



Fixed points of uniformly lipschitzian mappings

S. Dhompongsa^a, W.A. Kirk^{b,*}, Brailey Sims^c

^a *Department of Mathematics, Chiang Mai University, Chiang Mai, Thailand*

^b *Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA*

^c *School of Mathematical and Physical Sciences, The University of Newcastle, NSW 2308, Australia*

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Abstract

Two fixed point theorems for uniformly lipschitzian mappings in metric spaces, due respectively to E. Lifšić and to T.-C. Lim and H.-K. Xu, are compared within the framework of the so-called CAT(0) spaces. It is shown that both results apply in this setting, and that Lifšić's theorem gives a sharper result. Also, a new property is introduced that yields a fixed point theorem for uniformly lipschitzian mappings in a class of hyperconvex spaces, a class which includes those possessing property (P) of Lim and Xu.

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1. Introduction

A mapping $T : M \rightarrow M$ of a metric space (M, d) is said to be uniformly lipschitzian if there exists a constant k such that $d(x, y) \leq kd(T^n x, T^n y)$, for all $x, y \in M$ and $n \in \mathbb{N}$. This class of mappings was introduced by Goebel and Kirk in [5], where it was shown that if C is a bounded closed convex subset of a uniformly convex Banach space X , then there exists a constant $k > 1$, depending on the modulus of convexity of X , such that every uniformly lipschitzian mapping $T : C \rightarrow C$ with constant k has a fixed point. Since then there have been a number of extensions of this result, typically in a Banach space setting (see, e.g., the discussion in [6]). However two results in a metric setting are noteworthy. The first is a result of Lifšić [11] and the second is

* Corresponding author.

E-mail addresses: sompongd@chiangmai.ac.th (S. Dhompongsa), kirk@math.uiowa.edu (W.A. Kirk), Brailey.Sims@newcastle.edu.au (B. Sims).

due to Lim and Xu [12]. Here we compare these results, taking as an underlying framework the so-called CAT(0) spaces. We show in particular that within this framework both the Lifšic and the Lim–Xu theorems apply, and that Lifšic’s theorem yields the sharper conclusion. This is an important feature of the paper because it provides a class of spaces which are **not** Banach spaces, but for which the Lifšic characteristic can be calculated, and which satisfy all of the assumption of the Lim–Xu theorem. This appears to be the first example of such a class of spaces.

We also introduce a new property that yields a fixed point theorem for uniformly lipschitzian mappings in certain hyperconvex spaces. The precise relationship of this new property to ones previously studied is not yet clear. However the proof is a departure from the usual methods, and the result yields the Lim–Xu theorem in a hyperconvex setting as a corollary.

We begin with some basic definitions and notation that will be needed later. Let (X, d) be a bounded metric space. For a nonempty subset D of X , set

$$\begin{aligned} r_x(D) &= \sup\{d(x, y) : y \in D\}, \quad x \in X; \\ r(D) &= \inf\{r_x(D) : x \in X\}; \\ C(D) &= \{x \in X : r_x(D) = r(D)\}; \\ \delta(D) &= \sup\{d(x, y) : x, y \in D\}; \\ \text{cov}(D) &= \bigcap \{B : B \text{ is a closed ball and } D \subset B\}. \end{aligned}$$

The number $r(D)$ is called the *Chebyshev radius* of D (in X) and $C(D)$ is called the *Chebyshev center* of D .

A subset A of X is said to be admissible if $\text{cov}(A) = A$. The number

$$\tilde{N}(X) := \sup \left\{ \frac{r(A)}{\delta(A)} \right\},$$

where the supremum is taken over all nonempty bounded admissible subsets A of X for which $\delta(A) > 0$ is called the *normal structure coefficient* of X . If $\tilde{N}(X) \leq c$ for some constant $c < 1$, then X is said to have *uniform normal structure*. (For some authors, $\tilde{N}(X)$ would be the inverse of the normal structure coefficient.)

The metric space (X, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha \in \Gamma} B(x_\alpha; r_\alpha) \neq \emptyset$$

for any collection of points $\{x_\alpha\}_{\alpha \in \Gamma}$ in X and positive numbers $\{r_\alpha\}_{\alpha \in \Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α, β in Γ . The classical spaces ℓ_∞ and L_∞ are examples of hyperconvex Banach spaces. Two facts are pertinent to what follows: $\tilde{N}(X) = 1/\sqrt{2}$ if X is a Hilbert space and $\tilde{N}(X) = 1/2$ if X is hyperconvex.

We now turn to the definition of the Lifšic characteristic of a metric space X . Balls in X are said to be *c-regular* if the following holds: for each $k < c$ there exist $\mu, \alpha \in (0, 1)$ such that for each $x, y \in X$ and $r > 0$ with $d(x, y) \geq (1 - \mu)r$, there exists $z \in X$ such that

$$B(x; (1 + \mu)r) \bigcap B(y; k(1 + \mu)r) \subset B(z; \alpha r). \quad (1.1)$$

The *Lifšic characteristic* $\kappa(X)$ of X is defined as follows:

$$\kappa(X) = \sup\{c \geq 1 : \text{balls in } X \text{ are } c\text{-regular}\}.$$

Theorem 1 (Lifšic [11]). *Let (X, d) be a bounded complete metric space. Then every uniformly k -lipschitzian mapping $T : X \rightarrow X$ with $k < \kappa(X)$ has a fixed point.*

In [12], Lim and Xu introduced the so-called property (P) for metric spaces. A metric space (X, d) is said to have *property (P)* if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X , there exists $z \in \bigcap_{n \geq 1} \text{cov}(\{z_j : j \geq n\})$ such that

$$\limsup_n d(z, x_n) \leq \limsup_j \limsup_n d(z_j, x_n).$$

The following theorem is the main result of [12].

Theorem 2 ([12, Theorem 7]). *Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure. Then every uniformly k -lipschitzian mapping $T : X \rightarrow X$ with $k < \tilde{N}(X)^{-\frac{1}{2}}$ has a fixed point.*

It is known that the Lifšic characteristic of a Hilbert space is $\sqrt{2}$, and in Section 3 we show that the Lifšic characteristic of an \mathbb{R} -tree is 2. Therefore in these spaces Lifšic’s theorem yields the sharper result. We also show that the same is true in the CAT(0) spaces, a class of spaces that includes these two spaces as extreme cases.

2. CAT(κ) spaces

Let (X, d) be a geodesic metric space in which each two points $x, y \in X$ are joined by a unique geodesic (metric) segment denoted $[x, y]$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

Denote by M_κ^2 the following classical metric spaces:

- (1) if $\kappa = 0$ then M_0^2 is the Euclidean plane \mathbb{E}^2 ;
- (2) if $\kappa < 0$ then M_κ^2 is obtained from the classical hyperbolic plane \mathbb{H}^2 by multiplying the hyperbolic distance by $1/\sqrt{-\kappa}$.

A metric space X is said to be a CAT(κ) space (the term is due to M. Gromov — see, e.g., [1, p. 159]) if it is geodesically connected, and if every geodesic triangle in X is at least as ‘thin’ as its comparison triangle in M_κ^2 . We make this precise below. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [1] or Burago et al. [2].

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d_{M_\kappa^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. The triangle inequality assures that comparison triangles always exists. If a point x is on an edge $[x_i, x_j]$ of Δ , then $\bar{x} \in \bar{\Delta}$ is called a comparison point of x if

$$d(x_i, x) = d_{M_\kappa^2}(\bar{x}_i, \bar{x}) \quad \text{and} \quad d(x_j, x) = d_{M_\kappa^2}(\bar{x}_j, \bar{x}).$$

A geodesic metric space is said to be a CAT(κ) space if all geodesic triangles of appropriate size satisfy the following CAT(κ) comparison axiom.

CAT(κ): Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset M_\kappa^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(κ) *inequality* if for all $x, y \in \Delta$,

$$d(x, y) \leq d_{M_\kappa^2}(\bar{x}, \bar{y}), \tag{2.1}$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the respective comparison points of x, y .

Of particular interest are the complete CAT(0) spaces, sometimes called *Hadamard spaces*. These spaces are uniquely geodesic and they include, as a very special case, the following class of spaces.

Definition 3. An \mathbb{R} -tree is a metric space T such that:

- (i) there is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in T$;
- (ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Proposition 4 ([1, Chapter II.1]). *The following relations hold:*

- (1) If X is a CAT(κ) space, then it is a CAT(κ') space for every $\kappa' \geq \kappa$.
- (2) X is a CAT(κ) space for all $\kappa < 0$ if and only if X is an \mathbb{R} -tree.

One consequence of (1) and (2) is that any result proved for CAT(0) spaces automatically carries over to any CAT(κ) spaces for $\kappa < 0$, and, in particular, to \mathbb{R} -trees.

Another fundamental property of CAT(0) spaces that we will need in the following is the so-called CN inequality. In fact a geodesic space is a CAT(0) space if and only if this inequality holds (see [1, p. 163]).

The CN inequality: for all $p, q, r \in X$ and all m with $d(q, m) = d(r, m) = d(q, r)/2$, one has

$$d(p, q)^2 + d(p, r)^2 \geq 2d(m, p)^2 + \frac{1}{2}d(q, r)^2. \tag{2.2}$$

All CAT(κ) spaces for $\kappa \leq 0$ have uniform normal structure with normal structure coefficient $c \leq 1/\sqrt{2}$. The precise values of c depend on κ . (See [10]; also the discussion in [8].)

3. The Lifšic characteristic of CAT(0) spaces

Theorem 5. *If (X, d) is a complete CAT(0) space, then $\kappa(X) \geq \sqrt{2}$. Moreover, if X is an \mathbb{R} -tree, $\kappa(X) = 2$.*

First a preliminary observation. Every bounded closed convex subset of a CAT(0) space has a unique Chebyshev center which is a singleton. Since closed convex subsets of a CAT(0) space are nonexpansive retracts of the space [1, p. 176], the unique minimal ball containing such a set must be centered at a point of the set. In other words, every bounded closed convex set contains its Chebyshev center. Thus in the definition of the Lifšic characteristic of such space, the inclusion (1.1) may be replaced with:

$$r \left(B(x; (1 + \mu)r) \cap B(y; k(1 + \mu)r) \right) \leq \alpha r, \tag{3.1}$$

where $r(\cdot)$ denotes the Chebyshev radius.

Proof of Theorem 1. We first show that in general $\kappa(X) \geq \sqrt{2}$. Let $r > 0$, choose $x, y \in X$ with $d(x, y) = r$ and let $\bar{x}, \bar{y} \in \mathbb{R}^2$ be any two points with $\|\bar{x} - \bar{y}\| = d(x, y)$.

Suppose $k = \kappa(X) < \sqrt{2}$. Then

$$r \left(B(x; r) \cap B(y; kr) \right) \leq \xi r$$

for some $\xi < 1$. (This is because the Lifšic characteristic of \mathbb{R}^2 is $\sqrt{2}$.) Now choose $\alpha \in (\xi, 1)$. Then for $\mu \in (0, 1)$ sufficiently near 0 and $\alpha \in (0, 1)$ sufficiently near 1,

$$r \left(B(\bar{x}; (1 + \mu)r) \cap B(\bar{y}; k(1 + \mu)r) \right) \leq \alpha r,$$

and we may assume in addition only that $d(x, y) \geq (1 - \mu)r$. Let

$$\bar{S} := B(\bar{x}; (1 + \mu)r) \cap B(\bar{y}; k(1 + \mu)r)$$

and

$$S := B(x; (1 + \mu)r) \cap B(y; k(1 + \mu)r).$$

The Chebyshev center \bar{c} of \bar{S} lies on the segment $[\bar{x}, \bar{y}]$. Also if $u \in S$ and if $\Delta(\bar{y}, \bar{x}, \bar{u})$ is a comparison triangle for $\Delta(y, x, u)$ in \mathbb{R}^2 , then $\bar{u} \in \bar{S}$. Therefore $\|\bar{u} - \bar{c}\| \leq \alpha r$. If c is the point of the segment $[x, y]$ for which $d(y, c) = \|\bar{y} - \bar{c}\|$, then (using the CAT(0) inequality)

$$d(u, c) \leq \|\bar{u} - \bar{c}\| \leq \alpha r.$$

Since this is true for any $u \in S$ it follows that $r(S) \leq \alpha r$, and since $k < \sqrt{2}$ was arbitrary, we have $\kappa(X) \geq \sqrt{2}$.

We now suppose X is an \mathbb{R} -tree, and we show that $\kappa(X) = 2$ by a direct calculation. Let $x, y \in X$ with $d(x, y) = r$, and let $k < 2$. Set

$$S := B(x; r) \cap B(y; kr).$$

We show that $\text{diam}(S) \leq 2(k - 1)r$. Let $u, v \in S$. There exist points $p, q \in [x, y]$ such that $d(x, v) = d(x, p) + d(p, v)$ and $d(x, u) = d(x, q) + d(q, u)$. Similarly, $(y, v) = d(y, p) + d(p, v)$ and $d(y, u) = d(y, q) + d(q, u)$. Without loss of generality we may assume $d(x, p) = d(x, q) + d(q, p)$. Therefore

$$d(u, v) = d(u, q) + d(q, p) + d(p, v).$$

Since $u, v \in B(y; kr)$ we now have

$$\begin{aligned} 2kr &\geq d(u, y) + d(y, v) \\ &= d(y, q) + d(q, u) + d(y, p) + d(p, v) \\ &= r - d(x, p) + d(p, v) + r - d(x, q) + d(q, v) \\ &= 2r + d(u, v). \end{aligned}$$

This implies $d(u, v) \leq 2(k - 1)r$. Therefore, for $\mu \in (0, 1)$ sufficiently small and $\alpha \in (0, 1)$ sufficiently near 1,

$$\text{diam} \left(B(x; (1 + \mu)r) \cap B(x; k(1 + \mu)r) \right) \leq 2\alpha r$$

when $d(x, y) \geq (1 - \mu)r$. Since X is hyperconvex (thus $\tilde{N}(X) = 1/2$) this in turn implies

$$r \left(B(x; (1 + \mu)r) \cap B(x; k(1 + \mu)r) \right) \leq \alpha r. \quad \square$$

In view of the Lifšic theorem we have the following result.

Theorem 6. *Let (X, d) be a bounded complete CAT(0) space. Then every uniformly k -lipschitzian mapping $T : X \rightarrow X$ with $k < \sqrt{2}$ has a fixed point.*

The case when X is an \mathbb{R} -tree is moot because every bounded (indeed every geodesically bounded) complete \mathbb{R} -tree has the fixed point property for continuous maps. This fact is a consequence of results of G.S. Young [13, cf. Theorem 16]. For a direct proof, see [9].

Remark 1. It seems reasonable to conjecture that the Lifšic characteristic of a $\text{CAT}(\kappa)$ space for $\kappa < 0$ is a continuous increasing function of κ which takes values in the interval $(\sqrt{2}, 2)$.

Remark 2. If $T : X \rightarrow X$ is uniformly k -lipschitzian, then T is nonexpansive relative to a metric r on X that satisfies

$$d(x, y) \leq r(x, y) \leq kd(x, y).$$

Also, if $T : X \rightarrow X$ is nonexpansive relative to a metric s on X with

$$\alpha d(x, y) \leq s(x, y) \leq \beta d(x, y),$$

then T is uniformly $\frac{\beta}{\alpha}$ -lipschitzian on (X, d) . (For the details, see [5].) While these observations might seem interesting, their usefulness in this context is mitigated by the fact that the $\text{CAT}(\kappa)$ inequality is not necessarily preserved under small perturbations of the metric.

4. $\text{CAT}(0)$ spaces and property (P)

In this section we show that every complete $\text{CAT}(0)$ space has property (P) . Let $\{x_n\}$ be a bounded sequence in a complete $\text{CAT}(0)$ space X and let K be a closed and convex subset of X . Define $\varphi : X \rightarrow \mathbb{R}$ by setting $\varphi(x) = \limsup_{n \rightarrow \infty} d(x, x_n)$, $x \in X$.

Proposition 7. *There exists a unique point $u \in K$ such that*

$$\varphi(u) = \inf_{x \in K} \varphi(x).$$

Proof. Let $r = \inf_{x \in K} \varphi(x)$ and let $\varepsilon > 0$. Then by assumption there exists $x \in K$ such that $\varphi(x) < r + \varepsilon$; thus for n sufficiently large $d(x, x_n) < r + \varepsilon$, i.e., for n sufficiently large $x \in B(x_n; r + \varepsilon)$. Thus

$$C_\varepsilon := \bigcup_{k=1}^\infty \left(\bigcap_{i=k}^\infty B(x_i; r + \varepsilon) \cap K \right) \neq \emptyset.$$

As the ascending union of convex sets, clearly C_ε is convex. Also the closure $\overline{C_\varepsilon}$ of C_ε is also convex (see [1, Proposition 1.4(1)]). Therefore

$$C := \bigcap_{\varepsilon > 0} \overline{C_\varepsilon} \neq \emptyset.$$

Clearly for $u \in C$, $\varphi(u) \leq r$. Uniqueness of such a u follows from the CN inequality (2.2). Specifically, suppose $u, v \in C$ with $u \neq v$. Then if m is the midpoint of the geodesic joining u and v ,

$$d(m, x_n)^2 \leq \frac{d(u, x_n)^2 + d(v, x_n)^2}{2} - \frac{1}{4}d(u, v)^2.$$

This implies $\varphi(m)^2 \leq r^2 - \frac{1}{4}d(u, v)^2$ — a contradiction. \square

In view of the above, X has property (P) if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X , there exists $z \in \bigcap_{n=1}^{\infty} \text{cov}\{z_j : j \geq n\}$ such that

$$\varphi(z) \leq \limsup_{j \rightarrow \infty} \varphi(z_j),$$

where φ is defined as above.

Theorem 8. A complete CAT(0) space (X, d) has property (P).

Proof. Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in X and define $\varphi(x) = \limsup_{n \rightarrow \infty} d(x, x_n)$, $x \in X$. For each n , let

$$C_n := \text{cov}\{z_j : j \geq n\}.$$

By Proposition 7 there exists a unique point $u_n \in C_n$ such that

$$\varphi(u_n) = \inf_{x \in C_n} \varphi(x).$$

Moreover, since $z_j \in C_n$ for $j \geq n$, $\varphi(u_n) \leq \varphi(z_j)$ for all $j \geq n$. Thus $\varphi(u_n) \leq \limsup_{j \rightarrow \infty} \varphi(z_j)$ for all n . We assert that $\{u_n\}$ is a Cauchy sequence. To see this, suppose not. Then there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $i, j \geq N$ such that $d(u_i, u_j) \geq \varepsilon$. Also, since the sets $\{C_n\}$ are descending, the sequence $\{\varphi(u_n)\}$ is increasing. Let $d := \lim_{n \rightarrow \infty} \varphi(u_n)$. Choose $\xi > 0$ so small that $\frac{2d\xi + \xi^2}{2} < \varepsilon/8$, and choose N so large that $|\varphi(u_i) - \varphi(u_j)| \leq \xi$ if $i, j \geq N$. Now choose $i > j \geq N$ so that $d(u_i, u_j) \geq \varepsilon$, let m_j denote the midpoint of the geodesic joining u_i and u_j , and let $n \in \mathbb{N}$. Then by the (CN) inequality

$$d(m_j, x_n)^2 \leq \frac{d(u_i, x_n)^2 + d(u_j, x_n)^2}{2} - \frac{\varepsilon}{4}.$$

This implies

$$\begin{aligned} \varphi(m_j)^2 &\leq \frac{\varphi(u_i)^2 + \varphi(u_j)^2}{2} - \frac{\varepsilon}{4} \\ &\leq \frac{(\varphi(u_j) + \xi)^2 + \varphi(u_j)^2}{2} - \frac{\varepsilon}{4} \\ &= \varphi(u_j)^2 + \frac{2\varphi(u_j)\xi + \xi^2}{2} - \frac{\varepsilon}{4} \\ &< \varphi(u_j)^2 - \frac{\varepsilon}{8}. \end{aligned}$$

Since $m_j \in C_j$, this contradicts the definition of u_j .

This proves that $\{u_n\}$ is a Cauchy sequence. Consequently there exists $z \in \bigcap_{n=1}^{\infty} C_n$ such that $\lim_{n \rightarrow \infty} u_n = z$ and, since φ is continuous, $\lim_{n \rightarrow \infty} \varphi(u_n) = \varphi(z)$. Since $\varphi(u_n) \leq \limsup_{j \rightarrow \infty} \varphi(z_j)$ for all n , we conclude that

$$\varphi(z) \leq \limsup_{j \rightarrow \infty} \varphi(z_j). \quad \square$$

5. Hyperconvex spaces

Since hyperconvex metric spaces have uniform normal structure, it is a consequence of Theorem 2 that if M is a bounded hyperconvex metric space with property (P), then every

uniformly k -lipschitzian $T : M \rightarrow M$ has a fixed point for $k < \sqrt{2}$. Here, by embedding the problem in a larger space, we show that uniformly lipschitzian mappings have fixed points under an assumption that appears to be weaker than property (P). Consequently we recover the Lim and Xu result in a hyperconvex setting.

Every metric space (X, d) can be embedded isometrically into a hyperconvex space. To see this let

$$\ell_\infty(X) = \left\{ \{m_x\}_{x \in X} : m_x \in \mathbb{R} \text{ for all } x \text{ and } \sup_{x \in X} |m_x| < \infty \right\}.$$

Define the distance d_∞ on $\ell_\infty(X)$ by $d_\infty(\{m_x\}, \{n_x\}) = \sup_{x \in X} |m_x - n_x|$. Thus the metric space $(\ell_\infty(X), d_\infty)$ is hyperconvex. Fix $a \in X$ and consider the map $I : X \rightarrow \ell_\infty(X)$ defined by $I(b) = \{d(b, x) - d(a, x)\}_{x \in X}$. It is easy to see that I is an isometry.

For a nonempty subset D of a bounded hyperconvex metric space (X, d) , it is known that $r(D) = \frac{1}{2}\delta(D)$ and

$$C(D) = C(\text{cov}(D)) = \bigcap_{x \in \text{cov}(D)} B(x; r(D)) \neq \emptyset$$

(see [4] for details).

Let $H = (H, d)$ be a bounded hyperconvex space. Embed H into $\ell_\infty(H)$ isometrically via the mapping $h \mapsto \{d(h, p) - d(a, p)\}_{p \in H}$, where a is a fixed element of H . Write $h_p = d(h, p) - d(a, p)$ for each $h, p \in H$. Let $R : \ell_\infty(H) \rightarrow H$ be a nonexpansive retraction.

Now let $\{e_n\}$ be the standard basis in the classical ℓ_∞ space, i.e., $\{e_n\} = \{\delta_{nj}\}_{j \in \mathbb{N}}$ where δ_{nj} is the Kronecker delta. Observe that $\bigcap_{n \geq 1} \text{cov}(\{e_j : j \geq n\}) = \{0\}$,

$$\limsup_n e_{nj} - \liminf_n e_{nj} < \varepsilon \text{ for all } j,$$

and

$$\limsup_n \sup_j |e_{nj} - 0| = 1 > 1 - \varepsilon \quad \text{for all } \varepsilon \in (0, 1).$$

Following this observation, we say that a sequence $\{x_n\}$ in H is a copy of $\{e_n\}$ if for each $\varepsilon \in (0, 1)$, there exists a sequence $\{p_n\}$ in H with $\limsup_n x_{np_j} - \liminf_n x_{np_j} \leq \varepsilon \delta(\{x_n\})$ for all j , and $\limsup_n \sup_j |x_{np_j} - z_{p_j}| > (1 - \varepsilon)\delta(\{x_n\})$ for all $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$.

Thus, if H does not contain a copy of $\{e_n\}$, given $\{x_n\}$ in H there exists $\varepsilon \in (0, 1)$, depending on $\{x_n\}$, such that if $\{p_n\}$ is any sequence in H for which

$$\limsup_n x_{np_j} - \liminf_n x_{np_j} < \varepsilon \delta(\{x_n\}) \text{ for all } j,$$

then there exists $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$ for which

$$\limsup_n \sup_j |x_{np_j} - z_{p_j}| \leq (1 - \varepsilon)\delta(\{x_n\}).$$

This prompts the following definition. We say that H has property (P_ε) if H does not contain a copy of $\{e_n\}$ in the following uniform sense: there exists an $\varepsilon \in (0, 1)$ such that, for a sequence $\{x_n\}$ and a collection $\{p_\lambda\}_{\lambda \in \Lambda}$ of points in H with $\limsup_n x_{np_\lambda} - \liminf_n x_{np_\lambda} < \varepsilon \delta(\{x_n\})$ for all λ , there exists $z \in \bigcap_{n \geq 1} \text{cov}(\{x_j : j \geq n\})$ satisfying

$$\limsup_n \sup_\lambda |x_{np_\lambda} - z_{p_\lambda}| \leq (1 - \varepsilon)\delta(\{x_n\}). \tag{5.1}$$

In proving the theorem of [12], Lim and Xu use property (P) to construct a sequence $\{x_n\}$ in X satisfying for each integer $j \geq 0$,

$$\limsup_n d(x_{j+1}, T^n x_j) \leq \tilde{c}\delta(\{T^n x_j\}), \tag{5.2}$$

$$d(x_{j+1}, y) \leq \limsup_n d(T^n x_j, y), \quad \text{for all } y \in X. \tag{5.3}$$

In the proof to follow, we shall construct such a sequence using property (P_ε) .

We first observe that if a hyperconvex space H has property (P), then it has property $(P_{1/2})$. To see this we let $\{x_n\}$ and $\{p_n\}$ be any sequences in H . Let $z_n \in C(\{x_j : j \geq n\})$. By property (P), we can take a point $z \in \bigcap_{n \geq 1} \text{cov}(\{z_j : j \geq n\})$ such that

$$\limsup_n d(z, x_n) \leq \limsup_j \limsup_n d(z_j, x_n) \leq \frac{1}{2}\delta(\{x_n\}). \tag{5.4}$$

Clearly,

$$\limsup_n \sup_j |x_{np_j} - z_{p_j}| \leq \limsup_n d(x_n, z) \leq \frac{1}{2}\delta(\{x_n\}).$$

We do not know whether the converse is true, that is, whether property $(P_{1/2})$ implies property (P). Indeed, if $\varepsilon' < \varepsilon$ then $P_\varepsilon \Rightarrow P_{\varepsilon'}$, but we see no reason why the reverse implication should be true.

Theorem 9. *Let H be a bounded hyperconvex metric space which has property (P_ε) for $\varepsilon \in (0, \frac{1}{2}]$. Then every uniformly k -lipschitzian mapping $T : H \rightarrow H$ with $k < \sqrt{\frac{1}{1-\varepsilon}}$ has a fixed point.*

Proof. We embed H in $\ell_\infty(X)$ as described above. Choose $\varepsilon' \in (0, \varepsilon)$ so that $k < \sqrt{\frac{1}{1-\varepsilon'}}$. Fix $x_0 \in H$ and consider the sequence $\{T^n x_0\}$. Define sets $A_n = \{T^j x_0 : j \geq n\}$ and $\mathcal{C} = \{p \in H : \limsup_n \{T^n x_0\}_p - \liminf_n \{T^n x_0\}_p < \varepsilon\delta(A_1)\}$. By (5.1) we see that with some $z \in \bigcap_{n \geq 1} \text{cov}(A_n)$,

$$\limsup_n \sup_{p \in \mathcal{C}} |\{T^n x_0\}_p - z_p| \leq (1 - \varepsilon)\delta(A_1). \tag{5.5}$$

Thus, since $0 < \varepsilon' < \varepsilon$, for all large n , we have

$$\sup_{p \in \mathcal{C}} |z_p - \{T^n x_0\}_p| < (1 - \varepsilon')\delta(A_1). \tag{5.6}$$

Let $p \in \mathcal{C}'$, where \mathcal{C}' is the complement of \mathcal{C} . Write

$$\begin{aligned} a &= \inf\{T^n x_0\}_p; & b &= \liminf_n \{T^n x_0\}_p; \\ c &= \limsup_n \{T^n x_0\}_p; & d &= \sup\{T^n x_0\}_p. \end{aligned}$$

Observe that we either have

$$a \leq d - \varepsilon\delta(A_1) \leq a + \varepsilon\delta(A_1) \leq d \quad \text{or} \quad a + \varepsilon\delta(A_1) < d - \varepsilon\delta(A_1),$$

from which we respectively have

$$[b, c] \cap [d - \varepsilon\delta(A_1), a + \varepsilon\delta(A_1)] \neq \emptyset \quad \text{or} \quad [b, c] \cap [a + \varepsilon\delta(A_1), d - \varepsilon\delta(A_1)] \neq \emptyset.$$

In either case, we can find a point $w_p \in [b, c]$ such that

$$|a - w_p| \leq \varepsilon\delta(A_1) \quad \text{and} \quad |d - w_p| \leq \varepsilon\delta(A_1) \quad \text{or} \\ |a - w_p| \leq (1 - \varepsilon)\delta(A_1) \quad \text{and} \quad |d - w_p| \leq (1 - \varepsilon)\delta(A_1).$$

Let $w = (w_p)_p$ where $w_p = z_p$ for $p \in \mathcal{C}$. Finally, let $x_1 = R(w)$, the image of w under the retraction R .

As $\varepsilon \leq \frac{1}{2}$, we have $\varepsilon \leq (1 - \varepsilon)$. Thus

$$|\sup_n \{T^n x_0\}_p - w_p| \leq (1 - \varepsilon)\delta(A_1) \\ |w_p - \inf_n \{T^n x_0\}_p| \leq (1 - \varepsilon)\delta(A_1).$$

So, for each j ,

$$|w_p - \{T^j x_0\}_p| \leq \max \left\{ \sup_n \{T^n x_0\}_p - w_p, w_p - \inf_n \{T^n x_0\}_p \right\} \leq (1 - \varepsilon)\delta(A_1),$$

and thus

$$\sup_{p \in \mathcal{C}'} |w_p - \{T^n x_0\}_p| \leq (1 - \varepsilon)\delta(A_1) \quad \text{for each } n. \tag{5.7}$$

Finally, (5.6) and (5.7) imply for $j \geq n$ where n is sufficiently large,

$$d(x_1, T^j x_0) = \sup_{p \in H} |x_{1p} - \{T^j x_0\}_p| \leq \sup_{p \in H} |w_p - \{T^j x_0\}_p| \\ \leq (1 - \varepsilon')\delta(A_1).$$

Consequently,

$$\limsup_n d(x_1, T^n x_0) \leq (1 - \varepsilon')\delta(A_1). \tag{5.8}$$

Consider a ball $B(y, r_y(A_n))$ in H . Since $|w_p - y_p| \leq r_y(A_n)$ for all p , bearing in mind that $z \in \text{cov}(A_n)$, we thus have

$$d(x_1, y) = d(R(w), R(y)) \leq d(w, y) \leq r(y, A_n).$$

This inequality holds for each y , therefore $x_1 \in \bigcap \text{cov}(A_n)$, and so

$$d(x_1, y) \leq \limsup_n d(T^n x_0, y) \quad \text{for all } y \in H. \tag{5.9}$$

By induction we can obtain a sequence $\{x_n\}$ in H satisfying (5.2) and (5.3). The proof now can be completed as in the proof of Theorem 7 in [12]. \square

Since hyperconvex spaces satisfy property $(P_{1/2})$, we have the following.

Corollary 10. *Let H be a bounded hyperconvex space which has property (P) . Then every uniformly k -lipschitzian mapping $T : H \rightarrow H$ with $k < \sqrt{2}$ has a fixed point.*

Corollary 11. *Let $T : H \rightarrow H$ be a uniformly k -lipschitzian mapping. Suppose that each orbit $\{T^n x\}_n$ of T is not a copy of $\{e_n\}$, i.e., all orbits $\{\{T^n x\}_n : x \in H\}$ satisfy (5.1) for some $\varepsilon \in (0, 1)$. If $k < \sqrt{\frac{1}{1-\varepsilon}}$, then T has a fixed point.*

Remark. In [7] it is shown that if M is bounded, hyperconvex, and satisfies property (P) , then every left reversible totally ordered uniformly k -lipschitzian semigroup of self-mapping of M has a common fixed point for $k < \sqrt{2}$. Extensions of the results of Lifšic and Lim–Xu to lipschitzian semigroups are given in [3].

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