2}\left(1 + 1 - \frac{a_1 + a_2}{b}\right) = 1 - \frac{a_1 + a_2}{2b}.$$

Now let X be uniformly convex with modulus of convexity

$$\delta(\varepsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon\right\}.$$

Then $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$ and $\|x - y\| < \varepsilon$ whenever $\|(x + y)/2\| > 1 - \delta(\varepsilon)$. Let $\varepsilon > 0$ be given. Choose positive sequences (ε_n) and (ε'_n) such that: ε_n is monotonically decreasing to 0; $\sum \varepsilon'_n < \varepsilon$ and

$$\frac{\varepsilon_n}{\varepsilon'_n} < \delta\left(\frac{1}{2^n}\right).$$

[For example, take $\varepsilon'_n = \varepsilon/2^n$ and $\varepsilon_n = (\varepsilon\delta(1/2^n))/2^{n+1}$.]

Using the above lemma we may inductively construct sequences (T_n) and $(x_n, f_n) \in \Pi$ as follows: $T_1 = T$ and (x_1, f_1) is such that $|f_1(T_1 x_1)| > \nu(T_1) - \varepsilon_1$; T_2 is such that $T_2 - T_1$ is a rank-one operator of norm ε'_1 and (x_2, f_2) is such that $|f_2(T_2 x_2)| > \nu(T_2) - \varepsilon_2$ and

$$\left\|\frac{x_1 + x_2}{2}\right\| > 1 - \frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon'_1} > 1 - \frac{\varepsilon_2}{\varepsilon'_1} > 1 - \delta\left(\frac{1}{2}\right).$$

So $\|x_2 - x_1\| < \frac{1}{2}$ [take $a_1 = \varepsilon_1$, $b = \varepsilon'_1$ and $a_2 = \varepsilon_2$]; and in general: T_n is such that $T_n - T_{n-1}$ is a rank-one operator of norm ε'_n and (x_n, f_n) is such that $|f_n(T_n x_n)| > \nu(T_n) - \varepsilon_n$ and $\|x_n - x_{n-1}\| < 1/2^n$. It now follows that (T_n) is a Cauchy sequence $[\|T_m - T_n\| \leq \sum_{k=n+1}^{m} \|T_k - T_{k-1}\| \leq \sum_{k=n+1}^{m} \varepsilon'_k \to 0$ as $m > n \to \infty]$ as is (x_n). Thus T_n converges to $T_\infty \in B(X)$ and $x_n \to x_\infty$ where $\|x_\infty\| = 1$ and $T_\infty - T$ is a compact operator of norm less than or equal to $\sum \varepsilon'_n < \varepsilon$. By passing to a subsequence if necessary we may assume that f_n converges weakly to f_∞ with $\|f_\infty\| \leq 1$. (X is reflexive, so the unit ball of X^* is weakly compact.)

We show $(x_\infty, f_\infty) \in \Pi$ and $|f_\infty(T_\infty x_\infty)| = \nu(T_\infty)$. Now,

$$\begin{aligned} f_\infty(x_\infty) &= \lim f_n(x_\infty) = \lim(f_n(x_n) + f_n(x_\infty - x_n)) \\ &= 1, \quad \text{as } f_n(x_n) = 1 \text{ and } \|x_\infty - x_n\| \to 0. \end{aligned}$$

So $(x_\infty, f_\infty) \in \Pi$. Further,

$$\begin{aligned}
\nu(T_\infty) &= \lim \nu(T_n) && \text{as } \|T_n - T_\infty\| \to 0 \\
&= \lim |f_n(T_n x_n)| && \text{as } \nu(T_n) - |f_n(T_n x_n)| \to 0 \\
&= \lim |f_n(T_\infty x_\infty)| && \text{as } \|T_n x_n - T_\infty x_\infty\| \to 0 \\
&= |f_\infty(T_\infty x_\infty)| && \text{as } f_n \xrightarrow{w} f_\infty,
\end{aligned}$$

and we have established:

THEOREM. *For X a uniformly convex Banach space, given $T \in B(X)$ and $\varepsilon > 0$, there exists a compact operator C with $\|C\| < \varepsilon$ and $(x, f) \in \Pi$ such that $\nu(T + C) = |f((T + C)x)|$.*

That is; T may be perturbed by a compact operator of arbitrarily small norm to obtain an operator which attains its numerical radius. In particular, the operators which attain their numerical radius are norm dense in $B(X)$. We remark that, as a consequence of a theorem of Weyl [Halmos, 1967, problem 43], in the case of a self-adjoint operator on a Hilbert space the necessary perturbation can be realized by a rank-one self-adjoint operator.

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