Chapter 2

EXAMPLES OF FIXED POINT FREE MAPPINGS

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1. Introduction

In this short chapter we collect together examples of fixed point free nonexpansive mappings in a variety of Banach spaces. These examples help delineate the class of spaces enjoying the *fpp*, the *w-fpp*, or the *w*-fpp*. We begin by recalling the relevant definitions.

Let X be a Banach space. A mapping $T : C \subseteq X \to X$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The *fixed point set* of T is $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$.

We say that the space X has the *fixed point property* (*fpp*) if for every nonempty closed bounded convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T) \neq \emptyset$.

Similarly, X is said to have the *weak fixed point property* (*w-fpp*) if for every nonempty weakly compact convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T) \neq \emptyset$.

If X is the dual space of a given Banach space E, $X = E^*$, we say that X has the *weak* fixed point property* (*w*-fpp*) if for every nonempty weak* compact (that is, $\sigma(X, E)$-compact) convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T) \neq \emptyset$. Which subsets of X are weak* compact depends on the choice of pre-dual. Thus, when discussing the *w*-fpp* it is important that we have a specific pre-dual E in mind.

Clearly, we have *fpp* $\Longrightarrow$ *w-fpp*, with the two properties coinciding if X is reflexive, and when $X = E^*$ we have *fpp* $\Longrightarrow$ *w*-fpp* $\Longrightarrow$ *w-fpp*. Finding characterizations of those spaces enjoying the *fpp*, the *w-fpp*, or the *w*-fpp* are perhaps the three most fundamental questions of metric fixed point theory. All three questions remain open.

Much of the effort expended on metric fixed point theory has gone into identifying widely applicable and easily verifiable sufficient conditions for either the *fpp*, the *w-fpp*, or the *w*-fpp*. The results of these efforts occupy a considerable portion of this handbook. This chapter approaches the questions from the opposite direction by identifying spaces which fail one or more of these properties.

W.A. Kirk and B. Sims (eds.), Handbook of Metric Fixed Point Theory, 35–48.

Unfortunately, known examples of fixed point free nonexpansive mappings are rather sparse. With the exception of Alspach's example (or modifications of it, see section 4), the mappings concerned are adaptations of affine maps (indeed, modified shifts), or minor variants thereof. This dearth of examples is a major impediment to a fuller understanding of metric fixed point theory and the discovery of informative new examples would be an important step forward.

In the following section we will document examples that demonstrate failure of the *fpp*. Subsequent sections will deal with more specialized examples that demonstrate failures of the *w*-fpp* in duals of certain Banach spaces and finally Alspach's famous demonstration that the *w-fpp* fails in $L_1[0,1]$.

2. Examples on closed bounded convex sets

Example 2.1 c_0 *fails the* fpp.

Let $C = B_{c_0}^{+} := \{(x_n) \in c_0 : 0 \leq x_n \leq 1, \text{ all } n\}$ and define two affine maps by

$$T_1(x_n) := (1,\ x_1,\ x_2,\ \ldots)$$

and

$$T_2(x_n) := (1 - x_1,\ x_1,\ x_2,\ \ldots).$$

Then for $i = 1, 2$ and any $x, y \in c_0$ we easily see that $\|T_i x - T_i y\| = \|x - y\|$. So, both T_1 and T_2 are nonexpansive, indeed metric isometries, and map C into C. On the other hand, the only possible fixed point for T_1 is $(1, 1, 1, \ldots)$ while the only possible fixed point for T_2 is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$ neither of which is in c_0.

It is possible to generalize the above examples in the way illustrated by the next example.

Example 2.2 c_0 *fails the* fpp *with a contraction*; that is a mapping T for which $\|Tx - Ty\| < \|x - y\|$ whenever $x \neq y$.

As an alternative to the presentation in example 2.1, we will describe the current example using the standard Schauder basis; $e_1, e_2, e_3, \cdots$ of c_0, where $e_n := (\delta_{n,i})$ with $\delta_{n,n} = 1$ and $\delta_{n,i} = 0$ for $i \neq n$.

Let (λ_n) be a decreasing sequence of real numbers converging 1. Define,

$$C := \left\{ \sum_{n=1}^{\infty} t_n \lambda_n e_n : (t_n) \in c_0 \text{ with } 0 \leq t_n \leq 1 \right\}$$

and an affine map T on C by,

$$T\left(\sum_{n=1}^{\infty} t_n \lambda_n e_n \right) := \lambda_1 e_1 + \sum_{n=1}^{\infty} t_n \lambda_{n+1} e_{n+1}.$$

Straight forward calculations show that T is a mapping of C into C that is always nonexpansive and a contraction, provided the sequence (λ_n) is strictly decreasing, whose only possible fixed point is $(\lambda_1, \lambda_2, \lambda_3, \cdots) \notin c_0$.

We would like to have examples of fixed point free *non-affine* nonexpansive maps on nonempty closed bounded convex subsets of c_0. Here is a simple example of such a map due to C. Lennard [private communication, 1995].

Example 2.3 *c_0 fails the* fpp *with a non-affine contraction.*

Let C be defined as in example 2.1 and let (p_n) be any real sequence that strictly decreases to 1. Define T by

$$Tx := \left(1, \frac{x_1^{p_1}}{p_1}, \ldots, \frac{x_k^{p_k}}{p_k}, \ldots\right) .$$

Then T is readily seen to be a non-affine contraction mapping C into C. Furthermore, if $p := \prod_{n=1}^{\infty} p_n$ is finite, then a simple calculation shows that T is fixed point free.

To put the next example into context it is important to recall that c_0 enjoys the *w-fpp*.

Example 2.4 *c_0 fails the* fpp *for a contraction and on a set which is compact in a topology only slightly coarser than the weak topology.*

The ideas underlying this somewhat interesting example should be clear to anyone familiar with properties of the summing basis for c_0. However, some of the details are both tedious and technical and will only be sketched. The interested reader is referred to [6] for a fuller account.

Let $a = \big(a(n)\big)$ be a strictly decreasing sequence of 'weights' in l_∞ satisfying $\alpha \leq a(n) \leq \beta$, for some $0 < \alpha \leq \beta < \infty$. Define elements of c_0 by: $a_0 := 0$ and

$$a_n := (a(1), \ldots, a(n), 0, 0, \ldots) \quad \text{for } n = 1, 2, 3, \ldots .$$

and let K be the closed convex hull of $\{a_n\}_{n=0}^{\infty}$. Thus, K consists of all vectors of the form

$$\sum_{n=0}^{\infty} \lambda_n a_n = \Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), a(3)\big(1-(\lambda_0+\lambda_1+\lambda_2)\big), \ldots\Big),$$

where $\lambda_n \geq 0$, for all n, and $\sum_{n=0}^{\infty} \lambda_n = 1$.

If T_a denotes the affine map defined on K by,

$$T_a\Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), \ldots\Big) := \Big(a(1), a(2)(1-\lambda_0), a(3)\big(1-(\lambda_0+\lambda_1)\big), \ldots\Big),$$

then we have the following.

Lemma 2.5 (i) *T_a maps K into K,*

(ii) *T_a is a contraction,*

(iii) *T_a is fixed point free in K.*

Proof. To establish (i) it suffices to note that for $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$, we have

$$\begin{aligned} T\Big(\sum_{n=0}^{\infty} \lambda_n a_n\Big) &= \Big(a(1), a(2)(1-\lambda_0), a(3)\big(1-(\lambda_0+\lambda_1)\big), \ldots\Big) \\ &= \sum_{n=1}^{\infty} \lambda_{n-1} a_n \in K. \end{aligned}$$

To verify (ii) note that for $x = \big(a(1)(1-\lambda_0), a(2)(1-(\lambda_0+\lambda_1)), \dots\big)$ and $y = \big(a(1)(1-\mu_0), a(2)(1-(\mu_0+\mu_1)), \dots\big)$ we have

$$\|x-y\| = \sup\{a(1)|\mu_0-\lambda_0|, a(2)|\mu_0-\lambda_0+\mu_1-\lambda_1|, \dots\},$$

while

$$\|T_a x - T_a y\| = \sup\{a(2)|\mu_0-\lambda_0|, a(3)|\mu_0-\lambda_0+\mu_1-\lambda_1|, \dots\}.$$

Since $a = \big(a(n)\big)$ is a strictly decreasing sequence, we now readily see that T_a is a contraction. Note: if the weights $\big(a(n)\big)$ were only required to be decreasing then T_a would be nonexpansive, but not necessarily contractive.

Finally, suppose that $x = \big(a(1)(1-\lambda_0), a(2)(1-(\lambda_0+\lambda_1)), \dots\big)$ were a fixed point of T_a; that is,

$$x = T_a x = \big(a(1), a(2)(1-\lambda_0), a(3)(1-(\lambda_0+\lambda_1)), \dots\big).$$

Then, we would have $\lambda_0 = 0, \lambda_1 = 0, \dots$ contradicting the requirement that $\sum_{n=0}^{\infty} \lambda_n = 1$, and so we have (iii). ■

We now introduce a topology $\mathcal{E}_a$ into c_0 which is only slightly coarser than the weak topology , but with respect to which K is compact.

To define this topology, we regard $a = \big(a(n)\big)$ as an element of ℓ_1^* and define

$$E_a := \ker(a) = \big\{(y(n)) \in l_1 : \sum y(n)a(n) = 0\big\}.$$

Thus, E_a is a norm closed, but not weak* closed (as $a \notin c_0$), co-dimension one subspace of $\ell_1 = c_0^*$. So, E_a is a weak*-dense, and hence, norming subspace for c_0. Indeed simple calculations show that for $x \in c_0$,

$$\frac{\alpha}{\alpha+\beta}\|x\| \le \sup\{f(x) : f \in E_a, \|f\| \le 1\} \le \|x\|.$$

We define $\mathcal{E}_a := \sigma(c_0, E_a)$. That is, $\mathcal{E}$ is the smallest locally convex linear topology on c_0 for which all the elements of E_a are continuous as linear functionals on c_0.

The topology $\mathcal{E}_a$ may be seen as only 'slightly' coarser than the weak topology, $\sigma(c_0, \ell_1)$, on c_0, being induced by a norming codimension one subspace of ℓ_1. None-the-less it displays some unusual, though not too pathological, properties. Here are some examples. A sequence (x_n) in c_0 is $\mathcal{E}_a$ convergent to $x \in c_0$ if and only if for every $f \in E_a$, we have $f(x_n) \to f(x)$. Closures are sequentially determined in the $\mathcal{E}_a$ topology. However, the norm is not $\mathcal{E}_a$-lower semi-continuous and Mazur's theorem is not valid for the $\mathcal{E}_a$ topology. The sequence a_n does not have any weakly convergent subsequences, but $a_n \xrightarrow{\mathcal{E}_a} a_0 = 0$. This will be used to show that K is $\mathcal{E}_a$-compact. However, first we need the following lemma.

Lemma 2.6 *K is $\mathcal{E}_a$-closed.*

Proof. For $n = 1, 2, \dots$ let

$$x_n = \sum_{k=0}^{\infty} \lambda_k^{(n)} d_k = \big(a(1)(1-\lambda_0^{(n)}), a(2)(1-(\lambda_0^{(n)}+\lambda_1^{(n)})), \dots\big),$$

where $\lambda_k^{(n)} \geq 0$ and $\sum_{k=0}^{\infty} \lambda_k^{(n)} = 1$, be such that $x_n \xrightarrow{\mathcal{E}_a} x = (\mu_1 a(1), \mu_2 a(2), \dots)$. Choosing $f := (1/a(1), -1/a(2), 0, 0, \dots) \in E_a$ we have

$$f(x_n - x) = (1 - \lambda_0^{(n)} - \mu_1) - (1 - \lambda_0^{(n)} - \lambda_1^{(n)} - \mu_2) \to 0.$$

That is

$$\lambda_1^{(n)} \to \mu_1 - \mu_2.$$

Similarly, choosing $f := (0, 1/a(2), -1/a(3), 0, 0, \dots)$ we obtain,

$$\lambda_2^{(n)} \to \mu_2 - \mu_3,$$

and in general,

$$\lambda_k^{(n)} \to \mu_k - \mu_{k+1}.$$

Thus, for $k = 1, 2, \dots$

$$\lambda_k := \mu_k - \mu_{k+1} = \lim_n \lambda_k^n \geq 0$$

and

$$x = \big(\mu_1 a(1), (\mu_1 - \lambda_1)a(2), (\mu_1 - \lambda_1 - \lambda_2)a(3), \dots\big) \in c_0.$$

So we must have

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k \geq 0,$$

and then, provided $\mu_1 \leq 1$,

$$x = \sum_{k=1}^{\infty} \lambda_k d_k \in K.$$

But, given $\epsilon > 0$ there exists N so that

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k < \sum_{k=1}^{N} \lambda_k + \epsilon/2,$$

and there exists n for which

$$|\lambda_k - \lambda_k^{(n)}| \leq \epsilon/2N, \quad \text{for } k = 1, 2, \dots, N.$$

Thus,

$$\mu_1 \leq \sum_{k=1}^{N} \lambda_k^{(n)} + \epsilon \leq 1 + \epsilon, \quad \text{as} \sum_{k=0}^{\infty} \lambda_k^{(n)} = 1,$$

and so $\mu_1 \leq 1$, as required. ■

Since $a_n \xrightarrow{\mathcal{E}_a} a_0$, we have that $\{a_n\}_{n=0}^{\infty}$ is $\mathcal{E}_a$-compact. The $\mathcal{E}$-compactness of K then follows from Lemma 2.6, the definition of $\mathcal{E}_a$, and the following general result from Banach space theory (see, for example, [6] for a proof).

Lemma 2.7 *Let X be a separable Banach space and let M be a closed norming subspace of X^*. If $D \subset X$ is $\sigma(X, M)$-compact then* co(D) *is $\sigma(X, M)$-precompact.*

This example suggests the following open question: *Does a nonempty closed bounded convex subset of c_0 have the fpp if* and only if *it is weakly compact?* See [6] for more evidence in support of this.

3. Examples on weak* compact convex sets

Example 3.1 *$l_1 = c_0^*$ with the equivalent dual norm $\|f\|' := \|f^+\| \vee \|f^-\|$ fails the w^*-fpp.*

This example is due to T. C. Lim [4] and provides us with a nonexpansive map T on a domain C that is a w^*-compact minimal invariant set for T of diameter 2.

We first show that $\|\cdot\|'$ is indeed an equivalent dual norm for l_1. To this end, for $x \in c_0$ define

$$\|x\|' := \|x^+\| + \|x^-\|$$

Then $\|\cdot\|'$ is an equivalent norm on c_0 satisfying $\|x\| \leq \|x\|' \leq 2\|x\|$ and so it suffices to show that for $f \in l_1$ we have

$$\|f\|' = \sup\{f(x) : x \in c_0,\ \|x\|' \leq 1\}.$$

Now for $x \in c_0$ with $\|x\|' \leq 1$ let

$$y_i = \begin{cases} x_i & \text{if } f_i x_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|y\|' \leq \|x\|' \leq 1$ and

$$\begin{aligned} f(x) &:= \sum_{i=1}^{\infty} f_i x_i \\ &\leq \sum_{i=1}^{\infty} f_i y_i \\ &\leq \|y^+\|\|f^+\| + \|y^-\|\|f^-\| \\ &= \left(\frac{\|y^+\|}{\|y\|'}\|f^+\| + \frac{\|y^-\|}{\|y\|'}\|f^-\| \right)\|y\|' \\ &\leq \left(\|f^+\| \vee \|f^-\|\right)\|y\|' \\ &\leq \|f\|'. \end{aligned}$$

To see the reverse inequality note that $\|f^+\|$ (or $\|f^-\|$) can be approximated arbitrarily well by $f(x)$ where the x_i are a suitable choice of 0 or 1 (0 or -1) and so $\|x\|' \leq 1$.

Now let $C = \{f \in l_1 : f_i \geq 0, \|f\|' \leq 1\}$ and define T by

$$Tf := (1 - \sum_{i=1}^{\infty} f_i,\ f_1,\ f_2,\ \ldots).$$

C is closed and bounded with respect to $\|\cdot\|$ and since the unit ball centred at 0 in the same norm is w^*-compact we have C is a weak*-compact convex subset of l_1. It is readily verified that T is a fixed point free affine mapping of C into C. Furthermore C is a minimal invariant set for T. To see this note that for any $f = (f_m) \in C$ the successive iterates are:

$$Tf = (1 - \sum_{1}^{\infty} f_m, f_1, f_2, \ldots)$$

$$T^2 f = (0, 1 - \sum_{1}^{\infty} f_m, f_1, f_2, \ldots)$$

$$T^3 f = (0, 0, 1 - \sum_1^\infty f_m, f_1, f_2, \dots)$$

$$\dots$$

So, $T^n f \rightharpoonup^{w^*} 0$. Thus 0 belongs to any nonempty T-invariant w^*-compact convex subset K of C. Hence the n'th basis vector, $e_n = T^n(0)$, is in K. It follows that $C = \overline{\text{co}}\{e_n\} \subseteq K \subseteq C$, so $K = C$.

We conclude by showing that T is a metric isometry (hence certainly a nonexpansive mapping) on C.

Given $f, g \in C$ let $P := \{i : f_i - g_i \geq 0\}$ and $N := \{i : f_i - g_i < 0\}$. In the case that $\sum\limits_{i \in P} (f_i - g_i) \geq \sum\limits_{i \in N} (g_i - f_i)$ we have

$$\begin{aligned}
\|Tf - Tg\|' &= \Big\| \Big(\sum_{i=1}^\infty (g_i - f_i),\ f_1 - g_1,\ f_2 - g_2,\ \dots \Big) \Big\|' \\
&= \Big\| \Big(\underbrace{\sum_{i \in N} (g_i - f_i) - \sum_{i \in P} (f_i - g_i)}_{\text{negative}},\ f_1 - g_1,\ f_2 - g_2,\ \dots) \Big) \Big\|' \\
&= \text{Max} \Big\{ \sum_{i \in P} (f_i - g_i),\ \sum_{i \in N} (g_i - f_i) \Big\} \\
&= \|f - g\|.
\end{aligned}$$

The equality follows similarly in the case when $\|f - g\|' = \sum\limits_{i \in N} (g_i - f_i)$.

Example 3.2 *$l_1 = c^*$ with its natural norm fails the w^*-ffp for an affine contraction.*

It will be convenient to take the dual action of l_1 on c to be

$$(f_n)(x_n) = f_1 x_1 + f_2 \lim_n x_n + f_3 x_2 + \dots,$$

where $(f_n) \in l_1$ and $(x_n) \in c$. In particular then, regarding $x = (-1, 1, 1, \dots) \in c$ as a weak* continuous linear functional over l_1. we see that,

$$\left\{ f \in l_1 : f_1 = \sum_{i=2}^\infty f_i \right\} = \ker x$$

is a w^*−closed hyperplane and consequently the set

$$C = \left\{ f : f_i \geq 0, f_1 = \sum_{i=2}^\infty f_i \leq 1, \right\}$$

being the intersection of $\ker x$ and weak* closed halfspaces is itself convex and weak* closed. Obviously, $C \subset 2B^+_{\ell_1}$, so C is weak* compact.

Now, let $\delta \in (0, 1]$ and let $(\epsilon_k) \subset [0, 1)$ be a sequence such that $\sum_{k=1}^\infty \epsilon_k < \infty$ and so $\prod_{k=1}^\infty (1 - \epsilon_k) > 0$. Define a mapping by

$$T(f) = \left(\delta (1 - f_1) + \sum_{k=1}^\infty (1 - \epsilon_k) f_{k+1}, \delta (1 - f_1), (1 - \epsilon_1) f_2, (1 - \epsilon_2) f_3, \dots \right),$$

then T is clearly an affine mapping. We claim that T is a fixed point free nonexpansive mapping of C into C and further, T is a contraction if all the ϵ_k are strictly positive.

To prove the T-invariance of K we need to show that $(Tf)_n \geq 0$, $(Tf)_1 = \sum_{n=1}^{\infty}(Tf)_k$ and $(Tf)_1 \leq 1$. The first two are obvious, for the third observe that,

$$(Tf)_1 = \delta(1-f_1) + \sum_{k=1}^{\infty}(1-\epsilon_k)f_{k+1} \leq \delta(1-f_1) + f_1 \leq \delta + (1-\delta)f_1 \leq \delta + (1-\delta) = 1.$$

We next show that T is always nonexpansive:

$$\begin{aligned}
&\|Tf - Tg\| \\
&= |\delta(g_1 - f_1) + \sum_{k=1}^{\infty}(1-\epsilon_k)(f_{k+1} - g_{k+1})| + |\delta(g_1 - f_1)| + \sum_{k=1}^{\infty}(1-\epsilon_k)|f_{k+1} - g_{k+1}| \\
&= |(\delta - 1)(g_1 - f_1) + \sum_{k=1}^{\infty}\epsilon_k(f_{k+1} - g_{k+1})| + |\delta(g_1 - f_1)| + \sum_{k=1}^{\infty}(1-\epsilon_k)|f_{k+1} - g_{k+1}| \\
&\leq (1-\delta)|g_1 - f_1| + \sum_{k=1}^{\infty}\epsilon_k|f_{k+1} - g_{k+1}| + \delta|(g_1 - f_1)| + \sum_{k=1}^{\infty}(1-\epsilon_k)|f_{k+1} - g_{k+1}| \\
&= \sum_{k=1}^{\infty}|f_k - g_k| \\
&= \|f - g\|.
\end{aligned}$$

Now suppose that $\epsilon_k > 0$, for all k, and that $\|Tf - Tg\| = \|f - g\|$, then the above contains only equalities. Hence

$$\begin{aligned}
&|(1-\delta)(g_1 - f_1) - \sum_{k=1}^{\infty}\epsilon_k(g_{k+1} - f_{k+1})| \\
&= |(1-\delta)(g_1 - f_1)| + |\sum_{k=1}^{\infty}\epsilon_k(g_{k+1} - f_{k+1})| \qquad (1)
\end{aligned}$$

and

$$|\sum_{k=1}^{\infty}\epsilon_k(f_{k+1} - g_{k+1})| = \sum \epsilon_k|f_{k+1} - g_{k+1}|. \qquad (2)$$

To satisfy (1) we must either have

$$(1-\delta)(g_1 - f_1) \geq 0$$

and

$$\sum_{k=1}^{\infty}\epsilon_k(g_{k+1} - f_{k+1}) \leq 0$$

or the reverse. Both cases follow a similar proof so we will prove the first case only. From (2) we see that the elements of the sum $\sum_{k=1}^{\infty}\epsilon_k(f_{k+1} - g_{k+1})$ are either all negative or all positive, so we must have

$$f_k \geq g_k, \quad \text{for } k \geq 2.$$

But also, $g_1 \geq f_1$, and hence

$$f_1 = \sum_{k=2}^{\infty} f_k \geq \sum_{k=2}^{\infty} g_k = g_1 \geq f_1.$$

Thus $f_k = g_k$ for $k \geq 1$; that is, $f = g$ and so T is a contraction.

Lastly we show that T is indeed fixed point free. Suppose there were an $f \in C$ with $Tf = f$. Then, for $n \geq 3$ we would have,

$$(f)_n = (1-\epsilon_{n-2})f_{n-1} = (1-\epsilon_{n-2})(1-\epsilon_{n-3})...(1-\epsilon_1)f_2.$$

Thus, if $f_2 = 0$, then $f_n = 0$ for $n \geq 3$ and also $\delta(1-f_1) = 0$, whence $f_1 = 1$ and we have the contradiction:

$$f_1 = (Tf)_1 = \delta(1-f_1) + \sum_{k=1}^{\infty}(1-\epsilon_k)f_{k+1} = 0 \neq f_1.$$

Consequently we must have $f_2 \neq 0$ and, since $f_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{f_n}{f_2} = 0.$$

This means $\prod_{k=1}^{\infty}(1-\epsilon_k) = 0$ which contradicts $\sum_{k=1}^{\infty} \epsilon_k < \infty$.

When T is a contraction there can be only one minimal invariant set for T, but we do not know if C is itself that minimal invariant set. However, when $\epsilon_k = 0$, for $k = 1, 2, ...$, this is not the case [2]. There is a smaller weak* closed convex T-invariant set; namely,

$$C' = \{f \in C : f_1 = 1\}$$

and a slightly more subtle variant of the argument used in example 3.1 shows that in this case C' is in fact the unique minimal invariant set for the nonexpansive map T. Indeed, simple calculations show that in these cases the orbit of any point of C under T converges weak* to $f_0 := (1, 1, 0, 0, \cdots)$. So, f_0 is in any set which is T-invariant and it suffices to note that the closed convex hull of the orbit of f_0 is C'. Computer experiments show that when the ϵ_k are not all zero C' need not be T-invariant.

Example 3.3 *A non-affine example in* $l_1 = c^*$.

In the same spirit as example 2.3 C. Lennard [private communication, 1995] has given a non-affine variant of example 3.2 in the case when $\delta = 1$ and $\epsilon_k = 0$, for all k.

Let C be defined as in example 3.2, and let (p_n) be any sequence of real numbers that strictly decreases to 1. Define T by

$$Tf := \left(1, 1 - \sum_{j=1}^{\infty} \frac{f_{j+1}^{p_j}}{p_j}, \frac{f_2^{p_1}}{p_1}, \ldots, \frac{f_{k+1}^{p_k}}{p_k}, \ldots\right).$$

Then one may verify that T is a non-affine contraction of C into C. Furthermore, if $p := \prod_{n=1}^{\infty} p_n$ is finite, then T is readily seen to be fixed point free.

4. Examples on weak compact convex sets

Although the question had been raised more than twenty years earlier it was not until 1981 that Dale Alspach gave an example, drawn from ergodic theory, showing that not all Banach spaces enjoy the w-FPP.

Example 4.1 *Alspach's example* [1]

Here we take C to be the set

$$C := \{f \in L_1[0,1] : 0 \leq f \leq 1, \int_0^1 f\,dx = \tfrac{1}{2}\}$$

As the intersection of an order interval with a hyperplane in an order continuous Banach lattice, C *is weak compact.*

The mapping T is essentially the *baker transform* of ergodic theory. Formally, for $f \in C$

$$Tf(t) := \begin{cases} (2f(2t)) \wedge 1 & \text{for } 0 \leq t \leq \frac{1}{2}; \\ (2f(2t-1)-1) \vee 0 & \text{for } \frac{1}{2} < t \leq 1. \end{cases}$$

It is clear from the above description that T is an isometry on C.

We now show that T is fixed point free and hence $L_1[0,1]$, and any space containing an isometric copy of it, fails to have the *w-fpp.*

Intuitively the idea is simple. First observe that the successive iterates of any point in C under T assume values closer to 0 or 1. Hence any fixed point for T must be a function which assumes only the values 0 or 1. By the 'ergodic' nature of T it then follows that such a function must be either constantly 0 or constantly 1, and neither of these functions lie in C.

The details follow.

For any $f \in C$ we have $Tf(t) = 1$ if and only if either

$$0 \leq t \leq \tfrac{1}{2} \quad \text{and} \quad \tfrac{1}{2} \leq f(2t) \leq 1$$

or

$$\tfrac{1}{2} < t \leq 1 \quad \text{and} \quad f(2t-1) = 1.$$

Furthermore if $\frac{1}{2} \leq t \leq 1$ and $Tf(t) = 1$, then $Tf(t - \frac{1}{2}) = 1$.

Now, suppose f is a fixed point for T then

$$\begin{aligned} A &:= \{t : f(t) = 1\} \\ &= \{t : Tf(t) = 1\} \\ &= \{t : 0 \leq t \leq \tfrac{1}{2} \text{ and } \tfrac{1}{2} \leq f(2t) \leq 1\} \cup \{t : \tfrac{1}{2} < t \leq 1 \text{ and } f(2t-1) = 1\} \\ &= \{\tfrac{1}{2}t : \tfrac{1}{2} \leq f(t) \leq 1\} \cup \{\tfrac{1}{2} + \tfrac{1}{2}t : f(t) = 1\} \\ &= \tfrac{1}{2}\{t : \tfrac{1}{2} \leq f(t) < 1\} \cup \tfrac{1}{2}A \cup (\tfrac{1}{2} + \tfrac{1}{2}A) \end{aligned}$$

Since the three sets in the above union are mutually disjoint and each of the last two sets has measure one half that of A it follows that:

$$B_1 := \{t : \tfrac{1}{2} \leq f(t) < 1\}$$

is a null set. But, then

$$\begin{aligned} B_1 &= \{t : \tfrac{1}{2} \leq Tf(t) < 1\} \\ &\supset \{\tfrac{t}{2} : \tfrac{1}{4} \leq f(t) < \tfrac{1}{2}\} \end{aligned}$$

and so $B_2 := \{t : \frac{1}{4} \leq f(t) < \frac{1}{2}\}$ is also a null set. Continuing in this way we have

$$B_n := \{t : \frac{1}{2^n} \leq f(t) < \frac{1}{2^{n-1}}\}$$

is a null set for $n = 1, 2, \ldots$, hence

$$\{t : 0 < f(t) < 1\} = \bigcup_{n=1}^{\infty} B_n$$

is null and

$$f \equiv \chi_A \quad (\text{where} \quad \text{meas}(A) = \int_0^1 \chi_A = \tfrac{1}{2}).$$

From the definition of T we have

$$T(\chi_A) = (\chi_{\frac{1}{2}A} + \chi_{(\frac{1}{2}+\frac{1}{2}A)})$$

so, up to sets of measure zero,

$$A = \tfrac{1}{2}A \cup (\tfrac{1}{2} + \tfrac{1}{2}A).$$

Continuing to iterate under T yields

$$A = \tfrac{1}{4}A \cup (\tfrac{1}{4} + \tfrac{1}{4}A) \cup (\tfrac{1}{2} + \tfrac{1}{4}A) \cup (\tfrac{3}{4} + \tfrac{1}{4}A)$$
$$A = \tfrac{1}{8}A \cup (\tfrac{1}{8} + \tfrac{1}{8}A) \cup (\tfrac{1}{4} + \tfrac{1}{8}A) \cup \ldots$$

et hoc genus omne.

Thus, the intersection of A with any dyadic interval (and hence any interval) has measure one half that of the interval, an impossibility for a set which is not of full measure.

Notice that, unlike the previous example, the domain C of the baker transform T is not a minimal invariant set. This follows since

$$\text{diam}(C) = 1,$$

as

$$1 \geq \text{diam}(C) \geq \|\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}\|_1 = 1,$$

while for any $f \in C$ we have $-\frac{1}{2} \leq f - \frac{1}{2} \leq \frac{1}{2}$ hence

$$\|f - \frac{1}{2}\chi_{[0,1]}\|_1 = \int_0^1 |f - \frac{1}{2}| \leq \frac{1}{2}.$$

Thus, C is not diametral and therefore not a minimal invariant set.

Indeed there seems to be no known explicit example of a non-trivial minimal invariant set for a nonexpansive map on a weak compact convex set.

Example 4.2 *Sine's modification of the Alspach example*

Robert Sine [9] gave the following modification to example 4.1 which allows us to take as the domain C of our fixed point free nonexpansive mapping the whole order interval of $0 \leq f \leq 1$.

For $f \in C := \{g : 0 \leq g \leq 1\}$ let $Sf := \chi_{[0,1]} - f$, then S defines a mapping of C onto C with $\|Sf - Sg\| = \|f - g\|$ for all $f, g \in C$.

An argument similar to that for Alspach's example shows that the composition ST, where T is the baker transform of 4.1, is an isometry on C with χ_A where $A = [0, 1]$ or 0 the only possible fixed points. However, the action of ST is to map each of these functions onto the other, hence *ST is fixed point free on the order interval* $0 \leq f \leq 1$.

Example 4.3 *Schechtman's construction.* Gideon Schechtman [8] gave a construction which leads to a greater variety of examples and is in some regards somewhat simpler than that of Alspach.

Suppose (Ω, Σ, μ) is a measure space for which there exists a measure preserving transformation $\tau : \Omega \to \Omega \times [0,1]$; that is, for any measurable $S \subseteq \Omega \times [0,1]$ we have $\mu(\tau^{-1}S) = \text{meas}(S)$ [3]. Then if C is the weak compact convex set

$$C := \{f \in L_1(\mu) : 0 \le f \le 1 \quad \text{and} \quad \int_\Omega f = \tfrac{1}{2}\}$$

we can define a mapping $T : C \to C$ by

$$Tf := \chi_{\tau^{-1}}\{(\omega, t) : 0 \le t \le f(\omega)\}$$

Clearly T is an isometry on C and $f \in C$ is a fixed point for T if and only if $f = \chi_A$ where $A \in \Sigma$ is such that $\mu(A) = \frac{1}{2}$ and $\hat{\tau}(A) := \tau^{-1}(A \times [0,1]) = A$ *a.e.*

Thus if τ is further chosen so that $\hat{\tau}$ is *ergodic* ; that is $\hat{\tau}(A) = A$ *a.e.* if and only if $A = \Omega$ or $A = \phi$, then T *is an example of a fixed point free nonexpansive mapping on* C.

Perhaps the simplest *example of an* (Ω, Σ, μ) *and* τ suitable for the above construction is the following.

Let $\Omega = [0,1]^{\aleph_0}$ with product Lebesgue measure and define τ by

$$\tau^{-1}\big((\omega_1, \omega_2, \ldots), t\big) := (t, \omega_1, \omega_2, \ldots) .$$

Clearly τ is measure preserving, further if $A \ne \phi$ and $\hat{\tau}(A) = A$, then for any $(\omega_1, \omega_2, \ldots) \in A$ we see that $(t, \omega_1, \omega_2, \ldots) \in A$ for any $t \in [0,1]$. Iterating under $\hat{\tau}$ gives
$(t_1, t_2, \ldots, t_n, \omega_1, \omega_2, \ldots) \in A$ for any $n \in \mathbf{N}$ and $t_1, t_2, \ldots, t_n \in [0,1]$, and so we have $A = \Omega$.

An alternative example with $\Omega = [0,1]$ is obtained by taking

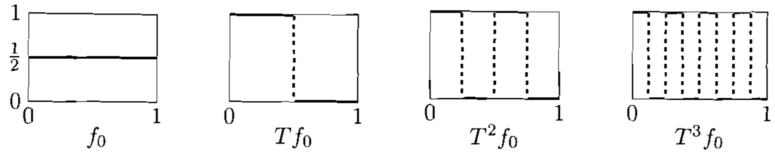
$$\tau^{-1}\Big(\sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}, \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}\Big) := \frac{\delta_1}{2} + \frac{\epsilon_1}{2^2} + \frac{\delta_2}{2^3} + \frac{\epsilon_2}{2^4} + \cdots ,$$

where $\epsilon_n, \delta_n \in \{0,1\}$ for $n = 1, 2, \ldots$ A good way to view this example is via the correspondence

$$[0,1] \longleftrightarrow \{0,1\}^{\aleph_0} : \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n} \longleftrightarrow (\epsilon_1, \epsilon_2, \ldots).$$

A set specified by prescribing precisely m of the ϵ_n's has measure $1/2^m$. From this it is clear that the product of two such sets has measure $1/2^{m_1+m_2}$ where $m_1 + m_2$ is also the number of digits prescribed for points in the τ^{-1} image of the product. It follows that τ is measure preserving. The ergodicity is established by iterating under $\hat{\tau}$ and an argument similar to that used for the conclusion of Alspach's example.

Remark 4.4 Schechtman's construction is both simpler and more versatile than that of Alspach and is of course also amenable to Sine's modification. None-the-less, the Alspach example has some advantages. The relatively simple action of the baker transform permits detailed calculations. For example, it is possible to determine the *orbit* $f_0, Tf_0, T^2f_0, T^3f_0, \ldots$, of certain starting functions f_0 under T. If $f_0 = \frac{1}{2}\chi_{[0,1]}$ we obtain the iterates depicted below.



Here we see that the sequence $T^n f_0 = \frac{1}{2}(r_n + 1)$ is an orbit under T, where r_n is the n'th *Rademacher function*. This may be combined with a result of Maurey ([7], also see the chapter entitled *Ultra-methods in metric fixed point theory*); that reflexive subspaces of $L_1[0,1]$ have the fixed point property, to show that the closed convex hull of an orbit of a nonexpansive mapping on a weakly compact convex set need not be invariant. Indeed, define D to be $\overline{\text{co}}\{T^n(f_0) : n \in \mathbf{N}\}$. Since the closed linear span of the Rademacher functions is isomorphic to $L_2[0,1]$ [5], D can not be invariant. Indeed, were it invariant, Maurey's result on the reflexive subspaces of $L_1[0,1]$, would imply that T possessed a fixed point in D and, a *fortiori*, in C.

5. Notes and Remarks

Example 2.1 is due to Kakutani, the modification presented in example 2.2 is due to Lennard. The presence of the λ_n allow one to compensate for slight perturbations of the e_n. Thus, the conclusion remains valid if the vectors e_n are replaced by vectors x_n which are 'asymptotic' to the basis vectors. This allows the example to be transported into spaces containing an 'asymptotically isometric copy' of c_0, thereby demonstrating that such spaces fail to have the fpp. Similarly, example 3.2 may be exploited to show that spaces containing an 'asymptotically isometric copy' of ℓ_1 also fail the fpp. Details of these exciting new ideas may be found in the chapter entitled *Renormings of ℓ_1 and c_0 and fixed point properties.*

Example 3.2 is also due to Lennard, the observation that it is in fact a contraction was made by Smyth who also extended it to the following broader result [10]: *Let Ω be an infinite compact Hausdorff topological space. Then $C(\Omega)^*$ fails the w^*-fpp with an affine contraction.*

In our example Ω is the one point compactification of $\mathbf{N}$, where '∞' is the extra point. So we can write $n \in \Omega$ in the form $n = (1, \infty, 2, 3, ...)$. Now, if for $z = (z_1, z_2, z_3, ...) \in c$ we write

$$z = (z_1, \lim_{n\to\infty} z_n, z_2, z_3, ...)$$

z is a continuous function on Ω. This is because

$$\lim_{n\to\infty} z(n) = \lim_{n\to\infty} z_n = z(\infty).$$

So c=$C(\Omega)$. If we let l_1 act on c by

$$x(z) = x_1 z_1 + x_2 \lim_{n\to\infty} + x_3 z_2 + ..., \quad \forall x \in l_1, z \in c$$

then $l_1 = c^* = C(\Omega)^*$ and T is the affine contraction for which the w^*-fpp fails.

The *fpp, w-fpp, or w*-fpp* relate to all mappings in a particular class having fixed points. This class of mappings depends on both which mappings are picked out as nonexpansive by the norm and which domains are admissible. Since $\ell_1 = c_0^*$ enjoys the

w^*-fpp in its natural norm, examples 3.1 and 3.2 taken together show that both of these factors are critical. Moving to an equivalent norm varies which mappings are picked out as nonexpansive, but not the admissible domains. On the other hand, for a dual space, changing the pre-dual does not affect the dual norm, nor alter which mapping are nonexpansive, but does change the class of admissible domains. These considerations also show that any characterization of the *w^*-fpp* will necessarily involve a condition on the pre-dual.

Chris Lennard [private communication, 1996] has given a wavelet construction of a fixed point free isometry, similar to that of Alspach, and also on the order interval $[0 \leq f \leq 1]$ in $L_1[0,1]$.

PROBLEMS.

The results of section 4 indicate an intimate connection between fixed point free isometries and ergodic transformations of the underlying measure space. In the true tradition of ergodic theory, we ask:

Is the set of fixed point free isometries on the order interval $[0 \leq f \leq 1]$ residual in an appropriate sense, at least among isometries which map into the set of $0, 1$-valued functions?

Clearly any space containing an isometric copy of $L_1(\mu)$ also fails the *w-fpp*. *Can one give an intrinsic description of examples demonstrating this failure for the spaces l_∞ and $C[0,1]$?*

Examples 3.1, 3.2 and those of section 4 also suggest the following question.

If a space X fails the (w, w^*)-fpp does it necessarily fail with an isometry?

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