

GEOMETRIC CONDITIONS SUFFICIENT FOR THE WEAK AND WEAK* FIXED POINT PROPERTY

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ABSTRACT

We look for geometric structures which underlie both the 'classical' geometric and more modern 'order theoretic' conditions for a Banach space (dual space) to have the weak (weak*) fixed point property; that is, for every nonexpansive self-mapping of a nonempty weak (weak*) compact convex subset to have a fixed point.

1. Introduction

Throughout X will denote a Banach space, and X^* its dual space. A mapping T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all x, y in its domain. We say that X (X^*) has the *weak-fixed point property*, w-fpp, (*weak*-fixed point property*, w*-fpp) if every nonexpansive self mapping of a nonempty weak (weak*) compact convex subset of X (X^*) has a fixed point.

The more *classical* sufficient conditions for the w (and w*)-fpp were of a 'geometric nature' while more recent results could be best described as 'order theoretic' in which the natural order resulting from a lattice or a basis structure have been exploited. After briefly surveying these two aspects of the theory we look for underlying commonalities and generalizations.

The opening gambit for almost all known existence results for fixed points of nonexpansive maps has been the following.

In order to work toward a contradiction we suppose that C is a nonempty weak (weak*) compact convex set and that $T : C \rightarrow C$ is a fixed point free nonexpansive mapping. By the weak (weak*) compactness and Zorn's lemma we may, and henceforth will, suppose that C is *minimal* in the sense that no proper weak (weak*) compact convex subset of C is invariant under T . Further, since C contains more than one point we may, by a dilation and redefinition of T if necessary, suppose

that

$$\text{diam } C := \sup\{\|x - y\| : x, y \in C\} = 1.$$

A basic result, due to Brodskii - Mil'man³, Garkarvi⁹ and Kirk¹⁹ is that such a 'minimal invariant' set C is *diametral* in the sense that the *radius function* is constant on C ; that is,

$$\begin{aligned} \rho(x) &:= \sup_{y \in C} \|x - y\| \\ &= \text{diam } C, \quad \text{for all } x \in C. \end{aligned}$$

Thus a space will enjoy the w-fpp (w^* -fpp) if it is known not to contain any non-trivial weak (w^*) compact convex diametral subsets. A space with this property is said to have *weak (w^*) normal structure*. While an excessively strong condition for the existence of fixed points, normal structure has become an important Banach space property and 'geometric' conditions implying it have been the subject of considerable study.

2. Geometric conditions

One of the earliest results was the observation that nontrivial diametral sets cannot live in *uniformly convex spaces*; that is, spaces in which the *modulus of convexity*

$$\delta(\epsilon) := \inf\{1 - \|\frac{x+y}{2}\| : \|x\|, \|y\| \leq 1, \text{ and } \|x - y\| \geq \epsilon\}$$

is strictly positive on $(0, 2]$. Thus such spaces have the fpp. More generally⁶ spaces which are ϵ_0 -*inquadrate*; that is $\delta(\epsilon_0) \geq 0$, for some $\epsilon \in (0, 1)$ have uniform normal structure and hence the fpp. It follows from Turett³⁰ that the duals of such spaces (which are necessarily reflexive) also have uniform normal structure. Recall, a Banach space has uniform normal structure if

$$\sup\{\text{rad } C : C \text{ is closed, bounded convex with } \text{diam } C = 1\} < 1,$$

where $\text{rad } C := \inf\{\rho(x) : x \in C\}$.

Garkavi⁹ has characterized spaces for which the radius function of every closed bounded convex set achieves its minimum at most once in the set as those spaces which are *uniformly convex in every direction*; that is, for each $z \in X$, whenever $\|x_n\|, \|x_n + \lambda_n z\| \leq 1$ and $\|x_n + (\lambda_n/2)z\| \rightarrow 1$ we have $\lambda_n \rightarrow 0$. So, such spaces have normal structure and hence the w-fpp.

The presence of a nontrivial diametral set can be refined as follows.

If C is a weak compact convex diametral subset with $\text{diam } C = 1$, starting with any choice for $x_1 \in C$ we may inductively construct a sequence by choosing $x_{n+1} \in C$ so that $\|x_{n+1} - \frac{1}{n} \sum_{k=1}^n x_k\| > 1 - \frac{1}{n}$. It follows that (x_n) satisfies

$$\text{dist}(x_{n+1}, \text{co}\{x_1, \dots, x_n\}) \rightarrow 1.$$

Since this property is retained by subsequences we can suppose that (x_n) is weak convergent and, by a translation if necessary, that

$$x_n \xrightarrow{w} 0.$$

Defining $K := \overline{\text{co}}\{x_n : n \in \mathbb{N}\}$ we see that X contains a closed convex diametral set for which (by Mazur's theorem) the sequence $(x_n) \subset K$ is *diameterizing* in the sense that

$$\lim_n \|x - x_n\| = \text{diam } K = 1,$$

in particular, since $0 \in K$ (again by Mazur's theorem), we have

$$\lim_n \|x_n\| = 1.$$

This shows that a space has weak normal structure, and hence the w -fpp, whenever it is ϵ_0 -weak uniformly Radon-Riesz (uRR) for some $\epsilon_0 \in (0, 1)$; that is, there exists $\delta > 0$ so that whenever W is a weak compact convex subset of the unit ball with $\gamma(W) \geq \epsilon_0$ we have $\min\{\|x\| : x \in W\} \leq 1 - \delta$, where $\gamma(W)$ is the index of noncompactness defined by

$$\gamma(W) := \sup\{\text{sep}(x_n) : (x_n) \subset W\} \quad \text{and} \quad \text{sep}(x_n) := \inf_{n \neq m} \|x_n - x_m\|.$$

Equivalently X is ϵ_0 -weak uRR if there exists $\delta > 0$ so that whenever $x_n \xrightarrow{w} x_\infty$, $\|x_n\| \rightarrow 1$, and $\text{sep}(x_n) \geq \epsilon_0$ we have $\|x_\infty\| \leq 1 - \delta$.

Spaces satisfying uRR conditions were introduced by Huff¹³ and studied in the context of the fpp by van Dulst and Sims⁵, and independently by Goebel and Sekowski¹¹.

The appeals to Mazur's theorem prevent our using the above arguments in the weak* case. None-the-less it is shown in van Dulst and Sims⁵ that if X^* is ϵ_0 -weak* uRR for some $\epsilon_0 \in (0, 1)$ then X^* has weak* normal structure and hence the w^* -fpp. Alternatively, at least in the case when C is separable and weak*

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sequentially compact, we can use the following device due to Lennard²⁰. Modify the construction of (x_n) by choosing x_{n+1} from C so that

$$\|x_{n+1} - \frac{1}{n} \sum_{k=1}^n x_k\|, \|x_{n+1} - y_1\|, \dots, \|x_{n+1} - y_n\| > 1 - \frac{1}{n},$$

where (y_n) is a dense sequence in C . Then extracting a weak* convergent subsequence and translating if necessary so that $x_n \xrightarrow{w^*} 0$ we see that (x_n) is diametrizing for $K := \overline{\text{co}}^{w^*} \{x_n : n \in \mathbb{N}\}$ with the properties required above.

Simple examples show that the properties considered in this section; inquadrate, UCED, and uRR, are in general mutually independent.

We now consider another, slightly less geometrical(?) and perhaps the most intriguing, 'classical' condition sufficient for the w-fpp.

3. Opial conditions

Given a bounded sequence (x_n) we define

$$\psi_{(x_n)}(x) := \limsup_n \|x - x_n\|.$$

A Banach space X (dual space X^*) is said to have the weak (weak*) Opial condition if whenever $x_n \xrightarrow{w} x_\infty$ ($x_n \xrightarrow{w^*} x_\infty$) we have

$$\psi_{(x_n)}(x_\infty) < \psi_{(x_n)}(x), \quad \text{for all } x \neq x_\infty.$$

By playing with subsequences it is readily seen that in the weak case this is equivalent to the original definition, introduced by Opial²³ in 1967, that $\liminf_n \|x_\infty - x_n\| < \liminf_n \|x - x_n\|$ whenever $x_n \xrightarrow{w} x_\infty$ and $x \neq x_\infty$. When the above inequalities are not required to be strict we shall refer to non-strict Opial conditions.

For (x_n) a diametrizing sequence for K as constructed in section 2 we see that $\psi_{(x_n)}(x) = \text{diam } K = 1$, for all $x \in K$ including $0 = w(w^*)\text{-}\lim_n x_n$. So spaces with the weak Opial condition have weak normal structure¹² and hence the w-fpp. Similarly dual spaces with the weak* Opial condition can contain no nontrivial separable weak* sequentially compact convex diametral sets, in particular separable duals with the weak* Opial condition have weak* normal structure and hence the w*-fpp. Using a simple argument due to van Dulst⁴ it is possible to say more.

$UC \Rightarrow URR$
 $w \text{ and } w^* \Rightarrow \text{w-fpp}$

We begin with the well known observation that a nonexpansive self mapping, T , of a bounded convex set admits an approximate fixed point sequence; that is, a sequence (a_n) in its domain, D , with $\|a_n - Ta_n\| \rightarrow 0$. Clearly any subsequence of an approximate fixed point sequence is itself an approximate fixed point sequence. [To construct such a sequence it suffices to use a density argument combined with the Banach contraction mapping principle to obtain a fixed point of the strict contraction $V_n x := (1 - \frac{1}{n})Tx + \frac{1}{n}x_0$, extended to the closure of D , where x_0 is any chosen point of D .]

Now suppose T is a nonexpansive self mapping of a weak (weak*) compact convex set and (a_n) is an approximate fixed point sequence for T with $a_n \xrightarrow{w(w^*)} a_\infty$, then

$$\begin{aligned} \psi_{(a_n)}(Ta_\infty) &= \limsup_n \|Ta_\infty - a_n\| \\ &= \limsup_n \|Ta_\infty - Ta_n\| \\ &\leq \limsup_n \|a_\infty - a_n\| = \psi_{(a_n)}(a_\infty), \end{aligned}$$

contradicting the weak (weak*) Opial condition unless $Ta_\infty = a_\infty$. Thus, spaces with the weak (weak*) Opial condition have the w-fpp (w*-fpp, provided weak* compactness is sequential) and any weak (weak*) limit of an approximate fixed point sequence for a nonexpansive map T is a fixed point of T .

While establishing weak normal structure for the James' space J in its 'isometric norm' Tingley²⁹ was led to consider the following weakened form of Opial condition.

We say that a Banach space X satisfies the *weakened Opial condition* (WO) if whenever $x_n \xrightarrow{w} x_\infty$ we have

$$\psi_{(x_n)}(x_\infty) < \sup \psi_{(x_n)}(\overline{\text{co}}\{x_n\}_1^\infty).$$

By the same argument as that used above for the weak Opial condition we see that spaces with WO have weak normal structure and hence the w-fpp²⁹.

LEMMA: The following are equivalent.

- (1) X has WO,
- (2) For $x_n \xrightarrow{w} 0$ and $\|x_n\| \rightarrow 1$ we have

$$\sup \psi_{(x_n)}(\overline{\text{co}}\{x_n\}) > 1.$$

- (3) There exists $x_0 \in \overline{\text{co}}\{x_n\}$ with $\psi_{(x_n)}(x_0) > 1$.

(4) There exists m with $\psi_{(x_n)}(x_m) > 1$.

[Essentially, the extreme points of $\overline{\text{co}}\{x_n\}$ are x_m 's or 0.]

(5) ~~If~~ $\sup_m \psi_{(x_n)}(x_m) > 1$.

[(1) – (4) remain true when we take any subsequence of (x_n) . Note: we always have a non-strict inequality in (4) from the nature of (x_n) .]

Perhaps somewhat surprisingly WO not only underpins the weak Opial condition, but also ϵ_0 -uRR.

PROPOSITION: If X is ϵ_0 -weak uRR for some $\epsilon_0 \in (0, 1)$, then X has WO.

This shows that WO is genuinely weaker than the Opial condition. Indeed $\mathcal{L}_p[0, 1]$ for $1 < p < \infty$, $p \neq 2$ has WO, but does not satisfy the weak Opial condition.

Proof. Let $\delta (> 0)$ be that from the definition of ϵ_0 -uRR, and suppose that X fails WO; that is, there exists $x_n \xrightarrow{w} 0$ with $\|x_n\| \rightarrow 1$ and $\limsup_n \|x_m - x_n\| = 1$, for all m .

We may choose m_0 so that $\|x_{m_0}\| > 1 - \delta$, and a subsequence (x_{n_k}) with $\|x_{m_0} - x_{n_k}\| \rightarrow 1$ and $\|x_{n_k} - x_{n'_k}\| > \epsilon_0$, for all $k \neq k'$.

Setting $y_k := x_{m_0} - x_{n_k}$ we have;

$$\begin{aligned} y_k &\xrightarrow{w} x_{m_0}, \\ \|x_{m_0}\| &> 1 - \delta, \\ \text{sep}(y_k) &> \epsilon_0, \end{aligned}$$

and

$$\|y_k\| \rightarrow 1.$$

Contradicting ϵ_0 -weak uRR. ■

Analogous results are true for the weak* version of WO in X^* , provided weak* compactness is sequential, and C is separable.

Before leaving this section we state one other result concerning types, $\psi_{(x_n)}(x)$, which has been basic to many of the more recent results; *viz*,

KARLOVITZ' LEMMA^{15,10}: If (a_n) is an approximate fixed point sequence for the nonexpansive map T of the 'minimal' domain C into itself, then

$$\psi_{(a_n)}(x) = \text{diam } C, \quad \text{for all } x \in C.$$

indeed, by considering subsequences,

$$\lim_n \|x - a_n\| = \text{diam } C, \quad \text{for all } x \in C.$$

Proofs of this result have directly, or indirectly, relied on the weak lower semi-continuity of $\psi_{(a_n)}$, and so only apply in the weak* case when such functions are weak* lower semi-continuous. While this is not generally valid it is none-the-less true in some useful classes of spaces^{17, 26}.

4. Order theoretic results

There are many such results dating from the work of Sine²⁷, and Soardi²⁸, but more fundamentally Maurey²². See for example Elton *et al*⁷, Borwein and Sims², Lin²¹, Sims²⁵, Khamsi and Turpin¹⁸, and the book by Aksoy and Khamsi¹. We will illustrate by presenting just one such result²⁵.

It is convenient to work in the space

$$\tilde{X} := \frac{\ell_\infty(X)}{c_0(x)},$$

with elements denoted by $\tilde{x} = [x_n]$ and the quotient norm given canonically by $\|[x_n]\| = \limsup_n \|x_n\|$.

Alternatively, in all that follows, we could take \tilde{X} to be the Banach space *ultra-power* of X with respect to some nontrivial ultrafilter U over \mathbb{N} . That is²⁴,

$$\tilde{X} := (X)_U := \ell_\infty(X) / \{(x_n) : x_n \in X, \lim_U \|x_n\| = 0\},$$

where the quotient norm is given by $\|[x_n]_U\| = \lim_U \|x_n\|$.

The mapping $J : X \rightarrow \tilde{X} : x \mapsto [x, x, x, \dots]$ is a natural isometric embedding of X into \tilde{X} .

Let

$$\tilde{C} := \{[x_n] : x_n \in C\},$$

and define

$$\tilde{T} : \tilde{C} \rightarrow \tilde{C} : [x_n] \mapsto [Tx_n].$$

Then \tilde{C} is a closed convex subset of \tilde{X} containing JC , and \tilde{T} is a well defined nonexpansive self mapping of \tilde{C} .

Observe that if (a_n) is an approximate fixed point sequence for T in C then $\tilde{a} := [a_n]$ is a fixed point of \tilde{T} and by Karlovitz' lemma

$$\|\tilde{a} - Jx\| = \text{diam } \tilde{C} = \text{diam } C = 1,$$

for all $x \in C$. In particular $\|\tilde{a}\| = 1$.

Further, if \tilde{K} is any nonempty convex subset of \tilde{C} which is invariant under \tilde{T} , then \tilde{K} contains approximate fixed point sequences for \tilde{T} and Karlovitz' lemma combined with a diagonalization argument shows that $\sup\{\|\tilde{y} - Jx\| : \tilde{y} \in \tilde{K}\} = 1$, for all $x \in C$. In particular \tilde{K} contains elements of norm arbitrarily near to 1.

So far our construction has been completely general. We now restrict our attention to the case when X is a Banach lattice which is weakly orthogonal²; that is, whenever $x_n \xrightarrow{w} 0$ we have

$$\| |x_n| \wedge |x| \| \rightarrow 0, \quad \text{for all } x \in X.$$

The notion of weak orthogonality was introduced in order to generalize Maurey's proof²² of the w-fpp for c_0 to a larger class of Banach lattices by Borwein and Sims² where it was shown that a wide class of 'sequential' lattices are weakly orthogonal; including $c_0(\Gamma)$, in both its original and Day's *lur* norm, $\ell_p(\Gamma)$, $1 \leq p < \infty$, but not the spaces $\mathcal{L}_p[0, 1]$ for $p \neq 2$. It was also shown that any weakly orthogonal Banach lattice with a Riesz angle $\alpha(X) := \sup\{\| |x| \vee |y| \| : \|x\|, \|y\| \leq 1\} < 2$ has the w-fpp. By adapting an argument of Lin²¹, it follows²⁵ that the assumption on Riesz angle is unnecessary.

THEOREM: Weakly orthogonal Banach lattices have the w-fpp.

Proof. Let (a_n) be an approximate fixed point sequence for T in the 'minimal' domain C . From the discussion above we may, by passing to subsequences if necessary, assume that

$$\begin{aligned} \|a_n\| &\rightarrow 1, \\ a_n &\xrightarrow{w} 0, \\ \|a_{n+1} - a_n\| &\rightarrow 1, \end{aligned}$$

and

$$\| |a_n| \wedge |a_{n+1}| \| \rightarrow 0.$$

Thus, defining $\tilde{a}_1 := [a_n]$ and $\tilde{a}_2 := [a_{n+1}]$, we have:

- i) $\|\tilde{a}_i - Jx\| = 1$ for all $x \in C$, and $i = 1, 2$. In particular $\|\tilde{a}_i\| = 1$, since $0 \in C$.

ii) $\|\tilde{a}_1 - \tilde{a}_2\| = 1$.

iii) For $i = 1, 2$ and $x \in C$ $|\tilde{a}_i| \wedge |Jx| = 0$, and $|\tilde{a}_1| \wedge |\tilde{a}_2| = 0$.

Let

$$\tilde{W} := \{\tilde{w} \in \tilde{C} : \text{for } i = 1, 2 \|\tilde{w} - \tilde{a}_i\| = 1/2 \text{ and } \text{dist}(\tilde{w}, JC) \leq 1/2\}.$$

Then \tilde{W} is a closed convex subset of \tilde{C} which is invariant under \tilde{T} and nonempty, since

$$\begin{aligned} \left\| \frac{\tilde{a}_1 + \tilde{a}_2}{2} - 0 \right\| &= \frac{1}{2} \|\tilde{a}_1 + \tilde{a}_2\| \\ &= \frac{1}{2} \|\tilde{a}_1 - \tilde{a}_2\|, \quad \text{as } |\tilde{a}_1| \wedge |\tilde{a}_2| = 0 \\ &= \frac{1}{2}, \end{aligned}$$

so $\frac{\tilde{a}_1 + \tilde{a}_2}{2} \in \tilde{W}$.

Hence, from above, \tilde{W} contains elements of norm arbitrarily near to 1.

To derive a contradiction, and establish the w-fpp, we pass to the countably order complete Banach lattice X^{**} where we may construct the principle band projections

$$P_{\tilde{a}_i} := \bigvee_{n=1}^{\infty} \left[(n|\tilde{a}_i|) \wedge \tilde{y}^+ \right] - \bigvee_{n=1}^{\infty} \left[(n|\tilde{a}_i|) \wedge \tilde{y}^- \right],$$

for $i = 1, 2$.

Then $P_{\tilde{a}_i}$, $I - P_{\tilde{a}_i}$ and $P_{\tilde{a}_1} + P_{\tilde{a}_2}$ are norm one projections with $P_{\tilde{a}_i}\tilde{a}_i = \tilde{a}_i$ and $P_{\tilde{a}_i}J = 0$. Thus for each $\tilde{w} \in \tilde{W}$, if $x \in C$ is such that $\|\tilde{w} - Jx\| \leq 1/2$, we have

$$\begin{aligned} \|\tilde{w}\| &= \frac{1}{2} \|(P_{\tilde{a}_1} + P_{\tilde{a}_2})(\tilde{w} - Jx) + (I - P_{\tilde{a}_1})(\tilde{w} - \tilde{a}_1) + (I - P_{\tilde{a}_2})(\tilde{w} - \tilde{a}_2)\| \\ &\leq \frac{1}{2} (\|\tilde{w} - Jx\| + \|\tilde{w} - \tilde{a}_1\| + \|\tilde{w} - \tilde{a}_2\|) \\ &\leq \frac{3}{4}. \end{aligned}$$

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As in Lin²¹ the discrepancy between 3/4 and 1 can be exploited to show that any Banach space whose Banach-Mazur distance from a weakly orthogonal Banach lattice is less than $(\sqrt{33} - 3)/2$ has the w-fpp.

The weak* analogue of the above theorem is also true, at least in duals where weak* compactness is sequential, since it is shown²⁶ that a weak* Karlovitz' lemma holds in weak* orthogonal dual lattices.

$$\begin{aligned} &1 + (\sqrt{33} - 3)/2 \\ &= \frac{\sqrt{33} + 1}{2} \end{aligned} \quad ?$$

5. Underlying commonalities

In \tilde{X} we define a subspace by

$$\mathcal{N} := \{[x_n] : x_n \xrightarrow{w} 0\}.$$

That X is a weakly orthogonal Banach lattice can now be interpreted as \mathcal{N} being lattice orthogonal to the natural embedding JX in \tilde{X} .

On the other hand, the weak lower semi-continuity of the norm implies that whenever $x_n \xrightarrow{w} 0$ we have for all $x \in X$ that

$$\begin{aligned} \|Jx\| &= \|x\| \\ &\leq \liminf_n \|x - x_n\| \\ &\leq \limsup_n \|x - x_n\| \\ &= \|Jx - [x_n]\|. \end{aligned}$$

That is, JX is orthogonal to \mathcal{N} in the sense of Birkhoff¹⁴.

By the construction in section 2, the failure of weak normal structure for X ensures the existence of norm one elements in X and \mathcal{N} for which the above inequality is in fact an equality. Thus spaces in which this is impossible have weak normal structure. This includes spaces which are ϵ_0 -inquadrate, for some $\epsilon_0 \in (0, 1)$, and weakly orthogonal Banach lattices in which for each norm one element x

$$\inf\{\|x + y\| : \|y\| = 1 \text{ and } |x| \wedge |y| = 0\} > 1$$

The weak Opial condition is precisely the requirement that

$$\|\tilde{n}\| < \|\tilde{n} - Jx\|, \quad \text{whenever } \tilde{n} \in \mathcal{N} \text{ and } 0 \neq x \in X.$$

So, spaces with the weak Opial condition are those for which in a strict sense \mathcal{N} and JX are symmetrically Birkhoff orthogonal. Such a connection with ‘symmetric orthogonality’ seems first to have been explored by Karlovitz¹⁶.

Similarly, WO corresponds to strict symmetric Birkhoff orthogonality between $[x_n]$ and at least one (in fact infinitely many) of the directions Jx_n .

Thus we see that ‘orthogonality conditions’ between elements of \mathcal{N} and JX lie at the heart of many fpp results.

A natural question is whether spaces satisfying the non-strict weak Opial condition; that is, spaces for which \mathcal{N} and JX are symmetrically Birkhoff orthogonal, have the

w-fpp. In an attempt to partially answer this I²⁵ asked whether the w-fpp holds for spaces with \mathcal{N} and JX orthogonal in the stronger sense of James¹⁴:

$$\|\tilde{n} + Jx\| = \|\tilde{n} - Jx\| \quad \text{for all } \tilde{n} \in \mathcal{N} \text{ and } x \in X.$$

Such spaces were said to have WORTH (weak orthogonality). Recently García Falset⁸ has shown that spaces with WORTH which are ϵ_0 -inquadrate for some $\epsilon_0 < 2$ have the w-fpp. This is somewhat reminiscent of the result for weakly orthogonal lattices with a Riesz angle $\alpha(X) < 2$, where the condition on Riesz angle was subsequently seen to be unnecessary.

6. References

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