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FIXED POINT THEORY FOR ALMOST CONVEX FUNCTIONS

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1. INTRODUCTION

Traditionally, metric fixed point theory has sought classes of spaces in which a given type of mapping (nonexpansive, assymptotically or generalized nonexpansive, uniformly Lipschitz, etc.) from a nonempty weakly compact convex set into itself always has a fixed point. In some situations the class of space is determined by the application while there is some degree of freedom in constructing the map to be used. With this in mind we seek to relax the conditions on the space by considering more restrictive types of mappings. Previous instances of this include:

- Strict contractions on complete metric spaces (the celebrated Banach contraction mapping principle). (See [1]).
- Affine selfmappings of nonempty weakly compact convex sets in a Banach space (which have fixed points by virtue of their weak-continuity and the Schauder–Tychonoff fixed point theorem).

We generalize the latter, and as we will shortly see, also to some extent the former.

Let C be a nonempty closed convex subset of a Banach space X. For a continuous strictly increasing function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ with $\alpha(0) = 0$ we say $T : C \to X$ is an *α-almost convex mapping* if for all $x, y \in C$ and all $\lambda \in [0, 1]$ we have

$$J_T(\lambda x + (1 - \lambda)y) \leq \alpha(\max\{J_T(x), J_T(y)\}),$$

where J_T is defined by

$$J_T(x) := \|x - Tx\|, \qquad \text{for all } x \in C.$$

In the case when $\alpha(t) = rt$, for some $r > 0$, we say T is *r-almost convex*, and simply refer to T as *almost convex* when $r = 1$. That is, T is almost convex whenever

$$J_T(\lambda x + (1 - \lambda)y) \leq \max\{J_T(x), J_T(y)\},$$

for all $x, y \in C$ and $\lambda \in [0, 1]$.

Affine maps are clearly almost convex, indeed they satisfy the seemingly stronger inequality,

$$J_T(\lambda x + (1 - \lambda)y) \leq \lambda J_T(x) + (1 - \lambda)J_T(y).$$

On the other hand, any α-almost convex map is of 'convex type' [2]; that is, if $J_T(x_n) \to 0$ and $J_T(y_n) \to 0$ then $J_T(\frac{1}{2}(x_n + y_n)) \to 0$, so the midpoint of two 'approximate fixed point

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sequences' is itself an approximate fixed point sequence for T. Khamsi initially showed that *nonexpansive* maps of convex type on nonempty weakly compact convex sets have fixed points when the space has the alternate Banach–Saks property, and later in [3] that the result is true in all Banach spaces. We shall show that a similar result holds for α-almost convex maps without the assumption of nonexpansivity.

α-almost (or quasi) convex functions have been considered in optimization theory [4, 5], where α is referred to as a 'forcing function' and is often also required to be convex.

2. EXAMPLES

Beyond the affine mappings already mentioned, instances of α-almost convex mappings include the following.

(1) $T:[0, 1] \to [0, 1]: x \mapsto x(1-x)$ is not affine, but $J_T(x) = |x - Tx| = x^2$ is a convex function, and so T is almost convex.

(2a) $T: B_{c_0} \to B_{c_0}$ defined by

$$T(x_n) := (x_1 - \operatorname{sgn}(x_1)\|(x_n)\|_\infty, x_2, x_3, \ldots)$$

is almost convex, as $J_T(x) = \|x\|_\infty$ is a convex function.

(2b) Let $(\phi_\nu : \mathbf{R} \to \mathbf{R})$ be a family of functions which are equicontinuous at 0 and satisfy

$$\phi_n(0) \to 0, \qquad \phi_n(x) \le x, \qquad \text{and } \phi_n'' \le 0,$$

then $T:(x_n) \mapsto (\phi_n(x_n))$ is an almost convex mapping from c_0 into c_0.

(3) A self mapping T of a metric space (M, d) is a contraction in the sense of Bianchini [1] whenever there exists a number h, $0 < h < 1$, such that, for each $x, y \in M$,

$$d(Tx, Ty) \le h \max\{d(Tx, x), d(Ty, y)\}$$

If M is a convex subset of a Banach space X, then this type of mapping is α-almost convex. Indeed,

$$\begin{aligned} J_T(\lambda x + (1-\lambda)y) &\le \lambda J_T(x) + (1-\lambda)J_T(y) \\ &\quad + \lambda h \max\{J_T(x), J_T(\lambda x + (1 - \lambda y))\} \\ &\quad + (1-\lambda)h \max\{J_T(y), J_T(\lambda x + (1 - \lambda y))\} \\ &\le 2(\lambda J_T(x) + (1-\lambda)J_T(y)) + hJ_T(\lambda x + (1 - \lambda y)). \end{aligned}$$

Therefore,

$$J_T(\lambda x + (1-\lambda)y) \le \frac{2}{1-h} \max\{J_T(x), J_T(y)\}.$$

(4) Let C be a convex nonempty subset of a Banach space X. Every k-Lipschitzian mapping $T: C \to X$ which satisfies

$$\|x - y\| \le \gamma(\max\{J_T(x), J_T(y)\})$$

where $\gamma: \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous strictly increasing function with $\gamma(0) = 0$ for all $x, y \in C$ is α-almost convex.

Indeed,

$$J_T(\lambda x + (1-\lambda)y) \le \lambda J_T(x) + (1-\lambda)J_T(y) + \lambda\|Tx - T(\lambda x + (1-\lambda)y)\| + (1-\lambda)\|Ty - T(\lambda x + (1-\lambda)y)\| \le \beta(\max\{J_T(x), J_T(y)\}),$$

where $\beta(t) = t + (k/2)\lambda(t)$.

(4a) Every strict contraction $T: C \to X$ where C is a convex nonempty subset of a Banach space X satisfies the above condition, and therefore it is an α-almost convex mapping.

Indeed, we can take $\lambda(t) = (2/1-k)t$ and hence $\beta(t) = (1/1-k)t$ where $0 < k < 1$ is the contraction constant of T.

(5) Similar, though more tedious, calculations to those of the last three examples establish that if $T: C \to X$ is a *generalized nonexpansive map*; that is,

$$\|Tx - Ty\| \le a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|)$$

where a, b and c are positive constants with $a + 2b + 2c \le 1$, and if either this last inequality is strict, or $b \ne 0$, then T is r-almost convex. Indeed,

$$J_T(\lambda x + (1-\lambda)y) \le \frac{(1+b+c)(1-c)}{(1-b-c)(1-a-2c)} \max\{J_T(x), J_T(y)\} \le \frac{3}{2b} \max\{J_T(x), J_T(y)\}.$$

(6) A mapping T of a closed convex subset of a Banach space X is said to be of *type* Γ [6] if there exists a continuous strictly increasing convex function $\gamma: \mathbf{R}^+ \to \mathbf{R}^+$ with $\gamma(0) = 0$ for which

$$\gamma(\|\lambda Tx + (1-\lambda)(Ty - T)\lambda x + (1-\lambda)y)\| \le |\|x - y\| - \|Tx - Ty\||.$$

Such maps are α-almost convex, where $\alpha(t) = t + \gamma^{-1}(2t)$.

To see this, note that γ^{-1} is strictly increasing and that

$$\begin{aligned} J_T(\lambda x + (1-\lambda)y &= \|\lambda x + (1-\lambda)y - T(\lambda x + (1-\lambda)y\| \\ &\le \|\lambda x + (1-\lambda)y - \lambda Tx + (1-\lambda)Ty\| \\ &\quad + \gamma^{-1}(|\|x - y\| - \|Tx - Ty\||) \\ &\le \lambda J_T(x) + (1-\lambda)J_T(y) + \gamma^{-1}(\|x - Tx\| + \|y - Ty\| \\ &\le \alpha(\max\{J_T(x), J_T(y)\}). \end{aligned}$$

As a consequence of this last example and [6] we have:

(6a) All nonexpansive selfmaps of closed bounded convex subsets in a uniformly convex space are α-almost convex.

At this point it is worth noting that the class of maps which are α-almost convex on a given domain C is stable under equivalent renormings. Indeed, if $m\|x\| \le \|x\|' \le M\|x\|$ and T is α-almost convex with respect to the norm $\|\cdot\|$ then it is α'-almost convex with respect to $\|\cdot\|'$, where $\alpha'(t) = M\alpha(t/m)$.

Many other examples of α-almost convex mappings are a consequence of the following elementary observation.

Alternative Principle: If C is a closed bounded convex set, and $T : C \to C$, then at least one of the following applies.

(i) T is r-almost convex, for some $r > 0$, or

(ii) $\inf \{J_T(x) : x \in C\} = 0$. That is, T admits approximate fixed points in C.

Proof. If T is not r-almost convex for any $r > 0$, then for each $n \in \mathbf{N}$, taking $r = n$, we see that there must exist points x_n and y_n in C and $\lambda_n \in [0, 1]$ such that

$$\infty > \operatorname{diam} C \geq J_T(\lambda_n x_n + (1 - \lambda_n)y_n) \geq n \max\{J_T(x_n), J_T(y_n)\},$$

so $J_T(x_n)$ and $J_T(y_n)$ tend to 0 as $n \to \infty$. ■

Combining this with the examples of maps with non-zero minimal displacement given in Chap. 20 of [7], we see that there exist r-almost convex self maps of weak compact convex sets (including B_{l_2}) with $\inf J_T(x) > 0$. In particular such maps are fixed point free, and so can not be weakly continuous. Indeed, example (2) above shows that unlike affine maps, almost convex maps need not be weakly continuous. To see this note that in c_0 the standard basis vectors $e_n \overset{w}{\rightharpoonup} 0$, but

$$\begin{aligned} T(e_n) &= (-1, 0, \cdots 1, 0, \cdots), \text{ where the 1 occurs in the nth position} \\ &\overset{w}{\rightharpoonup} (-1, 0, 0, \cdots) \neq T(0) = 0. \end{aligned}$$

None-the-less we have the following.

PROPOSITION 2.1. Let X be a Banach space and let C be a nonempty closed convex subset of X. If $T : C \to X$ is norm continuous and almost convex then $J_T(x) := \|x - Tx\|$ is weak lower semi-continuous.

Proof. Suppose that (x_n) is a sequence in C such that $x_n \overset{w}{\rightharpoonup} x$. Given $\varepsilon > 0$, choose a subsequence (x_{n_k}) such that $J_T(x_{n_k}) < \liminf J_T(x_n) + \varepsilon/2$, for all k, and let $\delta > 0$ be such that $|J_T(y) - J_T(x)| < \varepsilon/2$ whenever $\|y - x\| < \delta$ (possible, as T and hence J_T is norm continuous at x). Since $x_{n_k} \overset{w}{\rightharpoonup} x$, by Mazur's theorem, there exists $x_{n_{k_1}}, x_{n_{k_2}}, \cdots, x_{n_{k_m}}$ and $\lambda_1, \lambda_2, \cdots, \lambda_m \in (0, 1]$ with $\sum \lambda_i = 1$ such that $\|x - \sum \lambda_i x_{n_{k_i}}\| < \delta$. But, then

$$\begin{aligned} J_T(x) < J_T\left(\sum_{i=1}^{m} \lambda_i x_{n_{k_i}}\right) + \frac{\varepsilon}{2} &= J_T\left(\lambda_1 x_{n_{k_1}} + (1 - \lambda_1) \sum_{i=2}^{m} \frac{\lambda_i}{(1 - \lambda_1)} x_{n_{k_i}}\right) + \frac{\varepsilon}{2} \\ &\leq \max\left\{J_T(x_{n_{k_1}}), J_T\left(\sum_{i=2}^{m} \frac{\lambda_i}{(1 - \lambda_1)} x_{n_{k_i}}\right)\right\} + \frac{\varepsilon}{2} \\ &\leq \cdots \\ &\leq \max\{J_T(x_{n_{k_1}}), \cdots, J_T(x_{n_{k_m}})\} + \frac{\varepsilon}{2} < \liminf_n J_T(x_n) + \varepsilon, \end{aligned}$$

and so we conclude that J_T is weak lower semi-continuous. ■

COROLLARY 2.2. For X, C and T as above, if in addition C is weak compact, then

$$M(T) := \left\{x \in C : J_T(x) = \inf_{y \in C} J_T(y)\right\}$$

is a nonempty weak compact convex subset of C. Indeed the same is true of any of the sublevel sets for J_T.

In particular, such a T has a fixed point if and only if

$$\inf_{x \in C} J_T(x) = 0;$$

that is, if and only if T admits an approximate fixed point sequence in C. And, in this case, $\mathrm{Fix}(T) = M(T)$ is a nonempty weak compact convex set.

3. GENERAL RESULTS

We do not know if $J_T(x)$ is weak lower semi-continuous for arbitrary α-almost convex maps, however, we do have the following demiclosedness result. Recall that $V: C \to X$ is *demiclosed* at 0 [8] if whenever $x_n \overset{w}{\rightharpoonup} x$ and $\|Vx_n\| \to 0$ we have $Vx = 0$.

PROPOSITION 3.1. Let X be a Banach space and let C be a nonempty closed convex subset of X. If $T: C \to X$ is norm continuous and α-almost convex then $I - T$ is demiclosed at 0.

Proof. Suppose $x_n \overset{w}{\rightharpoonup} x_0$ and $J_T(x_n) = \|(I - T)(x_n)\| \to 0$. We may assume without loss of generality that

$$J_T(x_n) > 0$$

for all positive integers n.

Fix $\varepsilon > 0$. Since T is continuous, there exists $\delta > 0$ such that

$$J_T(x_0) < J_T(y) + \frac{\varepsilon}{2},$$

whenever $y \in C$ and $\|y - x_0\| < \delta$.

On the other hand, since α is continuous at 0 and $\alpha(0) = 0$, there exists a positive integer n_1 such that

$$0 < \alpha(J_T(x_{n_1})) < \frac{\varepsilon}{2}.$$

As $J_T(x_n) \to 0$ and $\alpha(J_T(x_n)) \to 0$, there exists $n_2 > n_1$ such that

$$0 < J_T(x_{n_2}) < \min\{J_T(x_{n_1}), \alpha(J_T(x_{n_1}))\}$$

and

$$0 < \alpha(J_T(x_{n_2})) < \min\{J_T(x_{n_1}), \alpha(J_T(x_{n_1}))\}.$$

Thus, by induction we can get a subsequence (x_{n_k}) of (x_n) satisfying

$$0 < J_T(x_{n_{k+1}}) < \min\{J_T(x_{n_k})), \alpha(J_T(x_{n_k}))\}$$

and

$$0 < \alpha(J_T(x_{n_{k-1}})) < \min\{J_T(x_{n_k}), \alpha(J_T(x_{n_k}))\}.$$

for all positive integer k.

We assert that if $m \geq 2$ and $\sum_{k=1}^{m} \lambda_k x_{n_k}$ is a convex combination of $x_{n_1}, x_{n_2}, \dots, x_{n_m}$ then

$$J_T\left(\sum_{k=1}^{m} \lambda_k x_{n_k}\right) \leq \alpha(J_T(x_{n_1})).$$

Indeed, for $m = 2$ we have

$$J_T(\lambda_1 x_{n_2} + \lambda_2 x_{n_2}) \leq \alpha(\max\{J_T(x_{n_1}), J_T(x_{n_2})\}) = \alpha(J_T(x_{n_1})).$$

If we suppose that the assertion is true for $k = m - 1$, then

$$J_T\left(\sum_{k=1}^{m} \lambda_k x_{n_k}\right) = J_T\left(\lambda_1 x_{n_1} + (1-\lambda_1)\sum_{k=2}^{m} \frac{\lambda_k}{(1-\lambda_1)} x_{n_k}\right)$$
$$\leq \alpha\left(\max\left\{J_T(x_{n_1}), J_T\left(\sum_{k=2}^{m} \frac{\lambda_k}{(1-\lambda_1)} x_{n_k}\right)\right\}\right)$$
$$\leq \alpha(\max\{J_T(x_{n_1}), \alpha(J_T(x_{n_2}))\}) = \alpha(J_T(x_{n_1})).$$

To complete the proof we need only observe that by Mazur's theorem, there exists a convex combination $\sum_{k=1}^{m} \lambda_k x_{n_k}$ such that

$$\left\|\sum_{k=1}^{m} \lambda_k x_{n_k} - x_0\right\| < \delta,$$

and then

$$J_T(x_0) < J_T\left(\sum_{k=1}^{m} \lambda_k x_{n_k}\right) + \frac{\varepsilon}{2} \leq \alpha(J_T(X_{n_1})) + \frac{\varepsilon}{2} < \varepsilon,$$

which concludes the proof. ■

As an immediate consequence we have the following fixed point theorem for α-almost convex maps.

PROPOSITION 3.2. Let X be a Banach space, let C be a nonempty weak compact convex subset of X, and let $T: C \to X$ be norm continuous and α-almost convex. Then T has a fixed point in C if and only if $\inf\{J_T(x): x \in C\} = 0$.

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$). Since $\inf\{J_T(x): x \in C\} = 0$, we can find an approximate fixed point sequence (x_n) in C which without loss of generality we can assume is weakly convergent to $x_0 \in C$. The above proposition now applies to yield the result. ■

Remark. For T affine and $x_N = (1/N)\sum_{n=0}^{N-1} T^n x_0$ we have $J_T(x_N) = \|x_0 - T^N x_0\|/N \leq (1/N)$ diam C. Thus the above result includes the existence of fixed points for affine selfmaps of nonempty weak compact convex sets. Similarly, for a strict contraction T with Lipschitz constant k and $x_n = T^n x_0$ we have $J_T(x_n) \leq k^n \|x_0 - Tx_0\| \leq k^n$ diam C. Combining this with Example (4a) we see that the above result also captures Banach's contraction mapping theorem at least for strictly contractive selfmaps of nonempty weak compact convex sets in a Banach space.

Recall, that a map $T: C \to C$ is *asymptotically nonexpansive* if there exists constants $k_n \to 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for $n = 1, 2, 3, \cdots$. Since asymptotically nonexpansive maps admit approximate fixed point sequences we have the following.

Corollary 3.3. Let C be a nonempty weak compact convex subset of the Banach space X, and let $T: C \to C$ be an asymptotically nonexpansive and α-almost convex map, then T has a fixed point in C.

This last corollary may be compared with the result of Khamsi [9]. From the definition of α-almost convexity and Proposition 3.1 we readily see that the following stronger version of Proposition 3.2 holds.

Corollary 3.4. For X, C, and T as in Proposition 3.2 with $\inf \{J_T(x): x \in C\} = 0$, we have that $\mathrm{Fix}(T)$ is a nonempty weak compact convex set.

Using fairly routine arguments, the last corollary enables us to see the following.

Corollary 3.5. Let C be a nonempty weak compact convex subset of the Banach space X and let $\mathcal{T}$ be a commuting family of maps from C to X such that each $T \in \mathcal{T}$ is norm continuous and α_T-almost convex with $\inf \{J_T(x): x \in D\} = 0$, whenever D is a nonempty closed convex T-invariant subset of C (as would be the case if the elements of $\mathcal{T}$ were a mixture of affine and α_T-almost convex asymptotically nonexpansive maps). Then the maps in $\mathcal{T}$ have a common fixed point.

As a further consequence of Proposition 3.1 we have the following.

Proposition 3.6. Let C be a nonempty weak compact convex subset of the Banach space X, and let $T: C \to X$ be norm continuous, α-almost convex, and asymptotically regular at $x_0 \in C$; that is, $J_T(T^n x_0) \to 0$ (for example, if $T = \frac{1}{2}(I + V)$, where V is α-almost convex and nonexpansive). Then the iterates $T^n x_0$ weakly converge to a fixed point of T if either

(i) T is a contraction; that is, $\|Tx - Ty\| < \|x - y\|$ whenever $x \neq y$, or

(ii) T is asymptotically nonexpansive and X has Opial's property: whenever $x_n \xrightarrow{w} 0$ and $x \neq 0$ we have $\liminf \|x_n\| < \liminf \|x - x_n\|$.

Proof. (i) Suppose this were not the case, then we can find subsequences $T^{n_k} x_0 \xrightarrow{w} y_0$ and $T^{m_k} x_0 \xrightarrow{w} z_0 \neq y_0$. By the demiclosedness both y_0 and z_0 are fixed points of T, a contradiction, since contractions can have at most one fixed point.

(ii) This follows from standard arguments similar to those used in the nonexpansive case [see for example, [10]]. ■

The following characterization of reflexivity follows from the theorem of Mil'man and Mil'man [11] and the above considerations.

Proposition 3.7. The Banach space X is reflexive if and only if whenever C is a nonempty closed bounded convex subset of X and $T: C \to C$ is norm continuous, α-almost convex with $\inf \{J_T(x): x \in C\} = 0$ it follows that T has a fixed point.

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