ERGODIC THEOREM AND STRONG CONVERGENCE OF AVERAGED APPROXIMANTS FOR NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. Let C be a bounded closed convex subset of a uniformly convex Banach space X and let T be an asymptotically nonexpansive in the intermediate mapping from C into itself. In this paper, we first provide a ergodic retraction theorem and a mean ergodic convergence theorem. Using this result, we show that the set $F(T)$ of fixed points of T is a sunny, nonexpansive retract of C if the norm of X is uniformly Gâteaux differentiable. Moreover, we discuss the strong convergence of the sequence $\{x_n\}$ defined by $x_n = a_n x + (1-a_n)T(\mu)x_n$ for $n = 0, 1, 2, \dots$, where $x \in C$, μ is a Banach limit on l^∞ and a_n is a real sequence in $(0, 1]$.

1. Introduction

Let C be a nonempty subset of a Banach space X. a mapping $T : C \mapsto C$ is said to be

(a) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for $x, y \in C$.
(b) asymptotically nonexpansive [19] if there exists a sequence $\{k_n\}$ such that $\limsup_{n\to\infty} k_n \le 1$ and $\|T^n x - T^n y\| \le k_n\|x - y\|$ for $x, y \in C$ and $n \in \mathcal{N}$.
(c) asymptotically nonexpansive in the intermediate if

$$\limsup_{n\to\infty}[\sup_{x,y\in C}[\|T^n x - T^n y\| - \|x - y\|] \le 0.$$

(d) asymptotically nonexpansive type [19] if for each x in C,

$$\limsup_{n\to\infty}\sup_{y\in C}[\|T^n x - T^n y\| - \|x - y\|] \le 0.$$

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It is easily seen that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ and that both the inclusions are proper (cf. [19, p. 112]). We denote $F(T)$ by the set of fixed points of T.

Let C be a bounded closed convex subset of a Banach space X. Let T be a nonexpansive mapping from C into itself and let x be an element of C and for each t with $0 < t < 1$, let x_t be the unique point of C which satisfies $x_t = tx + (1-t)x_t$. Browder [5] showed that $\{x_t\}$ converges trongly to the element of $F(T)$ which is nearest to x in $F(T)$ as $t \downarrow 0$ in the case when X is a Hilbert space. Reich [30] extended Browder's result to the case when X is a uniformly smooth Banach space and he showed that $F(T)$ is a sunny, nonexpansive retract of C, i.e., there is a nonexpansive retraction P from C onto $F(T)$ such that $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. Recently, using an idea of Browder [5], Shimizu and Takahashi [32] studied the convergence of another approximating sequence for an asymptotically nonexpansive mapping in a Hilbert space. This result was extended to a Banach space by Shioji and Takahashi [33].

On the other hand, Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then the Cesáro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to a fixed point y of T for each $x \in C$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$.

In recent years much effort has devoted to studying nonlinear ergodic theory for (asymptotically) nonexpansive mappings and semigroups. See [1-3, 15-18, 20-29, 34]. Most of the work was carried out in a uniformly convex Banach space X whose norm is either Frechet differentiable or satisfies Opial's condition. In this paper, we first prove an ergodic retraction theorem and an mean ergodic convergence theorems for non-lipschitzian mapping in a uniformly convex Banach space without using the Frechet differentiable norm, which includes many known results as special cases. Using this result, we show that the set $F(T)$ is a sunny, nonexpansive retract of C if the norm of X is uniformly Gâteaux differentiable. Moreover, we discuss the strong convergence of the sequence $\{x_n\}$ defined by $x_n = a_n + (1 - a_n)T(\mu)x_n$ for $n = 0, 1, 2, \ldots$, where $x \in C$, μ is a Banach limit on l^∞ and a_n is a real sequence in $(0, 1]$.

2. Preliminaries and Notations

Let X be a Banach space. We recall that the modulus of convexity of X is the function δ_X defined on $[0,2]$ by

$$\delta_X(\epsilon)=\inf\Big\{1-\frac{1}{2}\|x+y\|:\|x\|\le 1,\|y\|\le 1,\ \text{and}\ \|x-y\|\ge\epsilon\Big\}.$$

A Banach space X is said to be uniformly convex if $\delta_X(\epsilon)>0$ for all $0<\epsilon\le 2$. We need the following characterization of uniform convexity for a Banach space.

Proposition 1 (cf. [36]). *Let $p>1$ and $r>0$ be two real numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g:[0,\infty)\to[0,\infty)$, depending on p and r, $g(0)=0$, such that*

$$\|\lambda x+(1-\lambda)y\|^p\le\lambda\|x\|^p+(1-\lambda)\|y\|^p-W_p(\lambda)g(\|x-y\|)$$

for all $x,y\in B_r$ and $0\le\lambda\le 1$, where $W_p(\lambda)=\lambda(1-\lambda)^p+\lambda^p(1-\lambda)$ and B_r is the closed ball centered at the origin and with radius r.

Throughout this paper X denotes a uniformly convex real Banach space, C a nonempty bounded closed convex subset of X, and T an asymptotically nonexpansive in the intermediate sense. Put

$$c_n=\sup_{x,y\in C}(\|T^nx-T^y\|-\|x-y\|)\vee 0,$$

we have

$$\lim_{n\to\infty}c_n=0. \tag{2.1}$$

We denote by $\triangle^n$ the set $\{\lambda=(\lambda_1,\dots,\lambda_n):\lambda_i\ge 0,\sum_{j=1}^n\lambda_j=1\}$ for $n\in\mathcal{N}$, the set of all nonnegative integers. For a subset D of X, we denote by coD and $\overline{co}D$, the convex hull and convex closed hull of D respectively.

Let μ be a continuous linear functional on l^∞ and let $a=(a_0,a_1,\dots)\in l^\infty$, we write $\mu(n)(a_n)$ instead of $\mu(a)$. For $n\in\mathcal{N}$, we can define a point evaluation δ_n by $\delta_n(a)=a_n$ for each $a\in l^\infty$. A convex combination of point evaluations is called a finite mean on $\mathcal{N}$. Let X^* be the dual space of X. The value of $y\in X^*$ at $x\in X$ will be denoted by $\langle x,y\rangle$. Since X is reflexive, for any continuous linear functional μ and $x\in C$ there exists a unique element $T(\mu)x$ in X such that

$$\langle T(\mu)x,x^*\rangle=\mu(n)\langle T^nx,x^*\rangle$$

for all $x^*\in X^*$. We write $T(\mu)x$ by $\mu(n)\langle T^nx\rangle$. Also, if μ is a finite mean on $\mathcal{N}$, say

$$\mu=\sum_{i=1}^n a_i\delta_{n_i}(t_i\in\mathcal{N},a_i\ge 0,i=1,2,\cdots,n,\sum_{i=1}^n a_i=1),$$

then

$$T(\mu)x = \sum_{i=1}^{n} a_i T^{n_i} x.$$

Now, for each $m \in \mathcal{N}$, we can defined bounded linear operator r_m in l^∞ by $(r_m)(a_n) = (a_{n+m})$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu = r_n^*\mu$ for each $n \in \mathcal{N}$, where r_n^* is the conjugate operator of r_n. For a Banach limit, we know that

$$\liminf_{n\to\infty} a_n \le \mu(n)(a_n) \le \limsup_{n\to\infty} a_n \text{ for all } (a_0, a_1, \dots) \in l^\infty \tag{2.2}$$

The duality mapping J from X into X^* will be defined by

$$J(x) = \{y \in X^* : \langle x, y\rangle = \|x\|^2 = \|y\|^2\},$$

for each $x \in X$. X is said to be smooth if for each $x, y \in B_1$, the limit

$$\lim_{t\to o} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists. The norm of X is said to be uniformly Gâteaux differentiable if for each $y \in B_1$, the limit (2.3) exists uniformly for $x \in B_1$. The norm of X is said to be uniformly Fréchet differentiable if for each $x \in B_1$, the limit (2.3) exists uniformly for $y \in B_1$. X is said to be uniformly smooth if (2.3) exists uniformly for $x, y \in B_1$. It is well known that if X is smooth then the duality mapping is single-valued and norm to weak star continuous. In the case when the norm of X is uniformly Gâteaux differentiable, we know the following [35, Lemma 1]:

Proposition 2. *Let C be a convex subset of a Banach space X whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded subset of X, let z be a point of C and let μ be a Banach limit. Then*

$$\mu(n)\|x_n - z\|^2 = \min_{y\in C} \mu(n)\|x_n - y\|^2$$

if and only if

$$\mu(n)\langle y - z, J(x_n - z)\rangle \le 0 \text{ for all } y \in C.$$

Let C be a convex subset of X, let K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \ge 0$ with $Px + t(x - Px) \in C$. If the sunny retraction P is also nonexpansive, then K is said to be a sunny, nonexpansive retract of C. Concerning sunny, nonexpansive retractions, we know the following [9, 29]:

Proposition 3. *Let C be a convex subset of a smooth space, let K be a nonempty subset of C and let P be a retraction from C onto K. Then P is sunny and nonexpansive if and only if*

$$\langle x - Px, J(y - Px)\rangle \le 0 \quad \text{for all } x \in C \quad \text{and } y \in K.$$

Hence there is at most one sunny, nonexpansive retraction from C onto K.

3. Main Theorems

In this section, we will state our main Theorems and some remarks. The proof of Theorems will be given in the next section.

Theorem 1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let T be an asymptotically nonexpansive in the intermediate sense mapping from C to itself. Then, for any Banach limit μ, the mapping P defined by $Px = T(\mu)x$ is a retraction from C onto $F(T)$ satisfying the following properties:*

(i) *P is nonexpansive;*
(ii) *$PT = TP = P$;*
(iii) *$Px \in \cap_m \overline{co}\{T^n x : n \ge m\}$ for all $x \in C$.*

From Theorem 1, if there exists a unique retraction from C onto $F(T)$ having properties $(i)-(iii)$ of Theorem 1. Then $T(\mu) = T(\nu)$ for any Banach limits μ and ν. By the proof of Theorem 2 of [16], we have following corollary.

Corollary 1. *Let X, C and T be as in Theorem 1. Let $Q = \{q_{n,m}\}_{n,m\in\mathcal{N}}$ is a strongly regular matrix. Suppose that there exists a unique retraction from C onto $F(T)$ having properties $(i)-(iii)$ of Theorem 1. Then for every $x \in C$,*

$$w - \lim_{n\to\infty} \sum_{m=0}^{\infty} q_{n,m} T^{m+k} x = y \in F(T) \quad \text{uniformly in } m \in \mathcal{N}.$$

Now, using Theorem 1, we shall give a new approximating sequence for an non-lipschitzian mapping.

Let $\{a_n\}$ be a real sequence such that

$$0 < a_n \le 1, \quad \lim_{n\to\infty} a_n = 0.$$

Let x be an element of C and let μ be a Banach limit, and let x_n be the unique point of C which satisfies

$$x_n = a_n x + (1 - a_n)T(\mu)x_n \tag{3.1}$$

We remark that (3.1) is well defined since the mapping T_n from C into itself defined by $T_n u = a_n x + (1 - a_n)T(\mu)u$ satisfies $\|T_n u - T_n v\| \le (1 - a_n)\|u - v\|$ for each $u, v \in C$.

Theorem 2. *Let C be a bounded convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let T be an asymptotically nonexpansive in the intermediate sense mapping from C into itself. Then $F(T)$ is a sunny, nonexpansive retract of C.*

Theorem 3. *Let C be a bounded convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let T be an asymptotically nonexpansive in the intermediate sense mapping from C into itself and let P be the sunny, nonexpansive retract from C onto $F(T)$. Let x be an element of C and let $\{x_n\}$ be sequence of C which satisfies (3.1). Then $\{x_n\}$ converges strongly to Px.*

4.Proof of Theorems

To simplify, in the following, for each $\varepsilon \in (0,1]$, we define

$$a(\varepsilon) = \frac{\varepsilon^2}{10R}\delta_X(\frac{\varepsilon}{R}) \tag{4.1}$$

and

$$\mathcal{N}_\varepsilon = \{n_\varepsilon \in \mathcal{N} : c_{n+n_\varepsilon} < a(\varepsilon) \ for\ each\ n \in \mathcal{N}\}, \tag{4.2}$$

where δ_X is the modulus of convexity of the norm, $d = 2\sup\{\|x\| : x \in C\}$, and $R = 4d+1$. Noting that from (2.1), $\mathcal{N}_\varepsilon$ is nonempty for each $\epsilon > 0$, and if $n_\varepsilon \in \mathcal{N}_\varepsilon$, then $n + n_\varepsilon \in \mathcal{N}_\varepsilon$ for each $n \in \mathcal{N}$.

The following lemma shall play a crucial role in the proof of our main theorems.

Lemma 4.1. *Let x be a element of C and let λ be a finite mean on $\mathcal{N}$ and let $\varepsilon_i \in (0,1] (i = 1,2)$ be positive numbers. Then there exists $n_{\varepsilon_2} \in \mathcal{N}$, where n_{ε_2} is independent of ε_1, such that*

$$\|T^lT(\lambda)T^nx - T(\lambda)T^{l+n}x\| < \varepsilon_1 + \varepsilon_2 \tag{4.3}$$

for all $n \geq n_{\varepsilon_2}$ and $l \in \mathcal{N}_{\varepsilon_1}$.

Proof. We shall prove the Lemma by mathematical induction.

If $\lambda = \delta_{m_1}, m_1 \in \mathcal{N}$, then the assertion is clear. Now suppose that the assertion holds for such $\lambda = \sum_{i=1}^{k-1} a_i\delta_{m_i}$ $(m_i \in \mathcal{N}, (a_1, a_2, \cdots, a_{k-1}) \in \triangle^{k-1})$. Let

$$\lambda = \sum_{i=1}^{k} a_i\delta_{m_i} \quad (m_i \in \mathcal{N}, (a_1, a_2, \cdots, a_k) \in \triangle^k).$$

Defining

$$\mu = \frac{1}{1-a_k}\sum_{i=1}^{k-1} a_i\delta_{m_i},$$

we claim that

$$\lim_{n\to\infty} \|T(\mu)T^n x - T^{n+m_k}x\| \;\text{ exists.} \tag{4.4}$$

Let $\varepsilon > 0$, from assumption of induction there exists $n_1 \in \mathcal{N}$ such that

$$c_n < \frac{1}{3}\varepsilon,$$

and

$$\|T(l)T(\mu)T^n x - T(\mu)T^{n+l}x\| < \frac{1}{3}\varepsilon$$

for all $n \geq n_1$ and $l \geq n_1$. It follows that, for all $n \geq n_1$ and $l \geq n_1$,

$$\begin{aligned}\|T(\mu)T^{n+l}x - T^{n+l+m_k}x\| &\leq \|T(\mu)T^{n+l}x - T^l T(\mu)T^n x\| \\ &\quad + \|T^l T(\mu)T^n x - T^{n+l+m_k}x\| \\ &\leq \|T(\mu)T^n x - T^{n+m_k}x\| + \varepsilon.\end{aligned}$$

For fixed $n \geq n_1$, taking $l \to \infty$, we get

$$\limsup_{l\to\infty} \|T(\mu)T^l x - T^{l+m_k}x\| \leq \|T(\mu)T^n x - T^{n+m_k}x\| + \varepsilon,$$

and hence

$$\limsup_{l\to\infty} \|T(\mu)T^l x - T^{l+m_k}x\| \leq \liminf_{n\to\infty} \|T(\mu)T^n x - T^{n+m_k}x\| + \varepsilon,$$

Since $\varepsilon > 0$ is arbitrary, this implies (4.4) holds.
Put

$$r = \lim_{n\to\infty} \|T(\mu)T^n x - T^{n+m_k}x\|.$$

By assumption of induction again, for given $\varepsilon_2 > 0$, there exists $n_2(= n_2(\lambda, \varepsilon_2))$ such that

$$\Big| \|T(\mu)T^n x - T^{n+m_k}x\| - r \Big| < \frac{1}{2}a(\varepsilon_2), \tag{4.5}$$

and

$$\|T^l T(\mu)T^n x - T(\mu)T^{n+l}x\| < \frac{1}{2}a(\varepsilon_2), \tag{4.6}$$

for all $l, n \geq n_2$. Now, we put $n_{\varepsilon_2} = 2n_2 \in \mathcal{N}$. Since for $n \geq n_\varepsilon$,

$$\begin{aligned}\|T^l T(\mu)T^n x - T(\mu)T^{l+n}x\| &\leq \|T^l T(\mu)T^n x - T^{l+n_2}T(\mu)T^{n-n_2}x\| \\ &\quad + \|T^{l+n_2}T(\mu)T^{n-n_2}x - T(\mu)T^{l+n}x\| \\ &\leq c_l + \frac{1}{2}a(\varepsilon_2) \\ &\quad + \|T(\mu)T^n x - T^{n_2}T(\mu)T^{n-n_2}\| \\ &\leq c_l + a(\varepsilon_2)\end{aligned}$$

it then follows from (4.2) and (4.5) that

$$\|T^l T(\mu)T^n x - T(\mu)T^{n+l}x\| < a(\varepsilon_1) + a(\varepsilon_2) \tag{4.7}$$

for each $l \in \mathcal{N}_{\epsilon_1}$ and $n \geq n_{\epsilon_2}$. Put

$$x = (1 - a_n)(T^l T(\lambda)T^n x - T(\mu)T^{n+l}x)$$

and

$$y = a_n(T^{n+l+m_k}x - T^l T(\lambda)T^n x).$$

It then follows from (4.4), (4.5), and (4.6) that, for $l \in \mathcal{N}_{\varepsilon_1}$ and $n \geq n_{\varepsilon_2}$,

$$\begin{aligned}\|x\| &\leq (1 - a_n)(\|T^l T(\lambda)T^n x - T^l T(\mu)T^n x\| \\ &\quad + \|T^l T(\mu)T^n x - T(\mu)T^n x\| \\ &\leq (1 - a_n)(a(\varepsilon_1) + a(\varepsilon_2) + c_l + \|T(\lambda)T^n x - T(\mu)T^n x\|) \\ &\leq a_n(1 - a_n)r + 2a(\varepsilon_1) + 2a(\varepsilon_2)\,(\leq R), \\ \|y\| &\leq a_n(c_l + \|T^{n+m_k}x - T(\lambda)T^n x\|) \\ &\leq a_n(1 - a_n)r + a(\varepsilon_1) + a(\varepsilon_2)\,(\leq R),\end{aligned}$$

and

$$\|x - y\| = \|T^l T(\lambda)T^n x - T(\lambda)T^{l+n}x\|.$$

Suppose that

$$\|x - y\| \geq \varepsilon_1 + \varepsilon_2$$

for some $l \in G_{\varepsilon_1}$ and $n \geq n_{\varepsilon_2}$. Then we shall give the contradiction in following two cases.

Case I. If $4a_n(1-a_n)r \le \max\{\varepsilon_1, \varepsilon_2\}$, then

$$\|x-y\| \le \|x\| + \|y\| \le 2a_n(1-a_n)r + 3a(\varepsilon_1) + 3a(\varepsilon_2) < \varepsilon_1 + \varepsilon_2.$$

This is a contradiction.

Case II. If $4a_n(1-a_n)r > \max\{\varepsilon_1, \varepsilon_2\}$, then we have

$$\|a_n x + (1-a_n)y\| \le (a_n(1-a_n)r + 2a(\varepsilon_1) + 2a(\varepsilon_2))(1 - 2a_n(1-a_n)\delta(\frac{\varepsilon_1+\varepsilon_2}{R})),$$

by Lemma in [14]. And hence

$$\begin{aligned} &a_n(1-a_n)\|T(\mu)T^{n+l}x - T^{n+l+m_k}x\| \\ &\qquad \le a_n(1-a_n)r + 2a(\varepsilon_1) + 2a(\varepsilon_2) - 2a_n^2(1-a_n)^2 r\delta(\frac{\varepsilon_1+\varepsilon_2}{R}). \end{aligned}$$

It then follows (4.5) that

$$0 \le 2a(\varepsilon_1) + 3a(\varepsilon_2) - 2a_n^2(1-a_n)^2 r\delta(\frac{\varepsilon_1+\varepsilon_2}{R}).$$

If $\varepsilon_1 \ge \varepsilon_2$, then $a(\varepsilon_1) \ge a(\varepsilon_2), 4a_n(1-a_n)r > \varepsilon_1$, and $a_n(1-a_n) > \frac{\varepsilon_1}{R}$. It follows that

$$0 < 5a(\varepsilon_1) - \frac{\varepsilon_1^2}{2R}\delta(\frac{\varepsilon_1}{R}),$$

this contradicts (4.1). If $\varepsilon_1 < \varepsilon_2$, then we also have a contradiction in the same way. This completes the proof. □

Since $\mathcal{N}$ is commutative semigroup, there exists a net $\{\lambda_\alpha : \alpha \in A\}$ of finite means on $\mathcal{N}$ such that

$$\lim_{\alpha \in A} \|\lambda_\alpha - r_n^* \lambda_\alpha\| = 0 \tag{4.8}$$

for every $n \in \mathcal{N}$, where A is a directed set (see [12]).

For each $\varepsilon > 0$ and $l \in \mathcal{N}$, we set

$$F_\varepsilon(T^l)) = \{x \in C : \|T^l x - x\| \le \varepsilon\}.$$

Lemma 4.2. *For each $0 < \epsilon < 1$, there exist $\delta > 0$ and $l_0 \in \mathcal{N}$ such that*

$$coF_\delta(T^l)) \subset F_\epsilon(T^l))$$

for each $l \geq l_0$.

Proof. Since X is uniformly convex, by [7, Theorem 1.1], for given $\epsilon > 0$ we can choose a positive integer p such that for each $M \subset C$,

$$\mathrm{co}M \subset \mathrm{co}_p M + B_{\epsilon/4}, \tag{4.9}$$

where $\mathrm{co}_p M$ denotes the set of sums $\lambda_1 x_1 + \cdots + \lambda_p x_p$ with $(\lambda_1, \ldots, \lambda_p) \in \triangle^p$ and $x_i \in M, 1 \leq i \leq p$. We first claim that

$$\mathrm{co}_2 F_{a(\frac{\epsilon}{4})}(T^l) \subset F_{\frac{\epsilon}{4}}(T^l)), \tag{4.10}$$

for each $l \in G_{a(\frac{\epsilon}{4})}$, where $a(\frac{\epsilon}{4})$ and $G_{a(\frac{\epsilon}{4})}$ are defined in (3.1) and (3.2). In fact, let $x_0, x_1 \in F_{a(\frac{\epsilon}{4})}(T^l)$ and $x_t = tx_0 + (1-t)x_1$ for some $0 < t < 1$. Put $x = (1-t)(T^l x_t - x_1)$ and $y = t(x_0 - T^l x_t)$. Then we have

$$\begin{aligned}
\|x\| &\leq (1-t)(\|T^l x_t - T^l x_1\| + \|T^l x_1 - x_1\|) \\
&\leq t(1-t)\|x_0 - x_1\| + 2(1-t)a(\frac{\epsilon}{4})\,(\leq R) \\
\|y\| &\leq t(1-t)\|x_0 - x_1\| + 2ta(\frac{\epsilon}{4})\,(\leq R)
\end{aligned}$$

and

$$\|x - y\| = \|T^l x_t - x_t\|$$

We show the claim in the following two cases.
Case I. If $t(1-t)\|x_0 - x_1\| \leq \dfrac{\epsilon}{10}$, then

$$\begin{aligned}
\|T^l x_t - x_t\| = \|x - y\| &\leq \|x\| + \|y\| \\
&\leq 2t(1-t)\|x_0 - x_1\| + 2a(\epsilon/4) \\
&< \frac{\epsilon}{4}.
\end{aligned}$$

Case II. If $t(1-t)\|x_0 - x_1\| > \dfrac{\epsilon}{10}$, then $t(1-t) > \dfrac{\epsilon}{5R}$. Therefore we have

$$\begin{aligned}
\|tx + (1-t)y\| &\leq (t(1-t)\|x_0 - x_1\| + 2a(\frac{\varepsilon}{4}))(1 - 2t(1-t)\delta_X(\frac{\|x-y\|}{R})) \\
&\leq t(1-t)\|x_0 - x_1\| + 2a(\frac{\varepsilon}{4}) - 2t^2(1-t)^2\|x_0 - x_1\|\delta_X(\frac{\|x-y\|}{R}) \\
&\leq t(1-t)\|x_0 - x_1\| + 2a(\frac{\varepsilon}{4}) - \frac{\varepsilon^2}{15R}\delta_X(\frac{\|x-y\|}{R})
\end{aligned}$$

That is

$$\delta_X\left(\frac{\|x-y\|}{R}\right) \le \frac{30R}{\epsilon^2}a(\frac{\epsilon}{4}) < \delta_X(\frac{\epsilon}{4R}).$$

It follows that

$$\|T^l x_t - x_t\| \le \frac{\epsilon}{4}.$$

This shows (4.10) holds. By induction, we also have

$$\mathrm{co}_p F_\delta(T^l) \subset F_{\frac{\epsilon}{4}}(T^l) \tag{4.11}$$

for $\delta = a^{(p-1)}(\epsilon/4)$ and $l \in G_{a^{(p-1)}(\epsilon/4)}$. From (4.9) and (4.11), we get

$$\mathrm{co}F_\delta(T^l) \subset F_{\frac{\epsilon}{4}}(T^l) + B_{\frac{\epsilon}{4}}.$$

But

$$C \cap (F_{\frac{\epsilon}{4}}(T^l) + B_{\frac{\varepsilon}{4}}) \subset F_\epsilon(T^l))$$

because

$$\begin{aligned}\|T^l x - x\| &\le \|x-y\| + \|y - T^l y\| + \|T^l y - T^l x\| \\ &\le 2\|x-y\| + \|y - T^l y\| + c_l.\end{aligned}$$

This completes the proof. □

Lemma 4.3. *For each $0<\varepsilon<1$ and $l \in \mathcal{N}_{\frac{\varepsilon}{4}}$, there exist $\alpha \in A$ and $n_\alpha \in \mathcal{N}$ such that*

$$T(\lambda_\alpha)T^{n+n_\alpha}x \subset F_\varepsilon(T^l) \quad \text{for all} \quad n \in \mathcal{N}$$

Proof. For $l \in \mathcal{N}_{\frac{\varepsilon}{4}}$, from (4.8), there exists $\alpha \in A$ such that

$$\|\lambda_\alpha - r_l^*\lambda_\alpha\| < \frac{\varepsilon}{R}.$$

By Lemma 4.1, there is an $n_\alpha \in \mathcal{N}$ such that

$$\|T^l T(\lambda_\alpha)T^{n+n_\alpha}x - T(\lambda_\alpha)T^{l+n+n_\alpha}x\| < \frac{\varepsilon}{2}$$

for all $n \in \mathcal{N}$. It follows that

$$\begin{aligned}\|T^l T(\lambda_\alpha)T^{n+n_\alpha}x - T(\lambda_\alpha)T^{n+n_\alpha}x\| &\le \|T^l T(\lambda_\alpha)T^{n+n_\alpha}x - T(\lambda_\alpha)T^{l+n+n_\alpha}x\| \\ &\quad + \|T(\lambda_\alpha)T^{l+n+n_\alpha}x - T(\lambda_\alpha)T^{n+n_\alpha}x\| \\ &\le \frac{\varepsilon}{2} + d\|\lambda_\alpha - r_l^*\lambda_\alpha\| \\ &< \varepsilon\end{aligned}$$

This completes the proof. □

Lemma 4.4. *Let μ be a Banach limit, and $x \in C$. Then*

$$T(\mu)x \in F(T) \bigcap \bigcap_{m \in \mathcal{N}} \overline{co}\{T^n x : n \geq m\}.$$

proof. We only need to prove that $T(\mu)x$ is the fixed point of T. Let $\varepsilon > 0$, then we can choose $l_0 \in \mathcal{N}$ such that

$$\overline{F_{\frac{\varepsilon}{2}}(T^l)} \subset \ F_\varepsilon(T^l) \ \text{ for all } \ l \geq l_0.$$

By Lemma 4.2, there exists an $\delta > 0$ and $l_1 \geq l_0$ such that

$$coF_\delta \subset F_{\frac{\varepsilon}{2}}(T^l) \ \text{ for all } \ \ l \geq l_1.$$

it follows that

$$\overline{co}F_\delta(T^l) \subset F_\varepsilon(T^l) \ \text{ for all } \ l \geq l_1.$$

By Lemma 4.3, there exist $l_2 \geq l_1$ and for each $l \geq l_2$, there exist $\alpha \in A$ and $n_\alpha \in \mathcal{N}$ such that

$$T(\lambda_\alpha)T^{n+n_\alpha}x \subset F_\delta(T^l)$$

for all $n \in \mathcal{N}$. It follows that

$$T(\mu)x = \mu_n\langle T(\lambda_\alpha)T^{n+n_\alpha}x\rangle \subset \overline{co}F_\delta(T^l) \subset F_\varepsilon(T^l)$$

This implies that $T^l T(\mu)x \to T(\mu)x$ strongly as $l \to \infty$. Since T^N is continuous for some $n \in \mathcal{N}$, we have $T^N T(\mu)x = \lim_{l\to\infty} T^N T^l T(\mu)x = T(\mu)x$. This implies that $T(T(\mu)x) = T^{1+lN}(T(\mu)x) \to T(\mu)x$ as $l \to \infty$. That is $T(\mu)x \in F(T)$. This completes the proof. □

Now we can give the proof of Theorem 1.

Proof of Theorem 1. Let μ be a Banach limit,for $x \in C$, put $Px = T(\mu)x$. It then follows from Lemma 4.4 that P is a retraction from C onto $F(T)$ and $Px \in \cap_m \overline{co}\{T^n x : n \geq m\}$ for all $x \in C$. For $x, y \in C$ and $m \in \mathcal{N}$, we have

$$\|Px - Py\| = \|\mu(n)T^{n+m}x - \mu(n)T^{n+m}y\| \leq \|x - y\| + c_m(x).$$

Which proves (i). Finally, since $Px \in F(T)$, $TPx = Px$ is obvious. That $PTx = Px$ follows from the following reasoning:

$$PTx = T(\mu)Tx = \mu(n)T^n Tx = \mu(n)T^{n+1}x = T(u)x = Px.$$

□

To continue the proof of Theorem 3, we also need some Lemmas.

Lemma 4.5. [11]. *Let X be a real Banach space, then for all $x, y \in X$*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

We now turn to the proofs of Theorem 2 and Theorem 3. In the rest of this section, let $x \in C$ and $\{a_n\}, \{x_n\}$ and μ be as in (3.1).

Lemma 4.6. *Let x_{n_i} be a subsequence of $\{x_n\}$ and μ be a Banach limit. Then there exists the unique element z of C satisfying*

$$\mu_i \|x_{n_i} - z\|^2 = \min_{y \in C} \mu_i \|x_{n_i} - y\|^2 \tag{4.15}$$

and the point z is a fixed point of T.

Proof. Let f be a real valued function on C defined by

$$f(y) = \mu_i \|x_{n_i} - y\|^2 \quad \text{for each} \quad y \in C.$$

Then we know from [31] that f is continuous and convex and satisfies $\lim_{\|y\| \to \infty} f(y) = \infty$. Therefore there exists a unique $z \in C$ such that $f(z) = \min\{f(y) : y \in C\}$. Now , we show that z is a fixed point of T. by the proof of Lemma 4.4 it is enough to show that $\lim_{l \to \infty} T^l z = z$. To this end, from Property 1 we have, for each $l \in \mathcal{N}$,

$$\|x_{n_i} - \frac{T^l z + z}{2}\|^2 \leq \frac{1}{2}\|x_{n_i} - T^l z\|^2 + \frac{1}{2}\|x_{n_i} - z\|^2 - \frac{1}{4} g(\|T^l z - z\|).$$

That is

$$g(\|T^l z - z\|) \leq 2(f(T^l z) - f(z)).$$

since we have from Lemma 4.4 and (3.2) that

$$\begin{aligned}
\|x_{n_i} - T^l z\| &\leq a_{n_i}\|x - T^l z\| + (1 - a_{n_i})\|T(\mu)x_{n_i} - T^l z\| \\
&\leq a_{n_i}\|x - T^l z\| + (1 - a_{n_i})(c_l + \|T(\mu)x - z\|) \\
&\leq a_{n_i}(\|x - T^l z\| + \|x - z\|) + c_l + \|x_{n_i} - z\|
\end{aligned}$$

It follows that

$$\begin{aligned}
g(\|T^l z - z\|) &\leq \mu_i (c_l + \|x_{n_i} - z\|)^2 - \mu_i \|x_{n_i} - z\|^2 \\
&\leq c_l \mu_i (c_l + 2\|x_{n_i} - z\|)
\end{aligned}$$

This implies that $T^l z \to z$ strongly. This completes the proof. □

Lemma 4.7. *Suppose that the norm of X is uniformly Gâteaux differentiable. Then*

$$\langle x_n - x, J(x_n - z)\rangle \le 0$$

for all $n \in \mathcal{N}$ and $z \in F(T)$.

proof. Let $z \in F(T)$. since $x_n - x = \frac{1-a_n}{a_n}(T(\mu)x_n - x_n)$, we have

$$\begin{aligned}
\langle x_n - x, J(x_n - z)\rangle &= \frac{1-a_n}{a_n}\langle T(\mu)x_n - x_n, J(x_n - z)\rangle \\
&= \frac{1-a_n}{a_n}(\langle T(\mu)x_n - z, J(x_n - z)\rangle + \langle z - x_n, J(x_n - z)\rangle) \\
&\le \frac{1-a_n}{a_n}(\|T(\mu)x_n - z\|\|x_n - z\| - \|x_n - z\|^2) \\
&\le 0
\end{aligned}$$

□.

Lemma 4.8. *Suppose that the norm of X is uniformly Gâteaux differentiable. Then the set $\{x_n : n \in \mathcal{N}\}$ is a relative compact subset of C and each strong limit point of $\{x_n\}$ is fixed point.*

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$, it then follow from Lemma 4.7 that there is unique element z of $F(T)$ satisfying (4.15). By Lemma 4.8, we get $\langle x_{n_i} - x, J(x_{n_i} - z\rangle \le 0$. This inequality and Proposition 2 yield

$$\mu_i\|x_{n_i} - z\|^2 \le \mu_i\langle x - z, J(x_{n_i} - z)\rangle \le 0.$$

By (2.2), there exists a subsequence of $\{x_{n_i}\}$ converging strongly to z. This completes the proof. □

Proof of Theorem 2. Put $a_n = \frac{1}{n}$. First we shall show that $\{x_n\}$ converges strongly to an element of $F(T)$. By Lemma 4.8, we know that $\{x_n : n \ge 1\}$ is a relative compact subset of C. Let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ converging strongly to y and z of $F(T)$, respectively. We shall show that $y = z$. From Lemma 4.7, we have $\langle y - x, J(y - z)\rangle \le 0$ and $\langle z - x, J(z - y)\rangle \le 0$. So we get $\|y - z\|^2 \le 0$, i.e., $y = z$. So $\{x_n\}$ converges strongly to an element of $F(T)$. Hence we can define a mapping P from C onto $F(T)$ by $Px = \lim_{n\to\infty} x_n$. Using Lemma 4.7 again, we have $\langle Px - x, J(Px - z)\rangle \le 0$ for all $x \in C$ and $z \in F(T)$. Therefore P is the sunny, nonexpansive retraction by Proposition 4. □

Proof of Theorem 3. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging strong to an element y of $F(T)$. we shall show $y = Px$. By Lemma 4.7, we have $\langle x_{n_i} - x, J(x_{n_i} - Px)\rangle \leq 0$. So we get $\langle y - x, J(y - Px)\rangle \leq 0$. Hence we get

$$\|y - Px\|^2 \leq \langle x - Px, J(y - Px)\rangle \leq 0$$

by Proposition 4. This completes the proof. □

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