Chapter 12

GEOMETRIC PROPERTIES RELATED TO FIXED POINT THEORY IN SOME BANACH FUNCTION LATTICES

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1. Introduction

The aim of this chapter is to present criteria for the most important geometric properties related to the metric fixed point theory in some classes of Banach function lattices, mainly in Orlicz spaces and Cesaro sequence spaces. We also give some informations about respective results for Musielak-Orlicz spaces, Orlicz-Lorentz spaces and Calderón-Lozanovsky spaces.

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1.1. Orlicz Spaces

Some general facts. We denote by $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{N}$ the sets of real numbers, nonnegative real numbers and natural numbers respectively.

A mapping $\Phi\colon \mathbf{R} \to \mathbf{R}_+$ is said to be an *Orlicz function* if:

(i) Φ is even, continuous, convex and vanishing only at zero,

(ii) $\lim_{u\to 0}(\frac{\Phi(u)}{u}) = 0$ and $\lim_{u\to\infty}(\frac{\Phi(u)}{u}) = \infty$.

An Orlicz function Φ is said to satisfy the Δ_2-condition at zero ($\Phi \in \Delta_2$ for short) if there are constants $K \geq 2$ and $a > 0$ such that $\Phi(a) > 0$ and $\Phi(2u) \leq K\Phi(u)$ for all real u with $|u| < a$.

It is well known (see [Lu 55], [Mal 89], [Mu 83] and [Ra-Re 91]) that Φ is an Orlicz function if and only if $\Phi(u) = \int_0^{|u|} p(t)dt$, where p is the right derivative of Φ satisfying the following conditions:

(iii) p is right-continuous and nondecreasing on $\mathbf{R}_+$,

(iv) $p(t) > 0$ whenever $t > 0$,

(v) $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$.

Hence it follows immediately that

$$\frac{1}{2}\,p\left(\frac{u}{2}\right) \leq \frac{\Phi(u)}{u} \leq p(u) \qquad (u > 0). \tag{1.1}$$

By the convexity of Φ and $\Phi(0) = 0$, we get

$$\Phi(\alpha u) < \alpha\Phi(u) \quad (0 < \alpha < 1,\ u > 0), \tag{1.2}$$

which yields

$$\frac{\Phi(u)}{u} < \frac{\Phi(v)}{v} \quad (0 < u < v).$$

For the function p satisfying conditions (iii), (iv) and (v), we define

$$q(s) = \sup\{t > 0\colon\ p(t) \leq s\} = \inf\{t > 0\colon\ p(t) > s\},$$

which we call the *right-inverse function of* p. It is easy to show that q also satisfies conditions (iii), (iv) and (v). If Φ is an Orlicz function with the right derivative p and q is the right-inverse function of p, then the function

$$\Psi(v) = \int_0^{|v|} q(s)ds$$

is called the *complementary function of* Φ (or the *Young conjugate of* Φ). It is well known (see [Ch 96], [Lu 55], [Mal 89], [Mu 83] and [Ra-Re 91]) that we have the Young inequality

$$uv \leq \Phi(u) + \Psi(v) \quad (u, v \geq 0)$$

and that the equality

$$uv = \Phi(u) + \Psi(v) \tag{1.3}$$

holds for $u \geq 0$ if and only if $v \in [p_-(u), p(u)]$, where p_- is the left derivative of Φ.

Sometimes Orlicz functions are defined only by condition (i). It is easy to see that $(\Phi(u)/u) \to 0$ as $u \to 0$ is equivalent to the fact that Ψ vanishes only at zero and $(\Phi(u)/u) \to \infty$ as $u \to \infty$ is equivalent to the fact that Ψ has only finite values.

Example 1.1 Let Φ be an Orlicz function. If Ψ_1 is the Young conjugate of the function Φ_1 defined on $\mathbf{R}$ by $\Phi_1(u) = a\Phi(bu)$, where a, b are fixed positive numbers and p is the right derivative of Φ on $\mathbf{R}_+$, then the right derivative of Φ_1, is $p_1(t) = abp(bt)$ and so its right-inverse function is

$$q_1(s) = \frac{1}{b}q\Big(\frac{1}{ab}s\Big),$$

where q is the right derivative of Ψ, and Ψ is the Young conjugate of Φ. Hence

$$\Psi_1(v) = \int_0^{|v|} q_1(s)ds = a\int_0^{\frac{|v|}{ab}} q(s)ds = a\Psi\left(\frac{|v|}{ab}\right).$$

Example 1.2 Let Ψ_1, Ψ_2 be the Young conjugates of Orlicz functions Φ_1 and Φ_2, respectively. Suppose that

$$\Phi_1(u) \le \Phi_2(u) \quad (u \ge u_0 \ge 0).$$

Consider the relationship between Ψ_1 and Ψ_2. By the Young inequality and equality (1.3), we have

$$\Phi_2(q_2(v)) + \Psi_2(v) = q_2(v)v \le \Phi_1(q_2(v)) + \Psi_1(v) \quad (v \ge 0).$$

Hence by

$$\Phi_2(q_2(v)) \ge \Phi_1(q_2(v)) \quad (q_2(v) \ge u_0),$$

we obtain

$$\Psi_2(v) \le \Psi_1(v) \quad (q_2(v) \ge u_0).$$

Let (T, Σ, μ) denote a nonatomic, complete and finite measure space and denote by $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) Σ-measurable real functions defined on T.

Given an Orlicz function Φ, we define on L^0 a *convex modular* I_Φ by

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu.$$

The Orlicz space L^Φ generated by Φ is the set of those $x \in L^0$ that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$. If l^0 is the space of all real sequences $x = (x(i))_{i=1}^\infty$, then the modular I_Φ is defined on l^0 by

$$I_\Phi(x) = \sum_{i=1}^\infty \Phi(x(i))$$

and the corresponding space $l^\Phi = \{x \in l^0\colon\ I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}$ is called the *Orlicz sequence space.* We also define

$$E^\Phi = \{x \in L^0\colon\ I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}, \text{ and}$$
$$h^\Phi = \{x \in l^0\colon\ I_\Phi(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

Lemma 1.3 *Let Φ be an Orlicz function and Ψ be its Young conjugate. Then the following are equivalent:*

(i) $\Phi \in \Delta_2$.

(ii) *There exist $l > 1, u_0 > 0$ and $K > 1$ such that*

$$\Phi(lu) \leq K\Phi(u) \quad (u > u_0). \tag{1.4}$$

(iii) *For any $l_1 > 1$ and $u_1 > 0$ there exists $K' > 0$ such that* (1.4) *holds for $l = l_1, u_0 = u_1$ and $K = K'$.*

(iv) *For any $l_2 > 1$ and $u_2 > 0$ there exists ε in the interval $(0,1)$ such that*

$$\Phi((1+\varepsilon)u) \leq l_2\Phi(u) \quad (u > u_2). \tag{1.5}$$

(v) *For any $l_3 > 1$ there exist $v_0 > 0$ and $\delta > 0$ such that*

$$\Psi(l_3 v) \geq (l_3 + \delta)\Psi(v) \quad (v \geq v_0). \tag{1.6}$$

(vi) *There exist $l_3 > 1$, $v_0 > 0$ and $\delta > 0$ such that* (1.6) holds.

Proof.

The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Given $l_1 > 1$, choose an integer α such that $l^\alpha > l_1$. Then by (1.4),

$$\Phi(l_1 u) \leq \Phi(l^\alpha u) \leq K^\alpha \Phi(u) \quad (u \geq u_0).$$

Hence, if $u_1 \geq u_0$, then K^α is a candidate for K'. If $u_1 < u_0$, then, we choose $K' = \max(K^\alpha, K_0)$, where

$$K_0 = \max\{\Phi(l_1 u)/\Phi(u)\colon\ u \in [u_1, u_0]\}.$$

(iii) $\Rightarrow$ (iv). For $l_2 > 1$ and $u_2 > 0$, by (iii), there exists $K' > l_2$ such that

$$\Phi(2u) \leq K'\Phi(u) \quad (u \geq u_2).$$

Take $\varepsilon = (l_2 - 1)/(K' - 1)$. Then $0 < \varepsilon < 1$ and by the convexity of Φ,

$$\begin{aligned}\Phi((1+\varepsilon)u) &= \Phi((1-\varepsilon)u + 2\varepsilon u) \leq (1-\varepsilon)\Phi(u) + \varepsilon\Phi(2u),\\ &\leq (1-\varepsilon)\Phi(u) + \varepsilon K'\Phi(u) = l_2\Phi(u) \quad (u \geq u_2).\end{aligned}$$

(iv) $\Rightarrow$ (v). For any $l_3 > 1$ and $v_0 > 0$, choose $u_2 \in (0, q(v_0)]$. Then inequality (1.5) and Examples 1.1 and 1.2 imply

$$\Psi(v) \leq \frac{1}{l_3}\Phi\Big(\frac{l_3}{1+\varepsilon}v\Big) \quad (v \geq v_0).$$

Hence it follows that

$$l_3\Psi(v) \leq \Psi\Big(\frac{l_3}{1+\varepsilon}v\Big) \leq \frac{1}{1+\varepsilon}\Psi(l_3(v)) \quad (v \geq v_0).$$

By setting $\delta = l_3\varepsilon$, we get (1.6).

The implication (v) $\Rightarrow$ (vi) is trivial.

(vi) $\Rightarrow$ (i). Let $\beta = (l_3 + \delta)/l_3$. Then (1.6) can be written in the form

$$\frac{1}{\beta l_3}\Psi(l_3 v) \geq \Psi(v) \quad (v \geq v_0).$$

Choose $n \in \mathbf{N}$ such that $\beta^n \geq 2$ and set $K = \beta^n l_3^n$. Then by Examples 1.1 and 1.2, we get

$$\Phi(2u) \leq \Phi(\beta^n u) \leq \beta^n l_3^n \Phi(u) = K\Phi(u) \quad (u \geq u_0).$$

■

2. Normal structure, weak normal structure, weak sum property, sum property and uniform normal structure

Let X be a Banach space and $\{x_n\}$ a sequence in X such that $x_i \neq x_j$ whenever $i \neq j$. If for any $x \in \operatorname{co}\{x_n\}$, the convex hull of $\{x_n\}$, the limit $\triangle(x) = \lim_n \|x - x_n\| > 0$ exists and $\triangle(x)$ is affine on $\operatorname{co}\{x_n\}$, then $\{x_n\}$ is called a *limit affine sequence.* If, in particular, $\triangle(x)$ is equal to a constant on $\operatorname{co}\{x_n\}$, then $\{x_n\}$ is called a *limit constant sequence.* If X contains no (weakly convergent) limit affine sequence $\{x_n\}$ satisfying $\triangle(x_n) \uparrow$, then it is said to have the (*weak*) *sum-property.* We say that X has (*weak*) *normal structure* (NS (WNS)) if it contains no (weakly) convergent limit constant sequence.

The original definition of the (weak) normal structure is given in the following equivalent way:

X has (weak) normal structure if for any nonsingleton (weakly compact) nonempty bounded closed convex subset C of X, there exists $x \in C$ such that

$$r_c(x) := \sup\{\|x - y\| : \ y \in C\} < \operatorname{diam}(C) := \sup\{\|u - v\| : \ u, v \in C\}.$$

Moreover, if there exists $h \in (0,1)$ such that for each nonsingleton nonempty bounded closed convex subset C, there exists $x \in C$ such that $r_c(x) \leq (1-h)\operatorname{diam}(C)$, then X is said to have *uniform normal structure* (UNS).

The above concepts are introduced as a powerful tool in fixed point theory. For instance, if X has weak normal structure, then it has weak fixed point property (w-FPP), that is, any nonexpansive self-mapping defined on a weakly compact convex nonempty subset of X has a fixed point (see [Ki 65] and [Go-Ki 90]).

In this section we will consider Orlicz spaces over a finite nonatomic measure space only.

Theorem 2.1 *Let X be equal to one of the space L_Φ, L_Φ^0, l_Φ or l_Φ^0. Then X has UNS if and only if it is reflexive.*

To prove this theorem, we need the following lemmas.

Lemma 2.2 *Suppose $\Phi \in \triangle_2$. Then for any $\beta > 1$ and $\varepsilon > 0$, there exists $K \geq 2$ such that for all $x \in L_\Phi$,*

$$I_\Phi(\beta x) \leq K I_\Phi(x) + \varepsilon.$$

Proof. Let $\alpha > 0$ satisfy $\Psi(\beta\alpha)\mu(T) < \varepsilon$. Then since $\Phi \in \triangle_2$, there exists $K \geq 2$ such that $\Phi(\beta u) \leq K\Phi(u)$ for all $u \geq \alpha$. For given $x \in L_\Phi$, set $F = \{t \in T : |x(t)| \geq \alpha\}$. Then

$$I_\Phi(\beta x) = I_\Phi(\beta x|_F) + I_\Phi(\beta x|_{T\setminus F}) \leq K I_\Phi(x|_F) + \Phi(\beta\alpha)\mu(T \setminus F) \leq K I_\Phi(x) + \varepsilon,$$

where $x|_F = x\chi_F$. ■

Lemma 2.3 *Assume $\Phi \in \Delta_2 \cap \nabla_2$. Then for any $\alpha > 0$, there exist $c > 1$ and $\delta > 0$ such that*

$$\Phi\left(\frac{u+v}{2}\right) \le \frac{1-\delta}{2}[\Phi(u) + \Phi(v)]$$

whenever $|u| \ge \alpha$ and $|u| \ge c|v|$, or $uv \le 0$.

Proof. By Lemma 1.3, there exist $\gamma > 0$,and $\varepsilon \in (0, 1/2)$ such that

$$\Phi\left(\frac{w}{2}\right) \le \frac{1-\gamma}{2}\Phi(w) \qquad (|w| \ge \alpha)$$

and

$$\Phi((1+\varepsilon)w) \le \frac{2}{2-\gamma}\Phi(w) \qquad (|w| \ge \alpha).$$

Set

$$c = \frac{1}{\varepsilon}, \delta = 1 - \frac{2-2\gamma}{2-\gamma}.$$

Then $|u| \ge \alpha, |u| \ge c|u|$ or $uv \le 0$ imply

$$\Phi\Big(\frac{u+v}{2}\Big) \le \Phi\Big(\frac{1+c^{-1}}{2}u\Big) \le \frac{1-\gamma}{2}\Phi\left(\Big(1+\frac{1}{c}\Big)u\right)$$
$$\le \frac{1-\gamma}{2}\frac{2}{2-\gamma}\Phi(u) \le \frac{1-\delta}{2}[\Phi(u) + \Phi(v)].$$

■

Lemma 2.4 *If a Banach space X does not have UNS, then for each $n \in \mathbf{N}$ and $\varepsilon > 0$, there exists a family $\{x_i : 1 \le i \le n+1\}$ in X such that*

$$\|x_j\| \le 1, \ \|x_i - x_j\| \le 1 \quad (1 \le i \le j \le n+1)$$

and

$$\left\| x_{m+1} - \frac{1}{m}\sum_{i=1}^{m} x_i \right\| > 1 - \varepsilon \qquad (m = 1, \cdots, n).$$

Proof. By the assumption, there exists a bounded nonempty closed convex subset C of X such that for each $z \in C$, there exists $x \in C$ satisfying $\|z - x\| > (1-\varepsilon)\operatorname{diam} C$. Without loss of generality ,we may assume that $0 \in C$ and $\operatorname{diam} C = 1$, that is, $\|x\| \le 1$ and $\|x - y\| \le 1$ for all $x, y \in C$.

Pick $x_1 \in C$ arbitrarily. Then, by the hypothesis, there exists $x_2 \in C$ such that $\|x_2 - x_1\| > 1 - \varepsilon$. Since C is convex, $(x_1 + x_2)/2 \in C$. Therefore, there exists $x_3 \in C$ such that $\|x_3 - (x_1 + x_2)/2\| > 1 - \varepsilon$, and so on, by induction, we can choose the desired system of elements. ■

Proof of Theorem 2.1 We only prove the theorem for $X = L_\Phi$. The proofs for other spaces are analogous. Since all Banach spaces with UNS are reflexive,we only need to show the sufficiency. By Lemmas 2.2 and 2.3, there exist $K \ge 2, b > 0, c > 1$ and $\delta > 0$ such that

$$I_\Phi(2x) \le KI_\Phi(x) + \frac{1}{8} \quad (x \in L_\Phi), \tag{2.1}$$

$$\Phi(b)\mu(T) \le \frac{1}{8K} \tag{2.2}$$

and

$$\Phi\left(\frac{u+v}{2}\right) \le \frac{1-\delta}{2}[\Phi(u)+\Phi(v)] \quad (|u| \ge b,\ |u| \ge c|v|, \ \text{ or } \ uv \le 0). \tag{2.3}$$

Select an integer $p > 16c^2K^2$ and set $n = 8p$. If L_Φ fails to have UNS, then Lemma 2.4 and $\Phi \in \Delta_2$ yield the existence of $\{x_i\}$, $1 \le i \le n+1$, such that

$$I_\Phi(x_i) \le 1,\ I_\Phi(x_i - x_j) \le 1 \quad (1 \le i \le j \le n+1) \ \text{ and} \tag{2.4}$$

$$1 \ge I_\Phi(x_{m+1} - \frac{1}{m}\sum_{i=1}^{m} x_i) > 1 - \frac{\delta}{4n^2K}. \tag{2.5}$$

Now, we first introduce some notation. Set $u_i(t) = x_{n+1}(t) - x_i(t)$ $(i \le n)$ and for each $t \in T$, rearrange $\{u_i(t)\}_{i \le n}$ into $\{y_s(t) = u_{i_{s(t)}}(t)\}$ such that $y_1(t) \le \cdots \le y_n(t)$. Then it is not difficult to check that each $y_s(t)$ is Σ-measurable. Moreover, define

$$x(t) = \frac{[y_{4p}(t) + y_{4p+1}(t)]}{2}, \qquad x_0(t) = \frac{2}{n}\sum_{i=1}^{n}|u_i(t)|,$$

$$I(t) = \{i \le n : |u_i(t)| > c|x(t)| \ \text{ or } \ c|u_i(t)| < x(t) \ \text{ or } \ u_i(t)x(t) \le 0\},$$

$$A = \{t \in T : I(t) \text{ contains at least } 4p \text{ elements}\}, \quad B = T \setminus A.$$

Then

$$|x(t)| \le \max\{|y_s(t)|, |y_{4p+s}(t)|\} \le x_0(t). \tag{2.6}$$

Moreover, (2.1), (2.4) and the convexity of Φ imply

$$I_\Phi(x_0) \le KI_\Phi\left(\frac{1}{n}\sum_{i=1}^{n}|u_i|\right) + \frac{1}{8} \le \frac{K}{n}\sum_{i=1}^{n} I_\Phi(u_i) + \frac{1}{8} < K + \frac{1}{8}. \tag{2.7}$$

In the first step, we show that

$$\int_B \Phi(\frac{x_1(t) - x_2(t)}{2})dt > \frac{1}{2K}. \tag{2.8}$$

Since (2.4) and (2.1) yield

$$\frac{7}{8} < 1 - \frac{\delta}{4n^2K} < I_\Phi(x_1 - x_2) \le KI_\Phi(\frac{x_1 - x_2}{2}) < K + \frac{1}{8}.$$

That is, $I_\Phi(\frac{x_1-x_2}{2}) > \frac{3}{4K}$, to verify (2.8) it suffices to show that

$$\int_A \Phi(\frac{x_1(t) - x_2(t)}{2})dt < \frac{1}{4K}.$$

For this purpose, we first check that $t \in A$ implies

$$|y_s(t)| > c|y_{4p+s}(t)| \ \text{ or } \ c|y_s(t)| < |y_{4p+s}(t)| \ \text{ or } \ y_s(t)y_{4p+s}(t) \le 0$$

for each $s \le 4p$. In fact, if there exist some $j \le 4p$ and $t \in A$ such that none of the above three inequalities holds, then we get

$$c^{-1}y_{4p+j}(t) \le y_j(t) \le cy_{4p+j}(t) \ \text{ or } \ c^{-1}y_{4p+j}(t) \ge y_j(t) \ge cy_{4p+j}(t).$$

Since $x(t)$ is between $y_j(t)$ and $y_{4p+j}(t)$, we derive that

$$c^{-1}x(t) \le y_s(t) \le cx(t) \quad \text{or} \quad c^{-1}x(t) \ge y_s(t) \ge cx(t)$$

for all $s = j, j+1, \cdots, 4p+j$, which contradicts the definition of A.

Hence if we define for each $s \le 4p$, $A(s) = \{t \in A : \max\{|y_s(t)|, |y_{4p+s}(t)|\} > b\}$, then (2.3) and the convexity of Φ imply

$$\begin{aligned}
1 - \frac{\delta}{4n^2K} &< I_\Phi\left(x_{n+1} - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= I_\Phi\left(\frac{2}{n}\sum_{s=1}^{4p}\frac{y_s + Y_{4p+s}}{2}\right) \\
&\le \frac{2}{n}\sum_{s=1}^{4p} I_\Phi\left(\frac{y_s + Y_{4p+s}}{2}\right) \\
&\le \frac{2}{n}\sum_{s=1}^{4p}\frac{1}{2}\int\limits_{G/A(s)}[\Phi(y_s(t)) + \Phi(y_{4p+s}(t))]dt \\
&\quad + \frac{2}{n}\sum_{s=1}^{4p}\frac{1}{2}(1-\delta)\int\limits_{A(s)}[\Phi(y_s(t)) + \Phi(y_{4p+s}(t))]dt \\
&= \frac{1}{n}\sum_{i=1}^{n} I_\Phi(u_i) - \frac{\delta}{n}\sum_{s=1}^{4p}\int\limits_{A(s)}[\Phi(y_s(t)) + \Phi(y_{4p+s}(t))]dt.
\end{aligned}$$

It follows from (2.4) that

$$\sum_{s=1}^{4p}\int\limits_{A(s)}[\Phi(y_s(t)) + \Phi(y_{4p+s}(t))]dt < \frac{1}{4nK} < \frac{1}{8K}. \tag{2.9}$$

Now, we define

$$\begin{aligned}
D_i &= \{t \in A : |u_i(t)| > b\} \quad (i = 1, 2), \text{ and} \\
B_i(s) &= \{t \in A : u_i(t) = y_s(t) \quad \text{or} \quad y_{4p+s}(t)\} \quad (i = 1, 2).
\end{aligned}$$

Then from (2.2), (2.9) and the fact that $\cup_{s=1}^{4p} B_i(s) = A$, $D_i \cap B_i(s) \subset A(s)$ $(i = 1, 2)$, we derive

$$\begin{aligned}
\int\limits_A \Phi\left(\frac{x_1(t) - x_2(t)}{2}\right)dt &= \int\limits_A \Phi\left(\frac{u_1(t) - u_2(t)}{2}\right)dt \le \frac{1}{2}\sum_{i=1}^{2}\int\limits_A \Phi(u_i(t))dt \\
&\le \frac{1}{2}\sum_{i=1}^{2}\int\limits_{D_i}\Phi(u_i(t))dt + \Phi(b)\mu(A) \\
&= \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{4p}\int\limits_{D_i \cap B_i(s)}\Phi(u_i(t))dt + \Phi(b)\mu(A) \\
&\le \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{4p}\int\limits_{D_i \cap B_i(s)}[\Phi(y_s(t) + \Phi(y_{4p+s}(t))]dt + \Phi(b)\mu(A)
\end{aligned}$$

$$< \frac{1}{8K} + \frac{1}{8K} = \frac{1}{4K}.$$

This ends the proof of inequality (2.8).

In the second step, we set for each $i = 3, \cdots, n-1$,

$$G(i) = \{t \in B : |x_s(t) - x_i(t)| \leq c|x(t)|/p \text{ for some } s > i \text{ and } s \leq n\}.$$

Then

$$\bigcup_{i=3}^{n-1} G(i) = B. \tag{2.10}$$

In fact, for any $t \in B = T \setminus A$, by the definition of A, there exist at least five $u_i(t)$ such that their distance from each other is no more than $c|x(t)|/p$, and thus, there exist i, j; $3 \leq i < j \leq n$, such that

$$|u_i(t) - u_j(t)| \leq c|x(t)| \leq c|x(t)|/p,$$

that is $t \in G(i)$. This proves (2.10).

Now, we define

$$D(3) = G(3), \quad D(i) = G(i) \Big/ \bigcup_{k=3}^{i-1} G(k) \quad (i = 4, \cdots, n-1).$$

Then $\{D(i)\}_{i=3}^{n-1}$ are pairwise disjoint and $\cup_{i=3}^{n-1} D(i) = B$. Let, for each $i = 3, \cdots, n-1$ and each $t \in D(i)$,

$$i'(t) = i, i''(t) = \max\{k \leq n : |x_k(t) - x_i(t)| \leq c|x_(t)|/p\}.$$

Then $i'(t)$, $i''(t)$ are well defined by the definition of $G(i)$ and $i'(t) < i''(t)$. Next, we construct two Σ-measurable functions as follows:

$$x'(t) = \sum_{i=3}^{n-1} x_{i'(t)}(t)\chi_{D(i)}(t), \quad x''(t) = \sum_{i=3}^{n-1} x_{i''(t)}(t)\chi_{D(i)}(t).$$

Then by (2.6) and the definition of $i'(t), i''(t)$,

$$|x'(t) - x''(t)| \leq c|x(t)|/p \leq cx_0(t)/p. \tag{2.11}$$

Since (2.8) and the convexity of M imply

$$\frac{1}{2}\int_B [\Phi(x''(t) - x_1(t)) + \Phi(x''(t) - x_2(t))]dt \geq \int_B \Phi(\frac{x_1(t) - x_2(t)}{2})dt > \frac{1}{2K},$$

without loss of generality, we assume that

$$\int_B \Phi(x''(t) - x_1(t))dt > \frac{1}{2K}. \tag{2.12}$$

Finally, let $E = \{t \in B : |x''(t) - x_1(t)| \geq \max\{b, c^2 x_0(t)/p\}\}$. Then by (2.11), $t \in E$ implies $|x''(t) - x_1(t)| \geq c^2 x_0(t)/p \geq |x'(t) - x''(t)|$. It follows from (2.3) that if $t \in E$, then

$$\Phi\left(\frac{x''(t) - x'(t) + x''(t) - x_1(t)}{2}\right) \leq \frac{1-\delta}{2}[\Phi(x''(t) - x'(t)) + \Phi(x''(t) - x_1(t))]. \tag{2.13}$$

Moreover, (2.12), (2.3), (2.1), (2.7) and the inequality $c^2/p < (4K)^{-2} < 1$ imply that

$$\begin{aligned}\int_E \Phi(x''(t)-x_1(t))dt &= \int_B \Phi(x''(t)-x_1(t))dt - \int_{B\setminus E} \Phi(x''(t)-x_1(t))dt \qquad (2.14)\\ &> \frac{1}{2K} - \left[\int_{B\setminus E} \Phi\left(\frac{c^2 x_0(t)}{p}\right) dt + \Phi(b)\mu(B/E)\right]\\ &\geq \frac{1}{2K} - \left[\frac{c^2}{p}\int_G \Phi(x_0(t))dt + \frac{1}{8K}\right]\\ &\geq \frac{1}{2K} - \left[\frac{c^2}{p}\left(K+\frac{1}{8}\right) + \frac{1}{8K}\right]\\ &> \frac{1}{2K} - \frac{1}{8K} - \frac{1}{8K} = \frac{1}{4K}.\end{aligned}$$

In light of (2.13) and the convexity of Φ, for all $t \in E$, we have

$$\begin{aligned}&\sum_{m=2}^{n} \Phi\left(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\right)\\ &= \sum_{\substack{2\leq m\leq n\\ m\neq i''(t)}} \Phi\left(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\right)\\ &\quad + \Phi\left(\frac{1}{i''(t)-1}\left[\sum_{\substack{2\leq k\leq i''(t)-1\\ k\neq i'(t)}} (x''(t)-x_k(t)) + 2\frac{x''(t)-x'(t)+x''(t)-x_1(t)}{2}\right]\right)\\ &\leq \sum_{\substack{2\leq m\leq n\\ m\neq i''(t)}} \Phi\left(\frac{1}{m-1}\sum_{k=1}^{m-1}(x_m(t)-x_k(t))\right)\\ &\quad + \frac{1}{i''(t)-1}\left[\sum_{\substack{2\leq k\leq i''(t)-1\\ k\neq i'(t)}} \Phi(x''(t)-x_k(t)) + 2\Phi\left(\frac{x''(t)-x'(t)+x''(t)-x_1(t)}{2}\right)\right]\\ &\leq \sum_{m=2}^{n}\frac{1}{m-1}\sum_{k=1}^{m-1}\Phi(x''(t)-x_k(t)) - \frac{\delta}{i''(t)-1}[\Phi(x''(t)-x'(t))+\Phi(x''(t)-x_1(t))].\end{aligned}$$

Combining this with (2.5), (2.14) and $x''(t)-1 \leq n-1$, we deduce

$$\begin{aligned}1-\frac{\delta}{4n^2K} &< \frac{1}{n-1}\sum_{m=2}^{n} I_\Phi\left(x_m - \frac{1}{m-1}\sum_{k=1}^{m-1} x_k\right)\\ &\leq \frac{1}{n-1}\sum_{m=2}^{n}\frac{1}{m-1}\sum_{k=1}^{m-1} I_\Phi(x_m-x_k)\\ &\quad - \frac{\delta}{n(n-1)}\int_E [\Phi(x''(t)-x'(t))+\Phi(x''(t)-x_1(t))]dt < 1-\frac{\delta}{4n^2K}.\end{aligned}$$

This contraction completes the proof. ∎

Lemma 2.5 *Suppose $x_n \in B(L_\Phi)$ for $n \in \mathbf{N}$ and $x_n \to x$ in measure. Then $x \in B(L_\Phi)$.*

Proof. Since $I_\Phi(x_n) \leq 1$ for all $n \in \mathbf{N}$, by the Fatou Lemma, we have $I_\Phi(x) \leq \liminf_n I_\Phi(x_n) \leq 1$. ∎

Theorem 2.6 *The spaces L^0_Φ and l^0_Φ have the weak sum-property, so they have WNS.*

We will prove this theorem together with the next theorem.

Lemma 2.7 *If $\{x_n\}$ is a bounded sequence in L^0_Φ, $k_n \in K(x_n)$ for $n = 1, 2, \ldots,$ and $k_n \to \infty$, then $x_n \to 0$ in measure.*

Proof. For each $\sigma > 0$, define $G_n = \{t \in T\colon\ |x_n(t)| \geq \sigma\}$. Then

$$\|x_n\|^0 = \frac{1}{k_n}[1 + I_\Phi(k_n x_n)] \geq \frac{1}{k_n}\Phi(k_n\sigma)\mu(G_n).$$

Applying the fact that $\Phi(u)/u \to \infty$ as $u \to \infty$, we get $\mu(G_n) \to 0$. ∎

Lemma 2.8 (i) *If $\{x_n\}$ is a bounded sequence in L_Φ and it converges to zero in measure, then $x_n \to 0$ E_Ψ-weakly, where Ψ is the Young conjugate of Φ.*

(ii) *If $\{x_n\}$ is a bounded sequence in l_Φ and it converges to zero coordinate-wise, then $x_n \to 0$ h_Ψ-weakly.*

Proof. We only prove (i) because the proof of (ii) is analogous. Suppose that $\|x_n\| \leq K$ for any $n \in \mathbf{N}$. For any $v \in E_\Psi$ and $\varepsilon > 0$, choose $\delta > 0$ such that $E \in \Sigma$ and $\mu(E) < \delta$ imply $\|x\chi_E\|^0_\Psi < \varepsilon$. Since $x_n \to 0$ in measure, we can find $G_n \in \Sigma$ with $\mu(G_n) < \delta$ such that $|x_n(t)| < \varepsilon$ on $T \setminus G_n$ for all large n. Hence, for such n,

$$\begin{aligned} |\langle v, x_n\rangle| &\leq \int_{T\setminus G_n} |x_n(t)v(t)|d\mu \\ &\leq \varepsilon\|\chi_T\|\,\|v\|^0_\Psi + \|x_n\|\,\|v\chi_{G_n}\|^0_\Psi \leq \varepsilon\|\chi_T\|\,\|v\|^0_\Psi + \varepsilon K. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that $\langle v, x_n\rangle \to 0$. ∎

In the following by SAI of Φ we denote an interval $[a, b]$ such that Φ is affine on $[a, b]$ but it is not affine neither on $[a - \varepsilon, a]$ nor on $[b, b + \varepsilon]$, where $\varepsilon > 0$.

Theorem 2.9 *Let X be one of the spaces $L^0_\Phi, E^0_\Phi, l^0_\Phi$ or h^0_Φ. Then the following are equivalent:*

(i) *X has the sum-property.*

(ii) *X has NS.*

(iii) *There exist $a > 0$ and $c > 1$ such that for any SAI $[u, v]$ of Φ,*

$$u > a \Rightarrow v \leq cu \ (\textit{when } X = L^0_\Phi \textit{ or } E^0_\Phi),$$

$$0 < u \leq a \Rightarrow v \leq cu \ (when\ X = l_\Phi^0\ or\ h_\Phi^0).$$

Proof of Theorems 2.6 and 2.9. We only prove the theorems for $X = L_\Phi^0$ and $X = E_\Phi^0$. Let $\{x_n\}$ be a limit affine sequence in L_Φ^0 and $k_{ij} \in K(x_i - x_j)$ $(i \neq j)$. First we show that there exists a subsequence N_1 of $\mathbf{N}$ such that for any $j \in N_1$, $\{k_{ij} : i \in N_1\}$ is bounded. Indeed, if $\{k_{ij} : i \in \mathbf{N}\}$ is bounded for all $j \in \mathbf{N}$, then we set $N_1 = N$. Otherwise, there exist some $m \in \mathbf{N}$ and a subsequence I of $\mathbf{N}$ such that $k_{im} \to \infty$ as $i(\in I) \to \infty$. This shows that $x_i : i \in I$ converges to x_m in measure according to Lemma 2.7. Since $x_i \neq x_j$ for all $i \neq j$, by the same reason, we get that $N_1 = I \setminus \{m\}$ satisfies our requirement.

By the diagonal method, we can pick a subsequence N_2 of N_1 such that $k_{ij} \to k < \infty$ as $i(\in N_2) \to \infty$ for each $j \in N_1$. We claim that $k_j \to \infty$ as $i(\in N_2) \to \infty$. In fact, if this is not true, then N_2 contains a subsequence N_3 such that $k_j \to k < \infty$ as $j(\in N_3) \to \infty$. Therefore, for all $n, i, j \in N_3$ with $n \neq i, j$,

$$\begin{aligned}
&\|x_n - x_i\|^0 + \|x_n - x_j\|^0 - \|2x_n - x_i - x_j\|^0 \\
\geq{}& \frac{1}{k_{ni}}[1 + I_\Phi(k_{ni}(x_n - x_i))] + \frac{1}{k_{nj}}[1 + I_\Phi(k_{nj}(x_n - x_j))] \\
&- \frac{k_{ni} + k_{nj}}{k_{ni}k_{nj}}\left[1 + I_\Phi\left(\frac{k_{ni}k_{nj}}{k_{ni} + k_{nj}}(2x_n - x_i - x_j)\right)\right] \qquad (2.15)\\
={}& \int_G \left\{\frac{1}{k_{ni}}\Phi(k_{ni}(x(t) - x_i(t))) + \frac{1}{k_{nj}}\Phi(k_{nj}(x_n(t) - x_j(t)))\right. \\
&\left. - \frac{k_{ni} + k_{nj}}{k_{ni}k_{nj}}\Phi\left(\frac{k_{ni}k_{nj}}{k_{ni} + k_{nj}}(2x_n(t) - x_i(t) - x_j(t))\right)\right\} dt.
\end{aligned}$$

Denote the last integrand in (2.15) by $f_n^{ij}(t)$. Then $f_n^{ij} \geq 0$ for all $t \in T$ since Φ is convex. Recall that $\triangle(x)$ is affine on co$\{x_k\}$. By letting $n \to \infty$ we get

$$\int_G f_n^{ij}(t)dt \to 0,$$

and thus $f_n^{ij}(t) \to 0$ in measure. Hence, the diagonal method allows us to find a subsequence N_4 of N_3 such that $f_n^{ij}(t) \to 0$ μ-a.e. on T as $n(\in N_4) \to \infty$ for all $i, j \in N_3$.

Now, for each $t \in T$, we pick a subsequence $\{n_\gamma = n_\gamma(t)\}$ of N_4 such that

$$|v(t)| = \liminf_{n \in N_4} |x_n(t)|, \text{ and } \lim_\gamma x_{n_\gamma}(t) = v(t). \qquad (2.16)$$

Then, by the Fatou Lemma, $|v(t)| < \infty$ μ-a.e. on T, analogously to the proof of Lemma (2.5). Let $\gamma \to \infty$. Then the convexity of Φ gives

$$\begin{aligned}
0 = \lim_\gamma f_{n_\gamma}^{ij}(t) ={}& \frac{1}{k_i}\Phi(k_i(v(t) - x_i(t))) + \frac{1}{k_j}\Phi(k_j(v(t) - x_j(t))) \qquad (2.17)\\
&- \frac{k_i + k_j}{k_ik_j}\Phi\left(\frac{k_ik_j}{k_i + k_{-j}}(2v(t) - x_i(t) - x_j(t))\right)
\end{aligned}$$

μ-a.e. on T. Since for μ-a.e. $t \in T$, (2.17) holds for all $i, j \in N_3$, by replacing j by n_γ in (2.17) and letting $\gamma \to \infty$, we have for μ-a.e. $t \in T$,

$$\frac{1}{k_i}\Phi(k_iv(t) - x_i(t))) = \frac{k_i + k_j}{k_ik_j}\Phi\left(\frac{k_ik_j}{k_i + k_j}(v(t) - x_i(t))\right). \qquad (2.18)$$

Since $0 < \frac{k}{K_i+k_j} < 1$, by (2.7), condition (2.18) holds only for $v(t) = x_i(t)$. This means $v = x_i$ for all $i \in N_3$, contradicting the assumption that $x_i \neq x_j$ whenever $i \neq j$. This contradiction proves that $k_j \to \infty$ as $j (\in N_3) \to \infty$.

Now, we prove (iii) $\Rightarrow$ (i) in Theorem 2.9. If L^0_Φ does not have the sum-property, then there exists a limit affine sequence $\{y_n\}$ in L^0_Φ such that $\triangle(y_n) \uparrow$. By the above discussion, $\{y_n\}$ contains a subsequence $\{x_n\}$ satisfying $k_{ij} \to k_j < \infty$ as $i \to \infty$, and $k_j \to \infty$ as $j \to \infty$, where $k_{ij} \in K(x_i - x_j), i \neq j$. Since $\Phi(u)/u \to \infty$ as $u \to \infty$, for the constant $a > 0$ in (iii), we can find $b > a$ such that

$$\Phi\left(\frac{a+b}{2}\right) < \frac{\Phi(a) + \Phi(b)}{2}.$$

Since Φ is convex, by (iii),

$$\Phi(\alpha u + (1-\alpha)v) < \alpha\Phi(u) + (1-\alpha)\Phi(v) \tag{2.19}$$

for all $0 < \alpha < 1$ and all $u \leq a, v \geq b$ or $u \geq a, v \geq cu$. If we define $v(t)$ as in (2.16), then by (2.16) and (2.19), for μ-a.e. $t \in T$, if $k_i|v(t) - x_i(t)| \leq \alpha$, then $k_j|v(t) - x_j(t)| \leq b$; if $k_i|v(t) - x_i(t)| \geq \alpha$, then $k_j|v(t) - x_j(t)| \leq ck_i|v(t) - x_i(t)|$. Therefore, for μ-a.e. $t \in T$,

$$k_j|v(t) - x_j(t)| \leq \max\{b, ck_j|v(t) - x_i(t)|\} =: u_i(t). \tag{2.20}$$

By the Fatou Lemma,

$$\triangle(x_j) \geq k_j^{-1}[1 + I_\Phi(k_j(v - x_j))] \geq \|c - x_j\|^0. \tag{2.21}$$

Thus $v - x_j \in L_\Phi$, whence $u_i \in L_\Phi$. Since $\triangle > 0$, $\liminf_j \|v - x_j\|^0 =: \gamma > 0$. It follows from (2.20) that $k_j = \|k_j(v - x_j)\|^0/\|v - x_j\|^0 \leq \|u_i\|^0/\|v - x_j\|^0$. Letting $j \to \infty$, we get a contradiction: $\infty = \|u_i\|^0/\gamma < \infty$.

Now, we turn to Theorem 2.6. If L^0_Φ does not have the weak sum-property, then by (2.15), there exists a weakly convergent (to zero) limit affine sequence (x_n) with $\|x_n\|^0 \to 1$ and $\triangle(x_n) \to 1$. By the first part of the proof, passing to a subsequence if necessary, we may assume that $k_{ij} \to k_j < \infty$ as $i \to \infty$ and $k_j \to \infty$ as $j \to \infty$, where $k_{ij} \in K(x_i - x_j)$. It follows from (2.21) that $x_j \to v$ in measure (verified as in the first part of the proof). Therefore, by Lemma 2.7, $v = 0$. We may also assume that $x_j \to 0$ μ-a.e. on T. We prove the theorem by showing that $\lim \triangle(x_j) \geq \frac{4}{3}$, which contradicts the assumption $\triangle(x_j) \to 1$.

For each $j \in \mathbf{N}$, we choose a set $G_j \in \sum$ such that x_j is bounded on G_j and

$$k_j^{-1}[1 + I_\Phi(k_j x_j \chi_{G_j})] > k_j^{-1}[1 + I_\Phi(k_j x_j)] - \frac{1}{k_j}.$$

Then by (2.21),

$$\begin{aligned}\triangle(x_j) \geq k_j^{-1}[1 + I_\Phi(k_j x_j)] &= k_j^{-1}[1 + I_\Phi(k_j x_j \chi_{G_j})] + k_j^{-1}[1 + I_\Phi(k_j x_j \chi_{T\setminus G_j})] - \frac{1}{k_j} \\ &> k_j^{-1}[1 + I_\Phi(k_j x_j)] - \frac{1}{k_j} + \|x_j \chi_{T\setminus G_j}\|^0 - \frac{1}{k_j} \\ &\geq \|x_j\|^0 + \|x_j \chi_{T\setminus G_j}\|^0 - \frac{2}{k_j},\end{aligned}$$

that is

$$\|x_j \chi_{T\setminus G_j}\| < \triangle(x_j) - \|x_j\|^0 - \frac{2}{k_j}. \tag{2.22}$$

It follows that

$$\begin{aligned}\|x_j\chi_{G_j}\|^0 &\geq \|x_j\|^0 - \|x_j\chi_{T\setminus G_j}\|^0 \\ &> 2\|x_j\|^0 - \triangle(x_j) - \frac{2}{k_j}.\end{aligned} \tag{2.23}$$

Since x_j is bounded on G_j, there exists $\delta = \delta(j) > 0$ such that

$$\|x_j\chi_E\|^0 < \frac{1}{k_j} \quad \text{whenever} \quad E \subset G_j \quad \text{and} \quad \mu(E) < \delta. \tag{2.24}$$

Since $x_i \to 0$ μ-a.e. on T, there exists $F \in \sum$ with $\mu(F) < \delta$ such that $x_i \to 0$ uniformly on $T \setminus F$. Hence, there exists $I = I(j) \in \mathbf{N}$ such that for all $i > I$,

$$\|x_i\chi_{T\setminus F}\|^0 < \frac{1}{k_j}. \tag{2.25}$$

It follows that

$$\|x_j\chi_F\|^0 \geq \|x_i\|^0 - \|x_i\chi_{T\setminus F}\|^0 > \|x_i\|^0 - \frac{1}{k_j}. \tag{2.26}$$

Hence, by (2.22)–(2.26),

$$\begin{aligned}\|x_i - x_j\|^0 &= k_{ij}^{-1}[1 + I_\Phi(k_{ij}(x_i - x_j)\chi_{T\setminus(T\setminus F)})] \\ &\quad + k_{ij}^{-1}[1 + I_\Phi(k_{ij}(x_i - x_j)\chi_{T/F})] - \frac{1}{k_{ij}} \\ &\geq \|(x_i - x_j)\chi_{T\setminus(T\setminus F)}\|^0 + \|(x_i - x_j)\chi_{T\setminus F}\| - \frac{1}{k_{ij}} \\ &\geq \|x_i\chi_{T\setminus(T\setminus F)}\|^0 - \|x_j\chi_{T\setminus(T\setminus F)}\|^0 \\ &\quad + \|x_j\chi_{G_j/F}\|^0 - \|x_i\chi_{G_j/F}\|^0 - \frac{1}{k_{ij}} \\ &= \|x_i\chi_{T\setminus(T\setminus F)}\|^0 - \|x_j\chi_{T\setminus G_j} + x_j\chi_{G_j\cap F}\|^0 \\ &\quad + \|x_j\chi_{G_j} - x_j\chi_{G_j\cap F}\|^0 - \|x_i\chi_{G_j\setminus F}\|^0 - \frac{1}{k_{ij}} \\ &> \left(\|x_i\|^0 - \frac{1}{k_j}\right) - \left(\triangle(x_j) - \|x_j\|^0 + \frac{2}{k_j} + \frac{1}{k_j}\right) \\ &\quad + \left(2\|x_j\|^0 - \triangle(x_j) - \frac{2}{k_j} - \frac{1}{k_j}\right) - \frac{1}{k_j} - \frac{1}{k_{ij}} \\ &= \|x_i\|^0 + 3\|x_j\|^0 - 2\triangle(x_j) - \frac{8}{k_j} - \frac{1}{k_{ij}}.\end{aligned}$$

Letting $i \to \infty$, we have $\triangle(x_j) \geq 1 + 3\|x_j\|^0 - 2\triangle(x_j) - \frac{9}{k_j}$. Hence, $\lim_j \triangle(x_j) \geq \frac{4}{3}$.

Finally, we prove the implication (ii)$\Rightarrow$(iii) of Theorem 2.9. If (iii) does not hold, then there exist sequences $\{u_j\}, \{v_j\}$ such that $\Phi(u_1)\mu(T) > 1, u_{j+1} > 2^j u_j, v_j > 2^j u_j$ and $p(u)$ is a constant on $[u_j, v_j]$, $j \in \mathbf{N}$. By the first two assumptions, we can choose disjoint sets $G_j \in \Sigma$ such that $\mu(T \setminus \bigcup_{j\in\mathbf{N}} G_j) > 0$ and

$$2^{-j} - u_j p(u_j)\mu(G_j) = [\Phi(u_j) + \Psi(p(u_j))]\mu(G_j). \tag{2.27}$$

Hence, we can find u_0 large enough so that there is $G_0 \subset T \setminus \bigcup_j G_j$ satisfying

$$\sum_{j\in\mathbf{N}} \Psi(p(u_j))\mu(G_j) + \Psi(p(u_0))\mu(G_0) = 1. \tag{2.28}$$

Define

$$v = \sum_{j\geq 0} p(u_j)\chi_{G_j}, \; x_n = u_0\chi_{G_0} + \sum_{j\leq n} v_j\chi_{G_j} + \sum_{j\geq n} u_j\chi_{G_j}.$$

Then by (2.28), $I_\Phi(v) = 1$, whence $v \in L^*_\Phi$ and $\|v\|_\Psi = 1$.

First we show that $x_n \in E_\Phi$ for any $n \in \mathbf{N}$. Given arbitrary $K > 1$, choose $J > n$ such that $2^J > K$. Then for all $j > J$, we have $v_j > 2^j u_j > Ku_j > u_j$. Therefore,

$$\begin{aligned}\sum_{j>J} \Phi(Ku_j)\mu(G_j) &= \sum_{j>J}[Ku_jp(Ku_j) - \Psi(p(Ku_j))]\mu(G_j)\\ &< \sum_{j>J} Ku_jp(Ku_j)\mu(G_j)\\ &= \sum_{j>J} Ku_jp(u_j)\mu(G_j) = K\sum_{j>J} 2^{-1} < \infty.\end{aligned}$$

This implies that $I_\Phi(Kx_n) < \infty$. Since $K > 1$ is arbitrary, we have $x_n \in E_\Phi$.

Let $k_n = \|x_n\|^0$ and $y_n = x_n/k_n$. Then $y_n \in E_\Phi$ and $\|y_n\|^0 = 1$. By (2.28),

$$\begin{aligned}\|y_n\|^0 \geq \langle v, y_n\rangle &= k_n^{-1}\left[u_0p(u_0)\mu(G_0) + \sum_{j\leq n} v_jp(u_j)\mu(G_j) + \sum_{j>n} u_jp(u_j)\mu(G_j)\right]\\ &= k_n^{-1}[\varrho_\Psi(v) + I_\Phi(k_ny_n)] \geq \|y_n\|^0 = 1.\end{aligned}$$

Moreover, since

$$k_n = \|x_n\|^0 \geq \langle v, x_n\rangle > \sum_{j\leq n} v_jp(u_j)\mu(G_j) \geq \sum_{j\leq n} 2^ju_jp(u_j)\mu(G_j) = n,$$

we have $k_n \to \infty$ as $n \to \infty$.

We complete the proof by showing that $\triangle = 2$ on $\mathrm{co}(y_n)$. Indeed, for any $y \in \mathrm{co}(y_n)$, there exist $\lambda_i \geq 0$ with $\sum_{i\leq m} \lambda_i = 1$ such that $y = \sum_{i\leq m} \lambda_i y_i$. Since $\langle v, y_n\rangle = 1$, we have $\langle v, y\rangle = \sum_{i\leq m} \lambda_i\langle v, y_n\rangle = 1$. For any $\varepsilon > 0$, since $y \in E_\Phi$, there exists $I > m$ such that $\|y\chi_F\|^0 < \varepsilon$, where $F = \cup_{i\leq I} G_i$. In view of $x_n(t) \leq \max\{v_I, u_0\}$ on $G \setminus F$ and $k_n \to \infty$ as $n \to \infty$, we can find $n_0 \in G_i$ such that $\|y_n\chi_{T\setminus F}\|^0 < \varepsilon$ for all $n > n_0$. Define $v_0 = v\chi_{T\setminus F} - v\chi_F$. Then $\|v_0\|_\Psi$ and for $n > n_0$,

$$\begin{aligned}2 \geq \|y\|^0 + \|y_n\|^0 &\geq \|y - y_n\|^0\\ &\geq \langle v_0, y - y_n\rangle\\ &= \langle v_0, y\chi_{T\setminus F}\rangle + \langle v_0, y\chi_F\rangle - \langle v_0, y_n\chi_{T\setminus F}\rangle - \langle v, y_n\chi_F\rangle\\ &= \langle v, y\chi_{T\setminus F}\rangle - \langle v, y\chi_F\rangle - \langle v, y_n\chi_{T\setminus F}\rangle + \langle v, y_n\chi_F\rangle\\ &= \langle v, y\rangle - 2\langle v, y\chi_F\rangle - 2\langle v, y_n\chi_{T\setminus F}\rangle + \langle v, y_n\rangle\\ &> 1 - 2\|y\chi_F\|^0 - 2\|y_n\chi_{T\setminus F}\| - 2\|y_n\chi_{T\setminus F}\|^0 + 1 > 2 - 4\varepsilon,\end{aligned}$$

which shows that $\triangle(y) = 2$. ■

Theorem 2.10 *Let X be one of the spaces L_Φ, E_Φ, l_Φ or h_Φ. Then the following are equivalent:*

(i) *X has the sum-property.*

(ii) *X has WNS.*

(iii) $M \in \Delta_2$.

Proof. This time, we prove the theorem for $X = l_\Phi$ and $X = h_\Phi$.

(i) $\Rightarrow$ (ii).This implication is trivial.

(ii)$\Rightarrow$ (iii). If $\Phi \notin \Delta_2$, then there exist $a_k \downarrow 0$ such that $\Phi(\alpha_1) < \varepsilon$ and $\Phi((1+\frac{1}{k})\alpha_k) > 2^k\Phi(\alpha_k)$ $(k \in \mathbf{N})$, where $0 < \varepsilon < 1$ is a given constant. For each $k \in \mathbf{N}$, choose an integer m_k such that

$$m_k\Phi(\alpha_k) \leq \frac{\varepsilon}{2^k}, \qquad (m_k+1)\Phi(\alpha_k) > \frac{\varepsilon}{2^k}$$

and define

$$x_n(i) = \alpha_n \sum_{i-1}^{m_n} e_{i+s_n} \quad (n \in \mathbf{N}),$$

where $\{e_i\}$ is the natural basis of c_0 and $s_n = \sum_{i=1}^{n-1} m_i$. Obviously, $\{x_n\}$ have mutually disjoint supports, and so, $I_\Phi(x_i - x_j) \leq \varepsilon/2^i + \varepsilon/2^j < 1$ $(i \neq j)$. Moreover, for any $v > 1$, it is easy to check that $I_\Phi(vx_n) \to \infty$ as $n \to \infty$. Therefore, for any $n \in \mathbf{N}, \Delta(x_n) = 1$ and $\Delta(x) = 1$ for all $x \in \mathrm{co}\{x_n\}$. Clearly, $x_n \to 0$ l_Φ-weakly, that is, $x_n \to 0$ weakly in h_Φ. This means that $\{x_n\}$ is a weakly convergent limit constant sequence, thus, h_Φ does not have the WNS.

(iii) $\Rightarrow$ (i). Assume that l_Φ has a limit affine sequence $\{x_n\}$ with $\Delta(x_n) \uparrow \Delta'$. By the diagonal method, we can find a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, such that $x_n \to x$ coordinate-wise. By Lemma 2.5, $x \in l_\Phi$. Hence, we may assume that $x_n \to 0$ coordinate-wise and that $\Delta' = \lim \Delta(x_n) > 0$.

For any $i, j \in \mathbf{N}$, since Δ is affine on $\mathrm{co}\{x_n\}$,

$$\lim_n \|2x_n - x_i - x_j\| = \Delta(x_i) + \Delta(x_j).$$

Hence, as $\Phi \in \Delta_2$,

$$\lim_n I_\Phi\left(\frac{x_n - x_i}{\Delta(x_i)}\right) = \lim_n I_\Phi\left(\frac{x_n - x_j}{\Delta(x_j)}\right) = \lim_n I_\Phi\left(\frac{2x_n - x_i - x_J}{\Delta(x_i) + \Delta(x_j)}\right) = 1. \tag{2.29}$$

Let $\lambda_{ij} = \frac{\Delta(x_i)}{\Delta(x_i)+\Delta(x_j)}$. Then by the convexity of Φ,

$$\begin{aligned} &\lambda_{ij} I_\Phi\left(\frac{x_n - x_i}{\Delta(x_i)}\right) + (1-\lambda_{ij}) I_\Phi\left(\frac{x_n - x_j}{\Delta(x_j)}\right) - I_\Phi\left(\frac{2x_n - x_i - x_j}{\Delta(x_i) + \Delta(x_j)}\right) \\ &\geq \lambda_{ij}\Phi\left(\frac{x_n(k) - x_i(k)}{\Delta(x_i)}\right) + (1-\lambda_{ij})\Phi\left(\frac{x_n(k) - x_j(k)}{\Delta(x_j)}\right) \\ &\quad - \Phi\left(\frac{2x_n(k) - x_i(k) - x_j(k)}{\Delta(x_i) + \Delta(x_j)}\right) \\ &\geq 0 \quad (k \in \mathbf{N}). \end{aligned} \tag{2.30}$$

Recall that $x_n \to 0$ coordinate-wise. By letting $n \to \infty$, we conclude from (2.19) and (2.30) that

$$\lambda_{ij}\Phi\left(\frac{x_i(k)}{\Delta(x_i)}\right) + (1-\lambda_{ij})\Phi\left(\frac{x_j}{\Delta(x_j)}\right) = \Phi\left(\frac{x_i(k) + x_j(k)}{\Delta(x_i) + \Delta(x_j)}\right) \quad (k \in \mathbf{N}).$$

Letting $j \to \infty$, this equality becomes the equality

$$\frac{\triangle(x_i)}{\triangle(x_i)+\triangle'}\Phi\left(\frac{x_i(k)}{\triangle(x_i)}\right) = \Phi\left(\frac{x_i(k)}{\triangle(x_i)+\triangle'}\right) \quad (k \in \mathbf{N}).$$

But $0 < \triangle(x_i)/(\triangle(x_i)+\triangle') < 1$ by (2.7), so the above equality holds only for $x_i(k) = 0$. This means that $x_i = 0$ $(i \in \mathbf{N})$, contradicting the assumption that $\{x_n\}$ is a limit affine sequence. ∎

To end this section, we present a different sufficient condition for h_Φ to have the weakly fixed point property.

Lemma 2.11 *The space h_Φ has the weak orthogonality property, that is for any sequence $\{x_n\}$ in h_Φ such that $x_n \to 0$ weakly, there holds*

$$\liminf_n \liminf_m \||x_n| \wedge |x_\Phi|\| = 0,$$

where $(x \wedge y)(t) = \min\{x(t), y(t)\}$ (see [S 88] and [S 92]).

Proof. The lemma results from obvious fact that the mapping $y \to |x| \wedge |y|$ is weak-norm continuous for every fixed $x \in h_\Phi$. In fact weak convergence of (y_n) to zero implies that $y_n \to 0$ coordinate-wise. So, if $x \in h_\Phi$ then $|y_n| \wedge |x| \le |x|$ and $|y_n| \wedge |x| \to 0$ coordinate-wise. By the dominated Lebesgue convergence theorem, we get $\| \, |y_n| \wedge |x| \, \| \to 0$. ∎

Lemma 2.12 *The Riesz angle $\alpha(l_\Phi) < 2$ if and only if $\Phi \in \nabla_2$, where*

$$\alpha(l_\Phi) = \sup\{\| \, |x| \vee |y| \, \| : \|x\| \le 1, \|y\| \le 1\}.$$

Proof. If $\Phi \notin \nabla_2$, then there exist $u_n \downarrow 0$ such that

$$2\Phi\left(\frac{u_n}{2}\right) > (1 - \frac{1}{n})\Phi(u_n) \quad (n \in N). \tag{2.31}$$

Let m_n be an integer satisfying $m_n\Phi(u_n) \le 1$ and $(m_n + 1)\Phi(u_n) > 1$. Define

$$x_n = u_n \sum_{i=1}^{m_n} e_i, \; y_n = u_n \sum_{i=m_n+1}^{2m_n} e_i.$$

Then it is easy to check that $1 \ge I_\Phi(y_n) \to 1$ and by (2.31),

$$I_\Phi\left(\frac{x_n \vee y_n}{2}\right) = 2m_n\Phi\left(\frac{u_n}{2}\right) > \left(1 - \frac{1}{n}\right) m_n\Phi(u_n) \to 1.$$

This shows that $\|x_n \vee y_n\| \to 2$.

Next we assume $\Phi \in \nabla_2$. That is, there exist $\delta > 0$ such that

$$\Phi((2-\delta)u) \ge 2\Phi(u) \quad (|u| \le \Phi^{-1}(1)).$$

Given $x, y \in B(l_\Phi)$, we have $|x(i)|$, $|y(i)| \le \Phi^{-1}(1)$, whence

$$I_\Phi\left(\frac{|x| \vee |y|}{2-\delta}\right) = I_\Phi\left(\frac{x}{2-\delta}\right) + I_\Phi\left(\frac{y}{2-\delta}\right) \le \frac{1}{2}\left[I_\Phi(x) + I_\Phi(y)\right] \le 1,$$

$$\text{that is, } \| |x| \vee |y| \| \le 2 - \delta.$$

■

Applying a result of Borwein and Sims [Bo-S 84] stating that every weakly orthogonal Banach lattice X with Riesz angle $\alpha(X) < 2$ has the weak fixed point property, from Lemmas 2.11 and 2.12 we deduce the following

Theorem 2.13 *If $\Phi \in \nabla_2$, then h_Φ has the weak fixed point property.*

Remark 2.14 Theorems 2.10 and 2.13 furnish a natural example of a space with the weakly fixed point property but without WNS.

Remark 2.15 Dowling, Lennard and Turett [Do-Le-T 96] investigated Orlicz spaces for which every nonexpansive self-mapping of a nonempty, closed, bounded, convex subset has a fixed point. This property is called the fixed point property (FPP). They proved that L^0_Φ has FPP if and only if it is reflexive. In fact, this can be obtained immediately from Theorem 2.1 presented above, Theorem 1.90 in [Ch 96] and the following two results given by Dowling and Lennard [Do-Le 97]:

(a) A Banach space X fails FPP if it contains an asymptotically isometric copy of l^1. That is, for every positive sequence (ε_n) decreasing to 0, there exists a sequences $\{x_n\}$ of norm-one elements in X such that $\sum_n (1-\varepsilon_n)|\alpha_n| \le \| \sum_n \alpha_n x_n \|$ for all sequences (α_n) of real numbers.

(b) If the dual of X contains an isometric copy of l^∞, then X contains an asymptotically isometric copy of l^1.

It is still an open problem whether the above conclusion is true or not for the Orlicz space L_Φ equipped with the Luxemburg norm. The only trouble is that one cannot prove the necessity of $\Phi \in \triangle_2$ in the same way as for the Orlicz norm.

Notes. Criteria for normal structure and uniform normal structure of Musielak-Orlicz spaces were given by Katirtzoglou [Kat 97]. In Orlicz-Lorentz spaces the criteria were presented by Kamińska, Lin and Sun [Ka-L-Sun 96].

3. Uniform rotundity in every direction

Recall that a Banach space X is said to be uniformly rotund in every direction (URED) if for any $z \in S(X)$ and every two sequences $\{x_n\}$ and $\{y_n\}$ in $S(X)$ such that $x_n - y_n = \varepsilon_n z$, where $\{\varepsilon_n\}$ is a sequence of reals, and $\|x_n + y_n\| \to 2$ we have $\|x_n - y_n\| \to 0$. If we change $S(X)$ into $B(X)$ in the above definition, we get the same property.

In the fixed point theory this geometric property is important because of the following well known theorem.

Theorem 3.1 *Any Banach space X which is uniformly rotund in every direction has normal structure.*

Now we will present criteria for URED of Orlicz spaces. We do not assume in this section that Orlicz functions Φ satisfy condition (ii) from the definition (see page 1).

Theorem 3.2 *Let (T, Σ, μ) be a nonatomic complete and σ-finite measure space and Φ be an Orlicz function. Then the Orlicz space L_Φ equipped with the Luxemburg norm is uniformly rotund in every direction if and only if Φ is strictly convex and Φ satisfies the Δ_2-condition on $\mathbf{R}_+$ if μ is infinite and the Δ_2-condition at infinity if it μ is finite.*

Proof. *Sufficiency.* Let $\{x_n\}, \{y_n\}$ be sequences in $S(L_\Phi)$, $z \in S(L_\Phi)$, $x_n - y_n = \varepsilon_n z$, where $\{\varepsilon_n\}$ is a sequence of reals and $\|(x_n + y_n)/2\| \to 1$. Then by the Δ_2-condition, we have $I_\Phi((x_n + y_n)/2) \to 1$ (see [Ch 96]). We will show that $x_n - y_n \to 0$ in measure. Assume for the contrary that it is not true. Then we can assume (passing to a subsequence if necessary) that for some ε, $\sigma > 0$ there holds $\mu(E)_n \geq \varepsilon$ for all $n \in \mathbf{N}$, where

$$E_n = \{t \in T\colon\ |x_n(t) - y_n(t)| \geq \sigma\}.$$

Choose $k > 1$ such that $F \in \Sigma$ and $\mu(F) = \varepsilon/4$ implies that $\|\chi_F\| = 1/k$ and define

$$A_n = \{t \in T\colon\ |x_n(t)| > k\}, \quad B_n = \{t \in T\colon\ |y_n(t)| > k\}.$$

Then we have $1 = \|x_n\|_\Phi \geq \|x_n \chi_{A_n}\|_\Phi > k\|\chi_{A_n}\|_\Phi$, whence $\|\chi_{A_n}\|_\Phi < \frac{1}{k}$ and consequently, $\mu(A_n) < \varepsilon/4$. Similarly, $\mu(B_n) < \varepsilon/4$ $(n \in \mathbf{N})$. By strict convexity of Φ there is $\delta > 0$ such that if $u, v \in [0, k]$ and $|u - v| \geq \sigma$, then

$$\Phi\Big(\frac{u+v}{2}\Big) \leq \frac{1-\delta}{2}\{\Phi(u) + \Phi(v)\}.$$

Denote $C_n = E_n \setminus (A_n \cup B_n)$. Then we have $\mu(C_n) \geq \mu(E_n) - (\mu(A_n) + \mu(B_n)) = \varepsilon - \varepsilon/2 = \varepsilon/2$. Moreover for any $t \in C_n$, we have $|x_n(t) - y_n(t)| \geq \sigma$ and $|x_n(t)| \leq k$, $|y_n(t)| \leq k$ for any $k \in \mathbf{N}$. Consequently

$$\Phi\left(\frac{x_n(t) + y_n(t)}{2}\right) \leq \frac{1-\delta}{2}\{\Phi(x_n(t)) + \Phi(y_n(t))\} \quad (t \in C_n),\ n = 1, 2, \ldots .$$

Consequently

$$\begin{aligned}
0 \leftarrow 1 - I_\Phi\Big(\frac{x_n + y_n}{2}\Big) &= \frac{1}{2}\{I_\Phi(x_n) + I_\Phi(y_n)\} - I_\Phi\Big(\frac{x_n + y_n}{2}\Big) \\
&\geq \frac{1}{2}\{I_\Phi(x_n)\chi_{C_n}) + I_\Phi(y_n)\chi_{C_n})\} - I_\Phi\Big(\frac{x_n + y_n}{2}\chi_{C_n}\Big) \\
&\geq \frac{1}{2}\{I_\Phi(x_n\chi_{C_n}) + I_\Phi(y_n\chi_{C_n})\} - \frac{1-\delta}{2}\{I_\Phi(x_n\chi_{C_n}) + I_\Phi(y_n\chi_{C_n})\} \\
&= \frac{\delta}{2}\{I_\Phi(x_n\chi_{C_n}) + I_\Phi(y_n\chi_{C_n})\} \geq \delta I_\Phi\Big(\frac{x_n - y_n}{2}\chi_{C_n}\Big) \\
&\geq \delta\Phi\Big(\frac{\sigma}{2}\Big)\mu(C_n) \geq \delta\Phi\Big(\frac{\sigma}{2}\Big)\frac{\varepsilon}{2},
\end{aligned}$$

a contradiction. Therefore $x_n - y_n \to 0$ in measure. Since $z \in S(L_\Phi)$ and so $z \neq 0$, we conclude from the equality $x_n - y_n = \varepsilon_n z$ $(n \in \mathbf{N})$ that $\varepsilon_n \to 0$. Consequently, there is $n_0 \in \mathbf{N}$ such that $|\varepsilon_n| \leq 1$ for $n \geq n_0$ and so $|x_n - y_n| \leq |z|$ for $n \geq n_0$. By the Lebesgue dominated convergence theorem, we get $(x_n - y_n) \to 0$ and by the suitable Δ_2-condition for Φ, we get $I_\Phi(\lambda(x_n - y_n)) \to 0$ as $n \to \infty$ for any $\lambda > 0$, which means that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Necessity. Assume that Φ does not satisfy the suitable Δ_2-condition. Then L^Φ contains an order isometric copy of l_∞ (see [Ch 96], [Ra-Re 91] and [T 76]). Since l_∞ is not URED, L_Φ is not URED, either.

Assume now that Φ is not strictly convex. We will show that L_Φ is not rotund and so is not URED. Since Φ is not strictly convex, there exists $u > v > 0$ such that $\Phi((u+v)/2) = \{\Phi(u) + \Phi(v)\}/2$. Choose two disjoint sets $A, B \in \Sigma$ and $a > 0$ such that $\mu(A) > 0$, $\mu(B) > 0$ and

$$\frac{1}{2}(\Phi(u) + \Phi(v))\mu(A) + \Phi(a)\mu(B) = 1.$$

Let $C, D \subset A$ be measurable sets such that $\mu(C) = \mu(D) = \frac{1}{2}\mu(A)$ and $C \cap D = \emptyset$. Define

$$x = u\chi_C + v\chi_D + a\chi_B,$$
$$y = v\chi_C + u\chi_D + a\chi_B.$$

Then $I_\Phi(x) = \Phi(u)\mu(C) + \Phi(v)\mu(D) + \Phi(a)\mu(B) = \frac{1}{2}(\Phi(u)+\Phi(v))\mu(A) + \Phi(a)\mu(B) = 1$. In the same way we can prove that $I_\Phi(y) = 1$. Moreover,

$$\begin{aligned} I_\Phi\Big(\frac{x+y}{2}\Big) &= \Phi\Big(\frac{u+v}{2}\Big)\mu(A) + \Phi(a)\mu(B) \\ &= \frac{1}{2}(\Phi(u) + \Phi(v))\mu(A) + \Phi(a)\mu(B) = 1. \end{aligned}$$

Consequently, $\|x\|_\Phi = \|y\|_\Phi = \|(x+y)/2\|_\Phi = 1$. Since, evidently, $x \neq y$, L_Φ is not rotund. This finishes the proof. ■

Notes. Kamińska [Ka 84] first gave criteria for URED of Musielak-Orlicz spaces of Bochner type. Theorem 3.2 can be easily deduced from her paper. The proof that we presented here is different.

4. *B*-convexity and uniform monotonicity

These properties are related to the fixed point theory by the following

Theorem 4.1 (see [Ak-K 90]) *If a Köthe function space X is B-convex and uniformly monotone, then it has the fixed point property.*

Recall that a Banach space X is said to be *B-convex* if no nonreflexive space Y is finitely represented in X (see [Ak-K 90] and [Ch 96]). Since UR implies nonsquareness, nonsquareness implies B-convexity and B-convexity is preserved by equivalent norms, we know that uniformly covexifiable Banach spaces are B-convex. The converse is also true.

Now, we will present criteria for B-convexity and uniform monotonicity of Orlicz spaces. We do not assume generally in this section that Orlicz functions that they must satisfy condition (ii) from the definition on page 1. First we will prove the following.

Lemma 4.2 *Let Φ be an Orlicz function such that its right derivative p on $\mathbf{R}_+$ satisfies the condition:*

For any $\varepsilon > 0$, there exists $K > 1$ such that $p((1+\varepsilon)t) \geq Kp(t) \quad (t \geq 0)$.

Then Φ is uniformly convex.

Proof. Given $\varepsilon \in (0,1)$, take $K > 1$ such that

$$p((1+\varepsilon/2)t) \geq Kp(t) \quad (t \geq 0).$$

We shall show that for any $u, v \in \mathbf{R}$ satisfying $|u-v| \geq \varepsilon \max\{|u|,|v|\}$, the inequality

$$\Phi\Big(\frac{u+v}{2}\Big) \leq (1-\delta)\frac{\Phi(u)+\Phi(v)}{2} \tag{4.1}$$

holds for $\delta = \varepsilon(1-1/K)/4 > 0$. We may assume without loss of generality that $u - v \geq \varepsilon u > \varepsilon v > 0$, that is $(1-\varepsilon)u \geq v > 0$. Define

$$\varphi(t) = \Phi(u) + \Phi(t) - 2\Phi\Big(\frac{u+t}{2}\Big) \quad (t \geq 0).$$

Then for almost all $t \in [0,u]$,

$$\varphi'(t) = p(t) - p\Big(\frac{u+t}{2}\Big) \leq 0.$$

Hence $\varphi(t)$ is nonincreasing on $[0,u]$. Therefore,

$$\begin{aligned}
\varphi(v) &= \Phi(u) + \Phi(v) - 2\Phi\Big(\frac{u+v}{2}\Big) \\
&\geq \Phi(u) + \Phi((1-\varepsilon)u) - 2\Phi((1-\varepsilon/2)u) \\
&= \int_{(1-\varepsilon/2)u}^{u} p(t)dt - \int_{(1-\varepsilon)u}^{(1-\varepsilon/2)u} p(t)dt = \int_{(1-\varepsilon/2)u}^{u} [p(t) - p((1-\varepsilon/2)t)]dt \\
&\geq \int_{(1-\varepsilon/2)u}^{u} (1-1/K)p(t)dt \geq (1-1/K)[\Phi(u) - (1-\varepsilon/2)\Phi(u)] \\
&> \frac{\varepsilon}{4}(1-1/K)[\Phi(u)+\Phi(v)]
\end{aligned}$$

for u and v as above, that is inequality (4.1) holds with $\delta = (\varepsilon/4)(1-1/K) > 0$. ■

Theorem 4.3 *For any Orlicz space L_Φ the following are equivalent:*

(i) *L_Φ is reflexive.*

(ii) *$\Phi \in \Delta_2$ and $\Psi \in \Delta_2$.*

(iii) *L_Φ is uniformly covexifiable.*

(iv) *L_Φ is B-convex.*

Proof. It is well known that (i) $\Leftrightarrow$ (ii) (see [Ra-Re 91] and [T 76]). Let us prove that (ii) $\Rightarrow$ (iii). We consider only the case of a nonatomic finite measure space, when $\Phi \in \Delta_2$ means that Φ satisfies the Δ_2-condition at ∞. By (ii) there exist $u_0 > 0$, $K > 2$ and $\delta > 0$ such that

$$(2+\delta)\Phi(u) \leq \Phi(2u) \leq K\Phi(u) \quad (u \geq u_0).$$

Since changing the value of Φ on $[0, u_0]$ does not affect the equivalence, we may assume that the above inequalities hold for all $u \in \mathbf{R}$. Let

$$\Phi_0(u) = \int_0^{|u|} \frac{\Phi(t)}{t}dt, \text{ and } \Phi_1 = \int_0^{|u|} \frac{\Phi_0(t)}{t}dt.$$

We claim that $\Phi_0 \sim \Phi_1 \sim \Phi$. Indeed, denoting by p the right derivative of Φ on $\mathbf{R}_+$, we have

$$p(u) \geq \frac{\Phi(u)}{u} \geq \frac{\Phi(2u)}{Ku} \geq \frac{1}{K}p(u) \quad (u > 0).$$

Integrating each term of the last inequalities from zero to u, we get $K\Phi(u) \geq K\Phi_0(u) \geq \Phi(u)$, that is $\Phi \sim \Phi_0$. Similarly $\Phi_0 \sim \Phi_1$. Next, we will show that Φ_1 is uniformly convex. Since

$$\begin{aligned} K^{-1}\Phi(u) \leq \Phi\Big(\frac{u}{2}\Big) = \int_0^{|u|/2} \frac{\Phi(u/2)}{|u/2|}dt \leq \int_{|u|/2}^{|u|} \frac{\Phi(t)}{t}dt \\ &\leq \Phi_0(u) \\ &= \int_0^{|u|/2} \frac{\Phi(t)}{t}dt + \int_{|u|/2}^{|u|} \frac{\Phi(t)}{t}dt \\ &\leq \Phi\Big(\frac{u}{2}\Big) + \frac{\Phi(u)}{2} \\ &\leq \Big(\frac{1}{2+\delta} + \frac{1}{2}\Big)\Phi(u) = \frac{1}{L}\Phi(u), \end{aligned}$$

where $L = \frac{4+2\delta}{4+\delta} > 1$, we obtain for $t > 0$,

$$L \leq \frac{\Phi(t)}{\Phi_0(t)} = \frac{t\Phi_0'(t)}{\Phi_0(t)} \leq K.$$

Hence we get

$$\frac{L}{t} \leq \frac{\Phi_0'(t)}{\Phi_0(t)} \leq \frac{K}{t}.$$

Integrating this inequality with respect to t from u to θu, where $\theta \geq 1$, we have

$$\theta^L\Phi_0(u) \leq \Phi_0(\theta u) \leq \theta^K\Phi(u) \quad (\theta \geq 1,\ u \in \mathbf{R}).$$

Set $p_1(t) = M_1'(t) = \Phi_0(t)/t$ for $t > 0$. We have for any $\varepsilon > 0$ and $u > 0$,

$$\begin{aligned} p_1((1+\varepsilon)u) = \frac{\Phi_0((1+\varepsilon)u)}{(1+\varepsilon)u} &\geq \frac{(1+\varepsilon)^L\Phi_0(u)}{(1+\varepsilon)u} \\ &= (1+\varepsilon)^{L-1}\Phi_0(u)/u = (1+\varepsilon)^{L-1}p_1(u), \end{aligned}$$

whence, by Lemma 4.2, Φ_1 is uniformly convex. Finally we will show that Ψ_1 is also uniformly convex, where Ψ_1 denotes the function complementary to Φ_1 in the sense of Young. Let $(\Psi_1)'(u) =: q_1(u)$ and

$$q_1((1+\varepsilon)v) = \alpha(v)q_1(v) \quad (v > 0).$$

Then $\alpha(v) > 1$ $(v > 0)$. Replacing v by $p_1(u)$, we get

$$\begin{aligned} (1+\varepsilon)p_1(u) = p_1(\alpha(v)u) &= \frac{\Phi_0(\alpha(v)u)}{\alpha(v)u} \\ &\leq \frac{\alpha^K(v)\Phi_0(u)}{\alpha(v)u} = \alpha^{K-1}(v)p_1(u). \end{aligned}$$

Hence, $\alpha(v) \geq (1+\varepsilon)^{1/(K-1)}$, and so $q_1((1+\varepsilon)v) \geq (1+\varepsilon)^{1/(K-1)}q_1(v)$, that is Ψ_1 is uniformly convex.

Therefore, applying the result of Kamińska [Ka 82b] and the fact that a Banach space X is uniformly rotund if and only if X^* is uniformly smooth, we get that both spaces L_{Φ_1} and L_Ψ are uniformly rotund and uniformly smooth for both the Luxemburg and Orlicz norms. Since $\Phi \sim \Phi_1$, the space L_Φ is B-convex, which finishes the proof of the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Let us prove that (iv) $\Rightarrow$ (i). We will show that if L_Φ is not reflexive, then L_Φ is not B-complex. L_Φ is nonreflexive if and only if $\Phi \notin \Delta_2$ or $\Psi \notin \Delta_2$. If $\Phi \notin \Delta_2$, then L_Φ equipped with the Luxemburg norm contains an order isometric copy of l_∞ (see [T 76]). Therefore L_Φ is not B-complex. Assume now that $\Psi \notin \Delta_2$ and $\Phi \in \Delta_2$. Then the dual space of L^0_Φ is L_Ψ, which contains an order isometric copy of l_∞ and consequently L^0_Φ contains an asymptotically isometric copy of l_1 (see [Ch-H-Sun 92]), so L^0_Φ (and consequently L_Φ) is not B-convex. This completes the proof. ■

Let us denote by $\Phi \in \Delta_2$ the fact that the Orlicz function Φ satisfies the suitable Δ_2-condition, which means the Δ_2-condition on $\mathbf{R}_+$ if μ is nonatomic and infinite, the Δ_2-condition at infinity if μ is non-atomic and finite and the Δ_2-condition at zero if μ is the counting measure.

Theorem 4.4 *An Orlicz space L_Φ equipped with the Luxemburg norm is uniformly monotone if and only if Φ vanishes only at zero and $\Phi \in \Delta_2$.*

Proof. *Sufficiency.* Assume that $a(\Phi) := \sup\{u \geq 0\colon\ \Phi(u) = 0\} = 0$ and $\Phi \in \Delta_2$. Let $x \in S(L_\Phi)$, $0 \leq y \leq x$ and $\|y\|_\Phi \geq \varepsilon$, where $\varepsilon \in (0,1)$. Then, by $\Phi \in \Delta_2$ and $I_\Phi(x) = 1$, there is $\delta(\varepsilon) \in (0,\varepsilon]$ such that $I_\Phi(y) \geq \delta(\varepsilon)$ (see [Ch 96]). Since Φ is superadditive on $\mathbf{R}_+$, we have

$$I_\Phi(x) = I_\Phi((x-y)+y) \geq I_\Phi(x-y) + I_\Phi(y),$$

whence $I_\Phi(x-y) \leq 1 - I_\Phi(y) \leq 1 - \delta(\varepsilon)$. Again by $\Phi \in \Delta_2$, there is a function $\sigma\colon\ (0,1) \to (0,1)$ such that $\|x\| \leq 1 - \sigma(\varepsilon)$ whenever $I_\Phi(x) \leq 1 - \varepsilon$ (see [Ch 96]). Consequently, $\|x-y\| \leq 1 - \sigma(\delta(\varepsilon))$, that is L_Φ is uniformly monotone.

Necessity. Assume that $a(\Phi) > 0$. First assume that the measure space is infinite. Then $x = \chi_T \in S(L_\Phi)$. Let $A \in \Sigma$ be such that $\mu(A) = \mu(T \setminus A) = \infty$ and define $y = \chi_A$. Then $0 \leq y \leq x$ and $\|x-y\|_\Phi = \|\chi_{T\setminus A}\|_\Phi = 1$, which means that L_Φ is not uniformly monotone. If μ is finite take $y \geq 0$ such that $\|y\|_\Phi = 1$ and $\mu(T \setminus \operatorname{supp} y) > 0$. Define $x = y + a(\Phi)\chi_{T\setminus \operatorname{supp} y}$. Then $0 \leq y \leq x$ and $\|x\|_\Phi = 1$, whence we get that L_Φ is not uniformly monotone.

Assume now that $\Phi \notin \Delta_2$. Then there exists $x \in S(L_\Phi)$, $x \geq 0$, such that $I_\Phi(\lambda x) = +\infty$ for any $\lambda > 1$ (see [Ch 96] and [Ra-Re 91]). Consequently, there exists $A \in \Sigma$ such that $\|x\chi_A\|_\Phi = \|x\chi_{T\setminus A}\|_\Phi = \|x - x\chi_A\|_\Phi = 1$, that is L_Φ is not uniformly monotone. This finishes the proof of the theorem. ■

Theorem 4.5 *Let Φ be an Orlicz function such that $\Phi(u)/u \to \infty$ as $u \to \infty$. The Orlicz space L^0_Φ equipped with the Orlicz norm is uniformly monotone if and only if $\Phi \in \Delta_2$ and Φ vanishes only at zero.*

The proof of this theorem is similar to that for Theorem 4.4 and so will be omitted.

Notes. B-convexity for Musielak-Orlicz spaces was characterized in [H-Ka 85] and for Orlicz-Lorentz spaces in [H-Ka-M 96]. Uniform monotonicity of Musielak-Orlicz spaces was characterized in [H-Ka-Ku 87], [Cu-H-W] and [Ku 92]. For Orlicz-Lorentz spaces it was done in [H-Ka 95]. In some Calderon-Lozanowsky spaces and Banach lattices monotonicity properties were considered in [C-H-M 95], [F-H 97], [F-H 99] and [H-Ka-M 00].

5. Nearly uniform convexity and nearly uniform smoothness

First we introduce the notions of nearly uniform convexity, k-nearly uniform convexity and nearly uniform smoothness.

For a given $\varepsilon > 0$ a sequence $\{x_n\}$ in a Banach space X is said to be *ε-separated* if

$$\operatorname{sep}(\{x_n\}) := \inf_{m \neq n} \|x_m - x_n\| > \varepsilon.$$

A Banach space X is called *nearly uniformly convex* (NUC) if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every sequence $\{x_n\}$ in $B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$ there is an element $x \in \operatorname{co}(\{x_n\})$ such that $\|x\| < 1 - \delta$. This notion was introduced by Huff [Hu 80], where it was also proved that a Banach space X is NUC if and only if it is reflexive and it has the *uniform Kadec-Klee property* (UKK). Recall that a Banach space X is said to have the UKK - property if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any sequence $\{x_n\}$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$ and any $x \in B(X)$, we have $\|x\| \leq 1 - \delta$ whenever $x_n \to x$ weakly.

It is well known that NUC Banach spaces have the FPP (see [Go-Ki 90]). The property NUC has also been defined by using the measure of noncompactness by Goebel and Sękowski [Go-Se 84].

Kutzarova [Kur 30] introduced the notion of *k-nearly uniform convexity* of Banach spaces (k-NUC). Let k be an integer, $k \geq 2$. A Banach space X is said to be k-NUC if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\}$ in $B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbf{N}$ such that $\|(x_{n_1} + x_{n_2} + \cdots + x_{n_k})/k\| < 1 - \delta$. Clearly k-NUC Banach spaces are NUC but the opposite implication does not hold in general (see [Cu-H-Li]).

The notion of *nearly uniform smoothness* (NUS) has been introduced by Sękowski and Stachura [Se-St 82]. The definition uses the notion of the measure of noncompactness. Prus [Pr 89], [Pr 99] used another (equivalent) definition of this property which is easier to formulate. Namely a Banach space X is said to be NUS if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each basic sequence $\{x_n\}$ in $B(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + t\varepsilon$$

for each $t \in [0, \delta]$. Prus [Pr 89] showed that a Banach space X is NUC if and only if its dual space X^* is NUS.

A natural generalization of NUS is WNUS where the condition "for every $\varepsilon > 0$" in the definition of NUS is replaced by "for some $\varepsilon \in (0, 1)$". Let A be a bounded set of X. Its *Kuratowski measure of non-compactness* $\alpha(A)$ is defined as the infimum of all numbers $d > 0$ such that A may be covered by finitely many sets of diameter smaller than d.

A Banach space X is said to be *nearly uniformly $*$-smooth* provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S(X)$, then

$$\alpha(S^*(x, \delta)) \leq \varepsilon,$$

where $S^*(x,\delta) = \{x^* \in B(X^*): x^*(x) \geq 1-\delta\}$. NUC and NUS have been also studied by Banaś [B 87], [B 91].

Also nearly uniform smoothness and weakly nearly uniform smoothness are related to the fixed point theory as it follows from the following.

Theorem 5.1 (see [Ga 97]) *If X is a WNUS Banach space, then X has the FPP. In particular, NUS Banach space have the FPP.*

In order to get criteria for *NUS* of Orlicz spaces it is natural to present first criteria for *NUC* of these spaces because *NUC* and *NUS* are dual properties (see [Pr 89]).

Since k-NUC implies NUC we present first criteria for k-NUC of Orlicz spaces given in [Cu-H-Li].

Theorem 5.2 *Let (T,Σ,μ) be a nonatomic and finite measure spaces and Φ be a Orlicz function satisfying $(\Phi(u)/u) \to \infty$ as $u \to \infty$ and X be equal to L_Φ or L^0_Φ. Then X is k-NUC if and only if Φ is a strictly convex and satisfies the Δ_2-condition at infinity and Φ is uniformly convex outside a neighbourhood of zero.*

Corollary 5.3 *Under the assumptions of Theorem 5.2 on μ and Φ, the spaces L_Φ and L^0_Φ are NUC if and only if both Φ and Ψ (where Ψ is the Young conjugate of Φ) satisfy the condition Δ_2 at infinity.*

Proof. It follows directly from the facts that k-NUC implies NUC, NUC implies reflexivity and reflexivity of L_Φ (respectively, L^0_Φ) is equivalent to the fact that both Φ and Ψ satisfy the suitable Δ_2-condition. ∎

Theorem 5.4 *The Orlicz sequence space l_Φ is k-NUC if and only if both Φ and Ψ satisfy the Δ_2-condition at zero, that is l_Φ is reflexive.*

Proof. We need only to prove the sufficiency of the theorem . Suppose that the implication is not true. Let any $\varepsilon > 0$ and $\{x_n\} \subset B(l_\Phi)$ with sep$(x_n)> \varepsilon$ be given. By $\Phi \in \Delta_2(0)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\inf\left\{I_\Phi\Big(\frac{x_n - x_m}{2}\Big) : \ n \neq m\right\} \geq \delta.$$

Next, we will show that for any $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that

$$\sum_{i=j}^{\infty} \Phi(x_{n_j}(i)) \geq \frac{\delta}{3}. \tag{5.1}$$

Otherwise, there exists $j_0 \in \mathbf{N}$ such that

$$\sum_{i=j}^{\infty} \Phi(x_{n_j}(i)) < \frac{\delta}{3}$$

for any $j \in \mathbf{N}$.

Defining $\overline{x}_n = (x_n(1), x_n(2), \cdots, x_n(j_0), 0, 0, \dots)$ for $n \in \mathbf{N}$, we easily see that there exists a subsequence $\{\overline{x}_{n_k}\}$ of $\{\overline{x}_n\}$ such that

$$I_\Phi\left(\frac{\overline{x}_{n_i} - \overline{x}_{n_j}}{2}\right) < \frac{\delta}{3}$$

for any $i \neq j$. Hence

$$
\begin{aligned}
& I_\Phi\left(\frac{x_{n_i}-x_{n_j}}{2}\right) \\
&= I_\Phi\left(\frac{\sum_{k=1}^{j_0}(x_{n_i}(k)-x_{n_j}(k))e_k}{2}\right)+I_\Phi\left(\frac{\sum_{k=j_0+1}^{\infty}(x_{n_i}(k)-x_{n_j}(k))e_k}{2}\right) \\
&\leq I_\Phi\left(\frac{\sum_{k=1}^{j_0}(x_{n_i}(k)-x_{n_j}(k))e_k}{2}\right)+\frac{1}{2}\sum_{k=j_0+1}^{\infty}\Phi(x_{n_i}(k))+\frac{1}{2}\sum_{k=j_0+1}^{\infty}\Phi(x_{n_j}(k)) \\
&= I_\Phi\left(\frac{\overline{x}_{n_i}-\overline{x}_{n_j}}{2}\right)+\frac{1}{2}\sum_{k=j_0+1}^{\infty}\Phi(x_{n_i}(k))+\frac{1}{2}\sum_{k=j_0+1}^{\infty}\Phi(x_{n_j}(k)) \\
&< \frac{\delta}{3}+\frac{\delta}{6}+\frac{\delta}{6}=\frac{2}{3}\delta<\delta.
\end{aligned}
$$

This contradiction shows that (5.1) holds.

Since Ψ satisfies the Δ_2-condition at zero, there is $\Theta \in (0,1)$ such that

$$
\Phi\left(\frac{u}{k}\right) \leq (1-\Theta)\frac{\Phi(u)}{k} \qquad (\forall 0 \leq u \leq \Phi^{-1}(1)) \tag{5.2}
$$

(see [Ay-D-Lo 97] and [C-H-Ka-M 98]). By $\Phi \in \Delta_2(0)$, there exists $\sigma > 0$ such

$$
|I_\Phi(x+y)-I_\Phi(x)| < \frac{\Theta\delta}{6k} \tag{5.3}
$$

whenever $I_\Phi(x) \leq 1,\ I_\Phi(y) \leq \sigma$ (see [Ay-D-Lo 97], [Ch 96]).

Take $n_1 < n_2 < \cdots < n_{k-1};\ n_1, n_2, \cdots, n_{k-1} \in \mathbf{N}$. Notice that

$$
I_\Phi\left(\frac{x_{n_1}+x_{n_2}+\cdots+x_{n_{k-1}}}{k}\right) \leq 1
$$

and $I_\Phi(x_{n_i}) \leq 1$ for $i = 1, 2, \cdots, k-1$. There exists $j_0 \in \mathbf{N}$ such that

$$
\sum_{i=j_0+1}^{\infty}\Phi\left(\frac{x_{n_1}(i)+x_{n_2}(i)+\cdots+x_{n_{k-1}}(i)}{k}\right) < \sigma \tag{5.4}
$$

and

$$
\sum_{i=j_0+1}^{\infty}\Phi\left(x_{n_j}(i)\right) < \frac{\delta}{3} \quad (j = 1, 2, \ldots, k-1).
$$

By (5.1), there exists $n_k \in \mathbf{N}$ such that

$$
\sum_{i=j_0+1}^{\infty}\Phi(x_{n_k}(i)) \geq \frac{\delta}{3}. \tag{5.5}
$$

So, in virtue of (5.2), (5.3), (5.4) and (5.5), we get

$$
I_\Phi\left(\frac{x_{n_1}+\cdots+x_{n_k}}{k}\right)
$$

$$
\begin{aligned}
&= \sum_{i=1}^{j_0} \Phi\left(\frac{x_{n_1}(i)+\cdots+x_{n_k}(i)}{k}\right) + \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_1}(i)+\cdots+x_{n_k}(i)}{k}\right) \\
&\le \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} \Phi(x_{n_j}(i)) + \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_k}(i)}{k}\right) + \frac{\Theta\delta}{6k} \\
&\le \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} \Phi(x_{n_j}(i)) + \frac{1-\Theta}{k}\sum_{i=j_0+1}^{\infty} \Phi\left(x_{n_k}(i)\right) + \frac{\Theta\delta}{6k} \\
&= \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{\infty} \Phi(x_{n_j}(i)) - \frac{\Theta}{k}\sum_{i=j_0+1}^{\infty} \Phi\left(x_{n_k}(i)\right) + \frac{\Theta\delta}{6k} \\
&\le 1 - \frac{\Theta\delta}{3k} + \frac{\Theta\delta}{6k} = 1 - \frac{\Theta\delta}{3k},
\end{aligned}
$$

which completes the proof. ■

Theorem 5.5 *For any Orlicz function Φ satisfying $(\Phi(u)/u) \to \infty$ as $u \to \infty$ the Orlicz sequence space l^0_Φ is k-NUC if and only if both Φ and Ψ satisfy the Δ_2-condition at zero, that is l^0_Φ is reflexive.*

Proof. We only need to prove the sufficiency. Let any $\varepsilon > 0$ and $\{x_n\} \subset B(l^0_\Phi)$ with sep$(\{x_n\}) > \varepsilon$ be given. By $\Phi \in \Delta_2(0)$, there exists $\delta > 0$ such that

$$\inf\left\{ I_\Phi\Big(\frac{x_n - x_m}{2}\Big) \,:\, n \ne m \right\} \ge \delta.$$

By the same argument as in Theorem 5.4, we have that for any $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that

$$\sum_{i=j}^{\infty} \Phi(x_{n_j}(i)) \ge \frac{\delta}{3}. \tag{5.6}$$

Take $k_n \ge 1$ satisfying

$$\|x_n\|_0 = \frac{1}{k_n}\left(1 + I_\Phi(k_n x_n)\right) \quad (n = 1, 2, \ldots).$$

Such numbers $k_n \ge 1$ exist by the assumption $(\Phi(u)/u) \to \infty$ as $u \to \infty$ (see [Ch 96]). Since Ψ satisfies the Δ_2-condition at zero, the number

$$k_0 = \sup\{k_n : n = 1, 2, \ldots\}$$

is finite (see [Ay-D-Lo 97]). Fix $n_1 < n_2 < \cdots < n_{k-1}$; $n_1, n_2, \ldots, n_{k-1} \in \mathbf{N}$. For any $n_k \in \mathbf{N}$, put

$$H = \prod_{i=1}^{k} k_{n_i}, \; h_j = \prod_{i \ne j} k_{n_i}, \; h = \prod_{i=1}^{k} \frac{k_{n_i}}{\sum_{j=1}^{k} h_j} \text{ and } \lambda = \frac{k_0^{k-1}}{k_0^{k-1}+1}.$$

By $\Phi \in \Delta_2(0)$, there exists $\Theta \in (0,1)$ such that

$$\Phi(\lambda u) \le (1-\Theta)\lambda\Phi(u) \quad \text{whenever} \quad 0 \le u \le \Phi^{-1}(k_0).$$

Since Φ is convex, for any $l \in [0, \lambda]$ and $u \in [0, \Phi^{-1}(k_0)]$, we have

$$\Phi(lu) = \Phi\left(\lambda \frac{l}{\lambda} u\right) \le (1-\Theta)\lambda\Phi\left(\frac{l}{\lambda}u\right)$$
$$\le \lambda(1-\Theta)\frac{l}{\lambda}\Phi(u) \le (1-\Theta)l\Phi(u).$$

Since $\frac{h_k}{\sum_{i=1}^{k} h_i} = \frac{h_k}{h_k + \sum_{i=1}^{k-1} h_i} \le \frac{k_0^{k-1}}{1+k_0^{k-1}} = \lambda$, the following holds

$$\frac{h_k}{\sum_{i=1}^{k} h_i} u \le (1-\Theta)\frac{h_k}{\sum_{i=1}^{k} h_i}\Phi(u) \tag{5.7}$$

whenever $0 \le u \le \Phi^{-1}(k_0)$. By $\Phi \in \Delta_2(0)$, there exists $\sigma > 0$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \frac{\Theta k_0^k}{1+k_0^{k-1}} \cdot \frac{\delta}{6}$$

if $I_\Phi(x) \le k_0$ and $I_\Phi(y) \le \sigma$ (see [Ay-D-Lo 97] and [Pr 89]). Consequently, we need only to prove that (ii) $\Rightarrow$ (iii). We will show that (ii) implies the Δ_2-condition at zero for Φ. If Φ does not satisfy the Δ_2-condition at zero, we can construct $x \in S(l_\Phi)$ such that $I_\Phi(x) \le 1$ and $I_\Phi((1+\frac{1}{n})x) = \infty$ for every $n \in \mathbf{N}$ (see [Ch 96] and [Ka 82a]). Take a sequence $\{i_k\}$ of natural numbers such that $i_k \uparrow$ and

$$\sum_{i=i_k+1}^{i_{k+1}} \Phi\left(\left(1+\frac{1}{k}\right)x(i)\right) \ge 1 \quad (k \in \mathbf{N}).$$

Put

$$x_k = (0, 0, \dots, 0, x(i_k+1), x(i_k+2), \dots, x(i_{k+1}), 0, 0, \dots) \quad (k \in \mathbf{N}).$$

Then it is obvious that

$$\frac{k}{k+1} \le \|x_k\| \le 1 \quad (k \in \mathbf{N}).$$

Moreover,

$$x_k \to 0 \quad \text{weakly}. \tag{5.8}$$

Indeed, for every $y^* \in (l_\Phi)^*$ we have $y^* = y_0^* + y_1^*$ uniquely, where y_0^* is the order continuous part of y^* and y_1^* is the singular part of y^*. That is $y_1^*(x) = 0$ for any $x \in h_\Phi$ (see [Ch 96]). The functional y_0^* is generated by some $y_0 \in l_\Psi$ by the formula

$$y_0^*(x) = \langle x, y_0 \rangle = \sum_{i=1}^{\infty} x(i)y_0(i) \quad (x \in l_\Phi).$$

Let $\lambda > 0$ be such that $\sum_{i=1}^{\infty} \Psi(\lambda y_0(i)) < \infty$. Since $x_k \in h_\Phi$ for any $k \in \mathbf{N}$, we have

$$\langle x_k, y^* \rangle = \langle x_k, y_0^* \rangle = \sum_{i=i_k+1}^{i_{k+1}} x(i)y_0(i)$$
$$\le \frac{1}{\lambda}\left(\sum_{i=i_k+1}^{i_{k+1}} \Phi(x(i)) + \sum_{i=i_k+1}^{i_{k+1}} \Psi(\lambda y_0(i))\right)$$

$$\to 0 \quad \text{as} \quad k \to \infty,$$

that is (5.8) holds.

Since the space l_Φ is nearly uniformly $*$-smooth, it has property A_2^ε, that is for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each weakly null sequence (z_n) in $B(l_\Phi)$ there is $m > 1$ such that

$$\|z_1 + tz_m\| \leq 1 + t\varepsilon$$

whenever $t \in [0,\delta]$ (see [Pr 89] and [Pr 99]). Take $k_0 \in \mathbf{N}$ such that $\frac{2}{k+1} < (1-\varepsilon)\delta$ if $k \geq k_0$. We have for $k \geq k_0$,

$$\begin{aligned} 1 + \delta\varepsilon \geq \|x + \delta x_k\| \geq \|(1+\delta)x_k\| \geq (1+\delta)\frac{k}{k+1} \\ = (1+\delta)\Big(1 - \frac{1}{k+1}\Big) > 1 + \delta - \frac{2}{k+1}, \end{aligned}$$

whence $\frac{2}{k+1} > (1-\varepsilon)\delta$. This is a contradiction which finishes the proof of the fact that (ii) implies the Δ_2-condition at zero for Φ.

Next, we will show that (ii) implies the δ_2-condition for Ψ. By the above part of the proof, we can assume that l_Ψ is nearly uniformly $*$-smooth and Φ satisfies the Δ_2-condition at zero. So, l_Φ is order continuous. Moreover, any Orlicz space l_Φ has the Fatou property and consequently, it is weakly sequentially complete. So, in view of Corollary 5.3, l_Φ is nearly uniformly smooth and consequently reflexive. This yields the Δ_2-condition at zero for Ψ. ∎

6. WORTH and uniform nonsquareness

Garcia-Falset [Ga 94] has proved that if a Banach space X has WORTH and is uniformly nonsquare, then X has the FPP.

So, we will present now criteria for uniform nonsquareness in Orlicz spaces and criteria for WORTH in Köthe sequence spaces. We say following Sims [S 88] that a Banach space X has *WORTH* if for any $x \in S(X)$ and any weakly null sequence (x_n) in X, we have

$$\lim_{n\to\infty}\Big|\, \|x_n + x\| - \|x_n - x\| \,\Big| \to 0.$$

Let ℓ^0 be an Orlicz sequence space. A Banach space $X \subset \ell^0$ is said to be a *Köthe sequence space* (or a *Banach sequence lattice*) if there is a sequence $x = (x(i))_{i=1}^\infty \in X$ with all $x(i) \neq 0$ and for every $x \in \ell^0$ and $y \in X$ with $|x(i)| \leq |y(i)|$ for all $i \in \mathbf{N}$ it follows that $x \in X$ and $\|x\| \leq \|y\|$.

Theorem 6.1 (see [Cu-H-P 99]) *A Köthe sequence space X has WORTH if and only if it is order continuous.*

In this section we do not assume that Φ satisfies condition (ii) from the definition of an Orlicz function.

Corollary 6.2 *Orlicz sequence spaces l_Φ equipped with the Luxemburg norm or with the Orlicz norm have WORTH if and only if $\Phi \in \Delta_2(0)$.*

Proof. Since order continuity of l_Φ and l_Φ^0 is equivalent to $\Phi \in \Delta_2(0)$, the corollary follows immediately from Theorem 6.1. ∎

The notion of uniform nonsquareness of a Banach space was introduced by James [J 64]. Recall that a Banach space X is said to be *uniformly nonsquare* (UNSQ) if there is $\varepsilon \in (0,1)$ such that for every $x, y \in B(X)$ there holds

$$\min\left(\left\|\frac{x+y}{2}\right\|, \left\|\frac{x-y}{2}\right\|\right) \leq 1-\varepsilon.$$

Theorem 6.3 (see [H 85], [H-Ka-Mu 88] and [Su 66])

(a) *In the case of a nonatomic infinite and σ-finite measure space as well as in the case of the counting measure space the Orlicz space L_Φ equipped with the Luxemburg norm is uniformly nonsquare if and only if it is reflexive.*

(b) *In the case of any finite nonatomic measure space the Orlicz space L_Φ equipped with the Luxemburg norm is uniformly nonsquare if and only if L_Φ is reflexive and $\Phi(b(\Phi))\mu(T) < 2$, where $b(\Phi) = \sup\{u \geq 0\colon\ \Phi$ is linear on the interval $[0,u]\}$.*

Note. Uniform nonsquareness of Musielak-Orlicz spaces was characterized in [H-Ka-Ku 87]. For Orlicz-Lorentz spaces it was done in [H-Ka-M 96], where uniform nonsquareness of some Calderon-Lozanowsky spaces was also considered.

Note. The characteristic of convexity of Orlicz function spaces equipped with the Luxemburg norm was calculated in [H-Ka-Mu 88] in the case when the measure spaces is nonatomic and infinite. Recall that this coefficient for a Banach space X is defined by

$$\varepsilon_0(X) = \inf\{\varepsilon \in (0,2]\colon\ \delta_X(\varepsilon) > 0\},$$

where δ_X denotes the modulus of convexity of X. In the case of nonatomic and finite measure space $\varepsilon_0(L^\Phi)$ was calculated in [H-W-Wa 92]. Lower and upper estimates for the characteristic of convexity of Köthe-Bochner spaces were given in [H-Lan 92].

7. Opial property and uniform Opial property in modular sequence spaces

In this section we will present some results on the uniform Opial property of modular sequence spaces. As a corollary we will obtain criteria for the Opial property and the uniform Opial property of Orlicz sequence spaces for both the Luxemburg and Orlicz norms.

Let X be a real vector space. A functional $m\colon X \to [0,\infty]$ is called a *modular* if (see [Mu 83] and [Mal 89]):

(i) $m(x) = 0$ if and only if $x = 0$,

(ii) $m(-x) = m(x)$ for all $x \in X$,

(iii) $m(\alpha x + \beta y) \leq \alpha m(x) + \beta m(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ (that is m is convex).

For any modular m on X, the space

$$X_m = \{x \in X\colon\ m(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\}$$

is called the *modular space* (generated by m). It is obvious that X_m is a vector space. The functional

$$\|x\| = \inf\{\lambda > 0 \colon \; m(x/\lambda) \le 1\}$$

is a norm on X_m, which is called the *Luxemburg norm* (see [Cu-H 98a] and [Cu-H 99a]).
A modular m is said to satisfy the Δ_2*-condition* ($m \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \ge 2$ and $a > 0$ such that

$$m(2x) \le Km(x) + \varepsilon$$

for all $x \in X_m$ with $m(x) \le a$.

If m satisfies the Δ_2-condition for any $a > 0$ with $K \ge 2$ dependent on a, we say that m satisfies the *strong* Δ_2-condition ($m \in \Delta_2^s$).

In this section a function $\Phi \colon (-\infty, \infty) \to [0, \infty)$ is said to be an Orlicz function if it is convex, even and $\Phi(0) = 0$ (see [Ch 96], [Lu 55], [Mal 89], [Mu 83], [Kr-R 61] and [Ra-Re 91]). For a given Orlicz function Φ one can define on the space l^0 of all real sequences $x = (x(i))$ the modular

$$m_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i)).$$

The modular space $(\ell^0)_{m_\Phi}$ is called an Orlicz sequence space (see [Ch 96], [Kr-R 61], [Lu 55], [Mal 89], [Mu 83] and [Ra-Re 91]).

It is easy to see that if Φ vanishes only at zero, then $m_\Phi \in \Delta_2^s$ whenever $\Phi \in \Delta_2(0)$.

Let X be a Banach sequence space (or Köthe sequence space), an element $x \in X$ is said to be *absolutely continuous* if

$$\lim_{n\to\infty} \|(0, \dots, 0, x(n+1), x(n+2), \dots)\| = 0.$$

The set of all absolutely continuous elements in X is denoted by X_a and it is a subspace of X. X is called absolutely continuous if $X_a = X$.

We say that a Banach sequence lattice X has the *Fatou property* if for any $x \in X$ and a sequence $\{x_n\}$ in X such that $0 \le x_n \le x$ and $x_n \uparrow x$, there holds $\|x_n\| \uparrow \|x\|$ (for the theory of Köthe sequence spaces we refer to [Kan-Aki 72]). A Banach space X is said to have the *Opial property* (see [O 67]) if for every weakly null sequence $\{x_n\}$ and every $x \ne 0$ in X there holds

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\|.$$

The Opial property is important because Banach spaces with this property have the weak fixed point property (see [G-La 72]).

Opial has proved in [O 67] that the Lebesgue sequence spaces ℓ_p $(1 < p < \infty)$ have this condition but $L_p[0, 2\pi]$ $(p \ne 2,\ 1 < p < \infty)$ do not have it. Franchetti [Fr 81] has shown that any infinite-dimensional Banach space admits an equivalent norm under which it has the Opial property. A Banach space X is said to be the *uniform Opial property* (see [Pr 92]) if for every $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence $\{x_n\}$ in $S(X)$ and $x \in X$ with $\|x\| \ge \varepsilon$ there holds

$$1 + \tau \le \liminf_{n\to\infty} \|x_n + x\|.$$

Let β be the *ball-measure* (that is the *Hausdorff measure*) *of noncompactness* in X, that is

$$\beta(A) = \inf\{\varepsilon > 0:\ \text{A can be covered by a finite family of sets of diameter} \leq \varepsilon\}$$

for any $A \subset X$. A Banach space X is said to have property (L) if $\lim_{\varepsilon\to 1^-} \Delta(\varepsilon) = 1$, where

$$\Delta(\varepsilon) = \inf\{1 - \inf\{\|x\|:\ x \in A\}\},$$

and the first infimum is taken over all closed sets A in the unit ball $B(X)$ of X with $\beta(A) \geq \varepsilon$.

The function Δ is called the *modulus of noncompact convexity* (see [Go-Se 84]). It has been proved in [Pr 92] that property (L) is useful to study the fixed point property and that a Banach space X has property (L) if and only if it is reflexive and has the uniform Opial property. We start with the following auxiliary lemma.

Lemma 7.1 *Assume that $m \in \Delta_2^s$. Then for every $L > 0$ and $\varepsilon > 0$ there exists $\delta = \delta(L, \varepsilon) > 0$ such that for all x, y in X_m with $m(x) \leq L$ and $m(y) \leq \delta$, there holds $|m(x + y) - m(y)| < \varepsilon$.*

Proof. Let $L > 0$ and $\varepsilon > 0$ be given. By $m \in \Delta_2^s$, we conclude that there is $K_0 \geq 2$ such that

$$m(2x) \leq K_0 m(x) + \varepsilon/8$$

for all $x \in X_m$ with $m(x) \leq L$. Set $\beta = \varepsilon/2K_0L$. Using again $m \in \Delta_2^s$, one can find $K_1 \geq 2$ such that

$$m\left(\frac{2}{\beta}x\right) \leq K_1 m(x) + \varepsilon/8$$

for all $x \in X_m$ with $m(x) \leq L$. Set $\delta = \varepsilon/2\beta K_1$ and assume that $m(x) \leq L$ and $m(y) \leq \delta$. Then

$$\begin{aligned}
m(x + y) &= m((1 - \beta)x + \beta(x + \beta^{-1}y)) \\
&\leq (1 - \beta)m(x) + \beta m(x + \beta^{-1}y) \\
&\leq m(x) + \beta m(2^{-1}(2x + 2\beta^{-1}y)) \\
&\leq m(x) + 2^{-1}\beta m(2x) + 2^{-1}\beta m(2\beta^{-1}y) \\
&\leq m(x) + 2^{-1}\beta K_0 L + \frac{\varepsilon}{8} + 2^{-1}\beta m(y) + \frac{\varepsilon}{8} \\
&< m(x) + \varepsilon.
\end{aligned}$$

In a similar way we can show that $m(x) - \varepsilon < m(x + y)$. Hence $|m(x + y) - m(x)| < \varepsilon$ whenever $m(x) \leq L$ and $m(y) \leq \delta$, which finishes the proof. ■

Corollary 7.2 *If $m \in \Delta_2^s$, then for any $x \in X_m$, $\|x\| = 1$ if and only if $m(x) = 1$.*

Proof. We only need to show that $\|x\| = 1$ implies $m(x) = 1$ because the opposite implication is obvious. Assume that $m \in \Delta_2^s$. We can easily get from Lemma 7.1 that the function f defined on $\mathbf{R}$ by $f(\lambda) = m(\lambda x)$ is continuous. Namely, it easily follows by $m \in \Delta_2^s$ that f is finitely valued, which yields that f is continuous. Take any $\varepsilon > 0$

and $\lambda_0 \in \mathbf{R} \setminus \{0\}$, and apply Lemma 7.1 which $L = m(\lambda_0 x)$ and $\delta = \delta(L, \varepsilon)$. We have $|m(x+y) - m(x)| < \varepsilon$ whenever $m(x) \leq L$ and $m(y) \leq \delta$. Hence

$$|m(\lambda x) - m(\lambda_0 x)| = |m((\lambda - \lambda_0)x + \lambda_0 x) - m(\lambda_0 x)| < \varepsilon$$

whenever $|\lambda - \lambda_0| < \delta$. So, we easily get that $m(x) = 1$ whenever $\|x\| = 1$. ■

Lemma 7.3 *If $m \in \Delta_2$, then for any sequence (x_n) in X_m the condition $\|x_n\| \to 0$ holds if and only if $m(x_n) \to 0$.*

Proof. It is easy to see that $\|x_n\| \to 0$ if and only if $m(\lambda x_n) \to 0$ for each $\lambda > 0$. By $m \in \Delta_2$ it follows from Lemma 7.1 that the property holds for sufficiently small positive L (say $L \leq L_0$). Assume that $m(x_n) \to 0$. There is $m \in \mathbf{N}$ such that $m(x_n) \leq L_0$ for all $n \geq m$. So, for any $\varepsilon \in (0,1)$ there is $K_\varepsilon > 0$ such that

$$m\left(\frac{x_n}{\varepsilon}\right) \leq K_\varepsilon m(x_n) + \frac{\varepsilon}{2}$$

for n sufficiently large. Let $n_0 \in \mathbf{N}$ be so large that $m(x_n) \leq 1/(2K_\varepsilon)$ for $n > n_0$. Hence $m(x_n/\varepsilon) \leq K_\varepsilon m(x_n) + \frac{1}{2} \leq 1$ for $n > n_0$ sufficiently large. This yields $\|x_n\| \leq \varepsilon$ for n sufficiently large. The opposite implication follows from the inequality $m(x) \leq \|x\|$ for x with $\|x\| \leq 1$. ■

Lemma 7.4 *If $m \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $m(x) \geq 1 + \varepsilon$.*

Proof. Suppose that there exist $\varepsilon_0 > 0$ and a sequence $\{x_n\}$ in X_m such that $\|x_n\| \downarrow 1$ and $m(x_n) \geq 1 + \varepsilon_0$. Since $m \in \Delta_2^s$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|m(x+y) - m(\varepsilon)| < \varepsilon$$

whenever $m(x) \leq 1$ and $m(y) \leq \delta$ (see Lemma 7.1). We may assume without loss of generality that $1 - 1/\|x_n\| < \delta$. Hence, applying the fact that $m(x_n/\|x_n\|) = 1$ for any $n \in \mathbf{N}$ (see the proof of Lemma 7.1), we get

$$\left| m\left(\left(1 - \frac{1}{\|x_n\|}\right)x_n + \frac{x_n}{\|x_n\|}\right) - m\left(\frac{x_n}{\|x_n\|}\right) \right| < \varepsilon.$$

That is $|m(x_n) - 1| < \varepsilon$. This contradiction shows that Lemma 7.4 is true. ■

Theorem 7.5 *Suppose that a Köthe sequence space X has the Fatou property. Then X is absolutely continuous whenever it has the uniform Opial property.*

Proof. Assume that X is not order continuous. Take $\varepsilon = 1/2$ and an arbitrary $\tau > 0$. Let $\theta = 1/(1 + \tau/2)$, whence $\theta > 1/(1+\tau)$. By Riesz's lemma (see [Ta-Lay 80], p. 64), for any $\theta \in (0,1)$ there is $x_0 \in S(X)$ such that $\|x_0 - x\| > \theta$ for any $x \in X_a$. Let x_0 corresponds to $\theta = 1/(1 + \tau/2)$. By the Fatou property of X,

$$\left\|\sum_{i=1}^{n} x_0(i)e_i\right\| \uparrow \|x_0\| = 1.$$

Let $n_1 = 0$. There is $n_2 \in \mathbf{N}$ such that

$$\left\|\sum_{i=1}^{n_2} x_0(i)e_i\right\| \geq \left(1 - \frac{1}{2}\right)\theta.$$

Since $\sum_{i=1}^{n_2} x_0(i)e_i \in X_a$, it follows that

$$\left\| \sum_{i=n_2+1}^{\infty} x_0(i)e_i \right\| \geq \theta.$$

So, there is $n_3 \in \mathbf{N}$, $n_3 > n_2$, such that

$$\left\| \sum_{i=n_2+1}^{n_3} x_0(i)e_i \right\| \geq \left(1 - \frac{1}{3}\right)\theta.$$

One can find by induction a sequence $(n_j)_{j=1}^{\infty}$ in $\mathbf{N}$ such that $n_1 = 0$, $n_1 < n_2 < \cdots$, and

$$\left\| \sum_{i=n_j+1}^{n_{j+1}} x_0(i)e_i \right\| \geq \left(1 - \frac{1}{j+1}\right)\theta > \left(1 - \frac{1}{j}\right)\theta \quad (j = 1, 2, \ldots).$$

Define

$$x_j = \sum_{i=n_j+1}^{n_{j+1}} x_0(i)e_i \quad (i = 1, 2, \ldots).$$

It is obvious that $x^*(x_j) = 0$, $j = 1, 2, \ldots$, for any singular functional $x^* \in X^*$. If $x^* \in X^*$ is order continuous, there is $y = (y(i))_{i=1}^{\infty} \in X'$ (the Köthe dual of X, see [Kan-Aki 72]) such that

$$x^*(z) = \sum_{i=1}^{\infty} y(i)z(i) \quad (\forall z = (z(i))_{i=1}^{\infty} \in X).$$

Since $\sum_{i=1}^{\infty} y(i)x_0(i)$ is convergent it follows that

$$x^*(x_j) = x^*\left(\sum_{i=n_j+1}^{n_{j+1}} x_0(i)e_i\right) = \sum_{i=n_j+1}^{n_{j+1}} y(i)x_0(i) \to 0$$

as $j \to \infty$. Therefore, $x_j \to 0$ weakly as $j \to \infty$. Moreover

$$\left(1 - \frac{1}{j}\right)\theta \leq \|x_j\| \leq 1 \quad (j = 1, 2, \ldots). \tag{7.1}$$

Define $y_j = x_j/\|x_j\|$. Then $\|y_j\| = 1$ for all $j \in \mathbf{N}$ and

$$\|x_0 - y_j\| \leq \left\| \frac{x_0 - x_j}{\|x_j\|} \right\| \leq \frac{\|x_0\|}{\|x_j\|} \leq \frac{1}{(1 - \frac{1}{j})\theta} \to \frac{1}{\theta}.$$

Since $1/\theta < 1 + \tau$, there is $j_0 \in \mathbf{N}$ such that

$$\|x_0 - y_j\| < 1 + \tau \quad (\forall j > j_0). \tag{7.2}$$

Since $x_j \to 0$ weakly, inequalities (7.1) yield that $y_j \to 0$ weakly. Hence and by (7.2) it follows that X does not have the uniform Opial property . This finishes the proof. ■

Theorem 7.6 *Assume that a modular $m \in \Delta_2^s$ and m is countably orthogonally additive and that the modular sequence space X_m is a Banach space. Then X_m has the uniform Opial property.*

Proof. Let $\varepsilon > 0$ be given. There is $\varepsilon_1 > 0$ such that (see Lemma 7.3) $m(x) \geq \varepsilon_1$ whenever $\|x\| \geq \varepsilon$. Since $m \in \Delta_2^s$, by Lemma 7.1, there is $\delta \in (0, \varepsilon_1/4)$ such that

$$|m(x+y) - m(x)| < \frac{\varepsilon_1}{8}$$

whenever $m(x) \leq 1$ and $m(y) \leq \delta$. By countable orthogonal additivity of m, there is $i_0 \in \mathbf{N}$ such that

$$m\left(\sum_{i=i_0+1}^{\infty} x(i)e_i\right) \leq \delta.$$

Let $\{x_n\}$ be a weakly null sequence in $S(X)$. It is obvious that $x_n \to 0$ coordinate-wise. Hence, there is $n_0 \in \mathbf{N}$ such that

$$m\left(\sum_{i=1}^{i_0} x_n(i)e_i\right) \leq \delta \quad (\forall n \geq n_0).$$

Therefore

$$\begin{aligned}
m(x_n + x) &= m\left(\sum_{i=1}^{i_i}(x_n(i) + x(i))e_i\right) + m\left(\sum_{i=1}^{\infty}(x_n(i) + x(i))e_i\right) \\
&\geq m\left(\sum_{i=1}^{i_0} x(i)e_i\right) - \frac{\varepsilon_1}{8} + m\left(\sum_{i=i_0+1}^{\infty} x_n(i)e_i\right) - \frac{\varepsilon_1}{8} \\
&\geq \frac{3}{4}\varepsilon_1 - \frac{\varepsilon_1}{8} - \frac{\varepsilon_1}{8} + m(x_n) - \frac{\varepsilon_1}{8} - \frac{\varepsilon_1}{8} \\
&= 1 + \frac{\varepsilon_1}{4}
\end{aligned}$$

for $n \geq n_0$. By Lemma 7.4, there is $\varepsilon_2 > 0$ that depends only on ε_1 and such that $\|x_n + x\| > 1 + \varepsilon_2$ whenever $n \geq n_0$. This means that X_m has the uniform Opial property. ■

In this section we write m in place of I_Φ in Orlicz spaces.

Corollary 7.7 *Orlicz sequence spaces ℓ^Φ equipped with the Luxemburg norm have the uniform Opial property if and only if $\Phi \in \Delta_2(0)$.*

Proof. *Sufficiency.* Orlicz spaces ℓ^Φ are Banach spaces and they are modular spaces $(\ell^0)_{m_\Phi}$, where

$$m_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i))$$

for $x = (x(i)) \in \ell^0$. If $\Phi \in \Delta_2(0)$, then m_Φ is countably orthogonally additive and $m_\Phi \in \Delta_2^s$. Therefore, by Theorem 7.6, ℓ^Φ have the uniform Opial property.

Necessity. If $\Phi \notin \Delta_2(0)$, then ℓ^Φ contains an order isometric copy of ℓ^∞ (see [Ka 82a]), so ℓ^Φ is not absolutely continuous. Since ℓ^Φ has the Fatou property (see [Lu 55]), by Theorem 7.5, ℓ^Φ does not have the uniform Opial property. ■

Corollary 7.8 *Orlicz sequence spaces l^Φ equipped with the Luxemburg norm have the Opial property if and only if $\Phi \in \Delta_2(0)$.*

Proof. If $\Phi \in \Delta_2(0)$, then by Corollary 7.7, l^Φ has the uniform Opial property, and hence has the Opial property as well.

Assuming that $\Phi \notin \Delta_2(0)$, one can find a sequence $\{x_n\}$ in $S(l^\Phi)$ such that $x_m \perp x_n$ for $m \neq n$ and

$$x = \sum_{n=1}^{\infty} x_n \in S(l^\Phi) \quad \text{(see [Ka 82a])}.$$

Then we easily get that $x_n \to 0$ weakly. However, $\|x_1 + x_n\| = 1$ for any $n > 2$. Consequently l^Φ fails to have the Opial property. ■

Corollary 7.9 *The Nakano sequence spaces $\ell^{(p_i)}$ with $1 \leq p_i < \infty$ for all $i \in \mathbf{N}$ have the uniform Opial property if and only if* $\limsup_{i\to\infty} p_i < \infty$.

Proof. The Nakano space $\ell^{(p_i)}$ is a Banach space and it is generated by the modular

$$m(x) = \sum_{i=1}^{\infty} |x(i)|^{p_i}$$

defined on ℓ^0 (see [Na 50] and [Mu 83]). If

$$\limsup_{i\to\infty} p_i < \infty$$

then $m \in \Delta_2^s$ and m is countably orthogonally additive. Therefore, by Theorem 7.6, $\ell^{(p_i)}$ has the uniform Opial property.

If

$$\limsup_{i\to\infty} p_i = \infty$$

then the Musielak-Orlicz function $\Phi = (\Phi)_{i=1}^{\infty}$, where $\Phi_i(u) = |u|^{p_i}$, does not satisfy the δ_2-condition (for the definition of $\Phi \in \delta_2$ see [Ka 82a] and [F-H 99]). Therefore (see [Ka 82a], [F-H 99] and [H 98]) $\ell^{(p_i)}$ contains an order isometric copy of ℓ^∞, whence $\ell^{(p_i)}$ is not absolutely continuous. Moreover, $\ell^{(p_i)}$ has the Fatou property whence, by Theorem 7.5, it follows that $\ell^{(p_i)}$ does not have the uniform Opial property, which finishes the proof. ■

For some other properties of $\ell^{(p_i)}$ we refer to [H-Wu-Y 94].

In the following we will consider the uniform Opial property for Orlicz spaces equipped with the *Amemiya norm*

$$\|x\|_\Phi^A = \inf_{k>0} \frac{1}{k}(1 + m_\Phi(kx)).$$

We write ℓ_A^Φ in place of $(\ell^\Phi, \|\cdot\|_\Phi^A)$. Denote by $K(x)$ the set of all $k > 0$ such that $\|x\|_\Phi^A = \frac{1}{k}(1 + m_\Phi(kx))$. It is known (see [Ch 96], [Ra-Re 91] and [Wu-Sun 91]) that $K(x) = [k_x^*, k_x^{**}]$, where $k_x^* = \inf\{k > 0\colon m_\Phi(p \circ kx) \geq 1\}$ and $k_x^{**} = \sup\{k > 0\colon m_\Phi(p \circ kx) \leq 1\}$ whenever $K(x) \neq \emptyset$ (that is $k_x^* < \infty$), where p denotes the right hand side derivative of Φ on $\mathbf{R}_+ = [0, \infty)$ and $p \circ kx$ denotes the composition of p and kx. It is also well known that $K(x) \neq \emptyset$ for all $x \in \ell_A^\Phi$ whenever $(\Phi(u)/u) \to \infty$ as $u \to \infty$ (see [Ch 96] and [Cu-H-N-P 99]).

The following lemma from [Cu-H-N-P 99] will be useful to get criteria for the uniform Opial property of ℓ_A^Φ.

Lemma 7.10 *If $x \in \ell^\Phi$ and $K(x) = \emptyset$, then $A := \lim_{u\to\infty}(\Phi(u)/u) < \infty$ and*

$$\|x\|_\Phi^A = A\sum_{i=1}^{\infty}|x(i)|.$$

Theorem 7.11 *The Orlicz space ℓ_A^Φ has the uniform Opial property if and only if $\Phi \in \Delta_2(0)$.*

Proof. Since ℓ_A^Φ is not absolutely continuous whenever $\Phi \notin \Delta_2(0)$, by Theorem 7.5, the necessity is obvious.

Sufficiency. Take any $\varepsilon > 0$ and $x \in \ell_A^\Phi$ with $\|x\|_\Phi^A \geq \varepsilon$. Let (x_n) be a weakly null sequence in $S(\ell_A^\Phi)$. By $\Phi \in \Delta_2(0)$ there is $\delta \in (0,\varepsilon)$ independent of x such that $m_\Phi\left(\frac{x}{2}\right) \geq \delta$. Take $j \in \mathbf{N}$ such that

$$\left\|\sum_{i=j+1}^{\infty} x(i)e_i\right\|_\Phi^A < \frac{\delta}{8}.$$

¿From $x_n \xrightarrow{w} 0$ it follows that $x_n(i) \to 0$ for any $i \in \mathbf{N}$. So, there exists $n_0 \in \mathbf{N}$ such that

$$\left\|\sum_{i=1}^{j} x_n(i)e_i\right\|_\Phi^A < \frac{\delta}{8} \quad (\forall n > n_0).$$

Hence

$$\begin{aligned}\|x + x_n\|_\Phi^A &= \left\|\sum_{i=1}^{j}(x(i) + x_n(i))e_i + \sum_{j=j+1}^{\infty}(x(i) + x_n(i))e_i\right\|_\Phi^A \\ &\geq \left\|\sum_{i=1}^{j}(x(i) + x_n(i))e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right\|_\Phi^A - \frac{\delta}{8} \\ &\geq \left\|\sum_{i=1}^{j} x(i)e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right\|_\Phi^A - \frac{\delta}{4} \qquad (7.3)\end{aligned}$$

whenever $n > n_0$. We will consider now two cases for $n > n_0$.

I. $K\left(\sum_{i=1}^{j} x(i)e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right) \neq \emptyset$. Then there exists $k_n > 0$ such that

$$\|x_n + x\|_\Phi^A = \frac{1}{k_n}\left(1 + m\left(k_n\left(\sum_{i=1}^{j} x(i)e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right)\right)\right).$$

Combining this with (7.3), we get

$$\begin{aligned}\|x_n + x\|_\Phi^A &\geq \frac{1}{k_n}\left(\sum_{i=1}^{j}\Phi(k_n x(i)) + \sum_{i=j+1}^{\infty}\Phi(k_n x_n(i))\right) - \frac{\delta}{4} \\ &= \frac{1}{k_n}\left(\sum_{i=1}^{\infty}\Phi(k_n x(i))\right) + \frac{1}{k_n}\sum_{i=j+1}^{j}\Phi(k_n x_n(i)) - \frac{\delta}{4}. \qquad (7.4)\end{aligned}$$

Moreover, from the inequalities

$$m\left(\sum_{i=j+1}^{\infty} x(i)e_i\right) \le \left\|\sum_{i=j+1}^{\infty} x(i)e_i\right\|_{\Phi}^{A} < \frac{\delta}{8} \quad \text{and} \quad m\left(\frac{x}{2}\right) \ge \delta,$$

it follows that

$$m\left(\sum_{i=1}^{j} x(i)e_i\right) > \frac{7\delta}{8}. \tag{7.5}$$

We may assume without loss of generality that $k_n \ge \frac{1}{2}$. Hence, inequalities (7.4) and (7.5) yield

$$\begin{aligned}\|x_n + x\|_{\Phi}^{A} &\ge \left\|\sum_{i=j+1}^{\infty} x_n(i)e_i\right\|_{\Phi}^{A} + 2\sum_{i=1}^{j}\Phi\left(\frac{1}{2}x(i)\right) - \frac{\delta}{4}\\ &\ge 1 - \frac{\delta}{8} + \frac{7\delta}{8} - \frac{\delta}{4} = 1 + \frac{\delta}{2}.\end{aligned}$$

II. $K\left(\sum_{i=1}^{m} x(i)e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right) = \emptyset$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{j} x(i)e_i + \sum_{i=j+1}^{\infty} x_n(i)e_i\right\|_{\Phi}^{A} &= A\sum_{i=1}^{j}|x(i)| + A\sum_{i=j+1}^{\infty}|x_n(i)|\\ &\ge \left\|\sum_{i=1}^{j} x(i)e_i\right\|_{\Phi}^{A} + \left\|\sum_{i=j+1}^{\infty} x_n(i)e_i\right\|\\ &> \frac{7\delta}{8} + 1 - \frac{\delta}{8} = 1 + \frac{3\delta}{4}.\end{aligned}$$

Therefore, by inequality (7.3) we get

$$\|x_n + x\|_{\Phi}^{A} \ge 1 + \frac{\delta}{2}.$$

So, in any case, $\|x_n + x\|_{\Phi}^{A} \ge 1 + \frac{\delta}{2}$ for $n > n_0$, which finishes the proof. ■

Corollary 7.12 *Orlicz spaces l_Φ generated by Orlicz functions Φ satisfying*

$$(\Phi(u)/u) \to \infty \quad as \quad u \to \infty$$

have normal structure if and only if $\Phi \in \Delta_2(0)$.

Proof. If $\Phi \in \Delta_2(0)$, then l_Φ^A has uniform normal structure, and so it has normal structure as well.

Assume that $\Phi \notin \Delta_2(0)$. Then there exist $x \in S(l_\Phi^0)$ and a sequence $\{x_n\}$ in $(l_\Phi^A)^+$ such that $x_m \perp x_n$ for $m \ne n$, $x_n \perp x$, $\{x_n\}$ has a majorant in $(l_\Phi^A)^+$ and $I_\Phi(k_0 x_n) \le 2^{-n}$ for any n, where $k_0 \ge 1$ satisfies $\frac{1}{k_0}(1 + m(k_0 x)) = \|x\|_\Phi^A = 1$, and $\|x_n\|_\Phi^A \to 1$ as $n \to \infty$. Then $x_n \to 0$ weakly. Moreover,

$$\frac{1}{k_0}(1 + m(k_0(x + x_n)) = \frac{1}{k_0}(1 + m(k_0 x)) + \frac{1}{k_0}m(k_0 x_n) \le 1 + 2^{-n}.$$

Therefore, $\lim_{n\to\infty} \|x_n + x_n\|_\Phi^A = \lim_{n\to\infty} \|x_n\|_\Phi^A = 1$. That is, l_Φ^0 does not have the Opial property. ■

8. Garcia – Falset coefficient

First we need to introduce some notation and definitions. Garcia-Falset [Ga 94] defined the coefficient

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n - x\|:\ x \in B(X),\ \{x_n\} \subset B(X),\ x_n \xrightarrow{w} 0\right\}$$

and proved in [Ga 97] that any Banach space X with $R(X) < 2$ has the weak fixed point property.

A Köthe sequence space X is said to have the *semi-Fatou property* ($X \in SFP$) if for every sequence $\{x_n\}$ in X and $x \in X$ such that $0 \le x_n \uparrow x$, we have $\|x_n\| \to \|x\|$.

Theorem 8.1 (see [H-M 93]) *If X is a Köthe sequence space with the semi-Fatou property and with the norm not being absolutely continuous, then X contains almost isometric copy of l_∞. That is, for any $\varepsilon > 0$ the exists a closed subspace Y of X and an isomorphism P of l_∞ onto Y which is a $(1+\varepsilon)$-isometry.*

Corollary 8.2 *If a Köthe sequence space with the semi-Fatou property is not absolutely continuous, then $R(X) = 2$.*

Proof. It is easy to see that $R(l_\infty) = 2$. Moreover, by Theorem 8.1, $R(X) = R(l_\infty)$. ■

Corollary 8.3 *If Φ does not satisfy the Δ_2-condition at zero, then $R(l_\Phi) = R(l_\Phi^0) = 2$.*

Proof. Each of the norms $\|\ \|$ and $\|\ \|^0$ have the semi-Fatou property (in fact they even have the Fatou property). Moreover, if $\Phi \notin \Delta_2(0)$, then l_Φ and l_Φ^0 are not absolutely continuous (see [Ch 96]). So, by Corollary 8.2, we get the desired conclusion. ■

Theorem 8.4 (see [Cu-H-Li 00]) *For any Orlicz function Φ, the equality*

$$R(h_\Phi) = \sup\left\{c_x\colon\ x = \sum_{i=1}^{m} x(i)e_i \in S(l_\Phi) \quad \textit{for some } m \in \mathbf{N}\right\}$$

holds, where c_x is positive number satisfying $I_\Phi(x/c_x) = 1/2$.

Remark 8.5 Note that $R(X) = R_1(X)$ for any Köthe sequence space with the semi-Fatou property and an absolutely continuous norm.

Corollary 8.6 *For any Lebesgue sequence space l_p $(1 < p < \infty)$, we have $R(l_p) = 2^{1/p}$.*

Proof. For any $x \in S(l_p)$ we have $c_x = 2^{1/p}$, which follows by the equalities

$$I_p\left(\frac{x}{c_x}\right) = \sum_{i=1}^{\infty}\left|\frac{x(i)}{c_x}\right|^p = \frac{1}{c_x^p}\sum_{i=1}^{\infty}|x(i)|^p = \frac{1}{c_x^p} = \frac{1}{2}.$$

■

To formulate the next corollary we need an equivalent definition of the Riesz angle for a Banach lattice X. It is defined by

$$\alpha(X) = \sup\{\||x| \vee |y|\| : x, y \in B(X), |x| \wedge |y| = 0\}.$$

Corollary 8.7 *For any Orlicz function Φ, the equality $R(h_\Phi) = \alpha(h_\Phi)$ holds.*

Proof. By the equality $R(h_\Phi) = d$ that was obtained in the proof of Theorem 8.4, we can easily get the inequality $R(h_\Phi) \leq \alpha(h_\Phi)$. On the other hand, for any $\varepsilon > 0$ there exist $x \in S(h_\Phi)$ and $y \in S(h_\Phi)$ such that $|x| \wedge |y| = 0$ and

$$\||x| \vee |y|\| > \alpha(l_\Phi) - \varepsilon.$$

For the sake of convenience, we may assume that $x \vee y = (x(1), y(1), x(2), y(2), \cdots)$. By the fact that h_Φ has an absolutely continuous norm, there exists $i_0 \in \mathbf{N}$ such that

$$\|(|x|(1), |y|(1), \cdots ; |x|(i_0), |y|(i_0), 0, 0, \cdots)\| \geq \alpha(h_\Phi) - \varepsilon.$$

Defining $x_0 = (x(1), x(2), \cdots, x(i_0), 0, 0, \cdots)$ and $y_n = (\overbrace{0, \cdots, 0,}^{(i_0+n)th} |y|(1), |y|(2), \cdots, |y|(i_0), 0, 0, \cdots)$ for all $n \in \mathbf{N}$, we get $y_n \stackrel{w}{\rightharpoonup} 0$ and

$$\liminf_{n\to\infty} \|y_n + x_0\| \geq \alpha(h_\Phi) - \varepsilon.$$

Hence $R(X) \geq \alpha(X) - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get $R(h_\Phi) \geq \alpha(h_\Phi)$ and consequently $R(h_\Phi) = \alpha(h_\Phi)$. ∎

Corollary 8.8 *For any Orlicz sequence space l_Φ, $R(l_\Phi) < 2$ if and only if $\Phi \in \Delta_2(0)$ and $\Psi \in \Delta_2(0)$.*

Corollary 8.9 *For any Orlicz function Φ, $R(h_\Phi) < 2$ if and only if $\Psi \in \Delta_2(0)$.*

Proof. By Corollary 8.2 and Theorem 3.11 in [Ch 96], which says that if $\Psi \in \Delta_2(0)$, then h_Φ has the $w - FPP$, both Corollaries 8.8 and 8.9 follow. ∎

Notes. It is known that property (β) which has been introduced by Rolewicz [Ro 87] is stronger than NUC and it implies normal structure of the dual space (see [Kut-Ma-Pr 92]). Property (β) has been considered in Orlicz-Bochner spaces, Musielak-Orlicz sequence spaces of Bochner type, Orlicz-Lorentz spaces and Calderón-Lozanovskiĭ spaces in [Ko-a], [Ko-b] and [Ko-c]. Properties UKK and NUC in Köthe-Bochner spaces have been considered in [Ko-d].

9. Cesaro Sequence Spaces

For $1 \leq p < \infty$ the Cesaro sequence space ces_p is defined by

$$\mathrm{ces}_p = \left\{ x \in l^0 \colon \|x\| = \left[\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i)|\Big)^p p\right]^{\frac{1}{p}} < \infty \right\}$$

(see [Lee 84] and [Sh 70]).

Lemma 9.1 (see [Cu-H 00b]) *For any $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that*

$$\Big|\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i) + y(i)|\Big)^p - \sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i)|\Big)^p\Big| < \varepsilon$$

whenever $1 \le p < \infty$, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p \le L$ *and* $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|y(i)|\right)^p \le \delta$.

Proof. It follows by the uniform continuity of the function $f(u) = u^p$ on any compact interval $[0, l]$. ■

Theorem 9.2 *For the Cesaro sequence space* ces_p $(1 < p < \infty)$ *we have* $R(\mathrm{ces}_p) = 2^{1/p}$.

Proof. We can apply Remark 8.5. Let $\varepsilon > 0$ be given. For any

$$\left\{x_n = \sum_{i=I_{n-1}+1}^{I_n} x_n(i)e_i\right\}_{n=2}^{\infty} \subset S(\mathrm{ces}_p),\ x_n \xrightarrow{w} 0,\ x = \sum_{i=1}^{I_1} x(i)e_i \in S(\mathrm{ces}_p), I_1 < I_2 < \cdots,$$

there exists $n_0 \in \mathbf{N}$ such that

$$\sum_{k=i_{n_0}+1}^{\infty}\left(\frac{a}{k}\right)^p < \min(\varepsilon, \delta), \quad \text{where } a = \sum_{i=1}^{I_1}|x(i)|$$

and $\delta > 0$ is the number corresponding to our $\varepsilon > 0$ and $L = 1$ in Lemma 9.1. Hence for any $m > n_0$ there holds

$$\begin{aligned}
\|x_m - x\|^p &= \sum_{k=1}^{I_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^p + \sum_{k=I_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(a + \sum_{i=1}^{k}|x_m(i)|\right)\right)^p \\
&\ge \sum_{k=1}^{I_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^p + \sum_{k=I_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_m(i)|\right)^p \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^p - \sum_{k=I_{m-1}+1}^{\infty}\left(\frac{a}{k}\right)^p + \sum_{k=I_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_m(i)|\right)^p \\
&> 1 - \varepsilon + 1 = 2 - \varepsilon.
\end{aligned}$$

That is, $\liminf_{n\to\infty}\|x_n - x\| \ge (2-\varepsilon)^{\frac{1}{p}}$. On the other hand, for any $m > n_0$,

$$\begin{aligned}
\|x_m - x\|^p &= \sum_{k=1}^{I_{m-1}}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(a + \sum_{i=1}^{k}|x_m(i)|\right)\right)^p \\
&\le \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\right)^p + \sum_{k=I_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}|x_m(i)|\right)^p + \varepsilon = 2 + \varepsilon.
\end{aligned}$$

That is, $\liminf_{n\to\infty}\left\|x_n - x\right\| \le (2+\varepsilon)^{\frac{1}{p}}$. By the arbitrariness of $\varepsilon > 0$ and by Remark 8.5, we get $R(\mathrm{ces}_p) = 2^{\frac{1}{p}}$. ■

Corollary 9.3 *Cesaro sequence spaces* ces_p $(1 < p < \infty)$, *have the fixed point property.*

Proof. For $1 < p < \infty$, ces_p is a reflexive space and since $R(\mathrm{ces}_p) = 2^{1/p} < 2$, ces_p has the weakly fixed point property. Therefore, ces_p has the fixed point property. ■

10. WCSC, uniform Opial property, k-NUC and UNS for ces_p

Our main aim in this section is to calculate the weakly convergence sequence coefficient for Cesáro sequence space ces_p and to prove that for any $p \in (1, \infty)$, ces_p is k-NUC for any integer $k \geq 2$ and has the uniform Opial property and property (L). The weakly convergence sequence coefficient, which is connected with normal structure, is an important geometric constant. It was introduced by Bynum [By 80].

For a sequence $\{x_n\} \subset X$, we consider

$$A(\{x_n\}) = \lim_{n\to\infty} \{\sup\{\|x_i - x_j\| : \ i, j \geq n, i \neq j\}\}$$
$$A_1(\{x_n\}) = \lim_{n\to\infty} \{\inf\{\|x_i - x_j\| : \ i, j \geq n, i \neq j\}\}.$$

The weakly convergence sequence coefficient of X, denoted by $WCS(X)$, is defined as follows:

$$WCS(X) = \sup\{k > 0 : \ \text{for each weakly convergent sequence } \{x_n\}, \text{ there is } y \in \text{co}(\{x_n\}) \ \text{ such that } \ k \cdot \limsup_{n\to\infty} \|x_n - y\| \leq A(\{x_n\})\},$$

see [B 91].

The number $M(X) = 1/WCS(X)$ for a reflexive Banach space is called the Maluta coefficient and it is known that $M(X) = 1$ for every non-reflexive Banach space X (see [Ma 84]). It is also well known that a Banach space X with $WCS(X) > 1$ has weak normal structure (see [Cu-H-Li]). A sequence $\{x_n\}$ is said to be an *asymptotic equidistant sequence* if $A(\{x_n\}) = A_1(\{x_n\})$ (see [Z 92]). The formula

$$\begin{aligned} WCS(X) &= \inf\{A(\{x_n\}) : \ \{x_n\} \subset S(X) \text{ and } x_n \xrightarrow{w} 0\} \\ &= \inf\{A(\{x_n\}) : \ \{x_n\} \text{ an asymptotic equidistant sequence in } S(X) \text{ and } x_n \xrightarrow{w} 0\} \end{aligned}$$

was obtained in [Z 92].

A Banach space X is said to have *weak uniform normal structure* if $WCS(X) > 1$. Recall that the functions α and β are the Kuratowski measure of noncompactness and the it Hausdorff measure of noncompactness in X, respectively. We can associate these functions with the notions of the set-contraction and the ball-contraction (see [De 85]). These notions are very useful in the study of nonlinear operator problems (see [De 85]).

The packing rate of a Banach space X is denoted by $\gamma(X)$ and it is defined by the formula

$$\gamma(X) = \delta(X)/\sigma(X),$$

where $\delta(X)$ and $\sigma(X)$ are defined as the supremum and the infimum, respectively, of the set

$$\left\{ \frac{\beta(A)}{\alpha(A)} : \ A \subset X, \ \ A \ \text{ is } \alpha\text{-minimal}, \ \ \alpha(A) > 0 \right\}.$$

Recall that $A \subset X$ is said to be *α-minimal* if $\alpha(B) = \alpha(A)$ for any infinite subset of A. For those definitions and for results concerning the existence of α-minimal and β-minimal sets we refer to [Ay-D-Lo 97], Chapter X.

Theorem 10.1 *If $1 < p < \infty$, then the space ces_p is k-NUC for any integer $k \geq 2$.*

Proof. Let $\varepsilon > 0$ be given. For every sequence $\{x_n\} \subset B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, we put $x_n^m = (0, 0, \dots, 0, x_n(m), x_n(m+1), \dots)$. For each $i \in \mathbf{N}$, the sequence $\{x_n(i)\}_{i=1}^{\infty}$ is bounded. Therefore, using the diagonal method one can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence $\{x_{n_k}(i)\}$ converges for each $i \in \mathbf{N}$. Therefore, for any $m \in \mathbf{N}$ there exists k_m such that $\operatorname{sep}(\{x_{n_k}^m\}_{k \geq k_m}) \geq \varepsilon$. Hence for each $m \in \mathbf{N}$ there exists $n_m \in \mathbf{N}$ such that

$$\left\| x_{n_m}^m \right\| \geq \frac{\varepsilon}{2}. \tag{10.1}$$

Write $I_p(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p$ and put $\varepsilon_1 = \frac{k^{p-1}-1}{2k^p(k-1)} \left(\frac{\varepsilon}{2} \right)^p$. Then $\exists\, \delta > 0$ such that

$$|I_p(x+y) - I_p(x)| < \varepsilon_1 \tag{10.2}$$

whenever $I_p(x) \leq 1$ and $I_p(y) \leq \delta$ (see Lemma 9.1).

There exists $m_1 \in \mathbf{N}$ such that $I_p(x_1^{m_1}) \leq \delta$. Next, there exists $m_2 > m_1$ such that $I_p(x_2^{m_2}) \leq \delta$. In such a way, there exists $m_2 < m_3 < \dots < m_{k-1}$ such that $I_p(x_j^{m_j}) \leq \delta$ for all $j = 1, 2, \dots, k-1$. Define $m_k = m_{k-1} + 1$. By condition (10.1), there exists $n_k \in \mathbf{N}$ such that $I_p(x_{n_k}^{m_k}) \geq (\varepsilon/2)^p$. Put $n_i = i$ for $1 \leq i \leq k-1$. Then in virtue of (10.1), (10.2) and convexity of the function $f(u) = |u|^p$, we get

$$\begin{aligned}
& I_p\Big(\frac{x_{n_1} + x_{n_2} + \dots + x_{n_{k-1}} + x_{n_k}}{k} \Big) \\
&= \sum_{n=1}^{m_1} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_1}(i) + \dots + x_{n_k}(i)}{k} \right| \right)^p \\
&\quad + \sum_{n=m_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_1}(i) + x_{n_2}(i) + \dots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k} \right| \right)^p \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} |x_{n_j}(i)| \right)^p + \sum_{n=m_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_2}(i) + \dots + x_{n_k}(i)}{k} \right| \right)^p + \varepsilon_1 \\
&= \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} |x_{n_j}(i)| \right)^p + \sum_{n=m_1+1}^{m_2} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_2}(i) + \dots + x_{n_k}(i)}{k} \right| \right)^p \\
&\quad + \sum_{n=m_2+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_2}(i) + x_{n_3}(i) + \dots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k} \right| \right)^p + \varepsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} |x_{n_j}(i)| \right)^p + \sum_{n=m_1+1}^{m_2} \frac{1}{k} \sum_{j=2}^{k} \left(\frac{1}{n} \sum_{i=1}^{n} |x_{n_j}(i)| \right)^p \\
&\quad + \sum_{n=m_3+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{n_3}(i) + x_{n_4}(i) + \dots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k} \right| \right)^p + 2\varepsilon_1 \\
&\leq \frac{I_p(x_{n_1}) + \dots + I_p(x_{n_{k-1}})}{k} + \frac{1}{k} \sum_{n=1}^{m_k-1} \left(\frac{1}{n} \Big(\sum_{i=1}^{n} |x_{n_k}(i)| \Big) \right)^p \\
&\quad + \sum_{n=m_{k-1}+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{|x_{n_k}(i)|}{k} \right)^p + (k-1)\varepsilon_1 \\
&\leq \frac{k-1}{k} + \frac{1}{k} \sum_{n=1}^{m_k-1} \left(\frac{1}{n} \sum_{i=1}^{n} |x_{n_k}(i)| \right)^p
\end{aligned}$$

$$+\frac{1}{k^p}\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{n_k}(i)|\right)^p+(k-1)\varepsilon_1$$
$$=1-\frac{1}{k}+\frac{1}{k}\left(1-\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{n_k}(i)|\right)\right)^p$$
$$+\frac{1}{k^p}\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{n_k}(i)|\right)^p+(k-1)\varepsilon_1$$
$$\leq 1+(k-1)\varepsilon_1-\left(\frac{k^{p-1}-1}{k^p}\right)\sum_{n=m_{k-1}+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{n_k}(i)|\right)^p$$
$$\leq 1+(k-1)\varepsilon_1-\left(\frac{k^{p-1}-1}{k^p}\right)(\frac{\varepsilon}{2})^p$$
$$=1-\frac{1}{2}\left(\frac{k^{p-1}-1}{k^p}\right)\left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, ces_p is $(k-NUC)$ for any integer $k\geq 2$. ■

Theorem 10.2 *For any $1<p<\infty$, the space ces_p has the uniform Opial property.*

Proof. For any $\varepsilon>0$ we can find a positive number $\varepsilon_0\in(0,\varepsilon)$ such that

$$1+\frac{\varepsilon^p}{2}>(1+\varepsilon_0)^p.$$

Let $x\in X$ and $\|x\|\geq\varepsilon$. There exists $n_1\in\mathbf{N}$ such that

$$\sum_{i=n_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p<(\frac{\varepsilon_0}{4})^p.$$

Hence we have

$$\left\|\sum_{i=n_1+1}^{\infty}x(i)e_i\right\|<\frac{\varepsilon_0}{4}<\frac{\varepsilon}{4},$$

where $e_i=(0,\dots,\overset{ith}{1},0,0,\dots)$. Furthermore, we have

$$\varepsilon^p\leq\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p+\sum_{n=n_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p$$
$$<\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p+\left(\frac{\varepsilon_0}{4}\right)^p$$
$$<\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p+\frac{\varepsilon^p}{4},$$

whence

$$\frac{3\varepsilon^p}{4}\leq\sum_{n=1}^{n_1}\left(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right)^p.$$

For any weakly null sequence $\{x_m\} \subset S(X)$, in virtue of $x_m(i) \to 0$ for $i = 1, 2, \dots$, there exists $m_0 \in N$ such that

$$\left\| \sum_{n=1}^{n_1} x_m(i) e_i \right\| < \frac{\varepsilon_0}{4}$$

when $m > m_0$. Therefore,

$$\begin{aligned} \|x_m + x\| &= \left\| \sum_{i=1}^{n_1} (x_m(i) + x(i)) e_i + \sum_{i=n_1+1}^{\infty} (x_m(i) + x(i)) e_i \right\| \\ &\geq \left\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \right\| - \left\| \sum_{i=1}^{n_1} x_m(i) e_i \right\| - \left\| \sum_{i=n_1+1}^{\infty} x(i) e_i \right\| \\ &\geq \left\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \right\| - \frac{\varepsilon_0}{2} \end{aligned}$$

when $m > m_0$. Moreover for $a := \sum_{i=1}^{n_1} |x(i)|$ there holds

$$\begin{aligned} \left\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \right\|^p &= \sum_{n=1}^{n_1} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i) e_i| \right)^p + \sum_{n=n_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} (a + |x_m(i)|) \right)^p \\ &\geq \sum_{n=1}^{n_1} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i) e_i| \right)^p + \sum_{n=n_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p \\ &\geq \frac{3\varepsilon^p}{4} + \left(1 - \frac{\varepsilon^p}{4} \right) \\ &= 1 + \frac{\varepsilon^p}{2} > (1 + \varepsilon_0)^p. \end{aligned}$$

Therefore, combining this with the previous inequality, we get

$$\begin{aligned} \|x_m + x\| &\geq \left\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \right\| - \frac{\varepsilon_0}{2} \\ &\geq 1 + \varepsilon_0 - \frac{\varepsilon_0}{2} = 1 + \frac{\varepsilon_0}{2}. \end{aligned}$$

This means that ces_p has the uniform Opial property. ∎

By the reflexivity of ces_p for $1 < p < \infty$, we get the following.

Corollary 10.3 *For $1 < p < \infty$ the space ces_p has property* (L) *and the fixed point property.*

Now, we will calculate the weakly convergence sequence coefficient of ces_p.

Theorem 10.4 *For $1 < p < \infty$, $WCS(\mathrm{ces}_p) = 2^{\frac{1}{p}}$.*

Proof. Take any $\varepsilon > 0$ and an asymptotic equidistant sequence $\{x_n\} \subset S(X)$ with $x_n \xrightarrow{w} 0$ and put $v_1 = x_1$. There exists $i_1 \in \mathbf{N}$ such that

$$\left\| \sum_{i=i_1+1}^{\infty} v_1(i) e_i \right\| < \varepsilon.$$

Since $x_n \to 0$ coordinate-wise, there exists $n_2 \in \mathbf{N}$ such that

$$\left\| \sum_{i=1}^{i_1} x_n(i) e_i \right\| < \varepsilon$$

whenever $n \geq n_2$.

Take $v_2 = x_{n_2}$. Then there is $i_2 > i_1$ such that

$$\left\| \sum_{i=i_2+1}^{\infty} v_2(i) e_i \right\| < \varepsilon.$$

Since $x_n(i) \to 0$ coordinate-wise, there exists $n_3 \in \mathbf{N}$ such that

$$\left\| \sum_{i=1}^{i_2} x_n(i) e_i \right\| < \varepsilon$$

whenever $n \geq n_3$.

Continuing this process in such a way by induction, we get a subsequence $\{v_n\}$ of $\{x_n\}$ such that

$$\left\| \sum_{i=i_n+1}^{\infty} v_n(i) e_i \right\| < \varepsilon \text{ and } \left\| \sum_{i=1}^{i_n} v_{n+1}(i) e_i \right\| < \varepsilon.$$

Put

$$z_n = \sum_{i=i_{n-1}+1}^{i_n} v_n(i) e_i$$

for $n = 2, 3, \dots$. Then

$$\begin{aligned} 1 \geq \|z_n\| &= \left\| \sum_{i=1}^{\infty} v_n(i) e_i - \sum_{i=1}^{i_{n-1}} v_n(i) e_i - \sum_{i=i_n+1}^{\infty} v_n(i) e_i \right\| \\ &\geq \left\| \sum_{i=1}^{\infty} v_n(i) e_i \right\| - \left\| \sum_{i=1}^{i_{n-1}} v_n(i) e_i \right\| - \left\| \sum_{i=i_n+1}^{\infty} u_n(i) e_i \right\| > 1 - 2\varepsilon. \end{aligned} \tag{10.3}$$

Moreover, for any $n, m \in \mathbf{N}$ with $n \neq m$, we have

$$\begin{aligned} \|v_n - v_m\| &= \left\| \sum_{i=1}^{\infty} u_n(i) e_i - \sum_{i=1}^{\infty} v_m(i) e_i \right\| \\ &\geq \left\| \sum_{i=i_{n-1}+1}^{i_n} v_n(i) e_i - \sum_{i=i_{m-1}+1}^{i_m} v_m(i) e_i \right\| - \left\| \sum_{i=1}^{i_{n-1}} v_n(i) e_i \right\| \\ &\quad - \left\| \sum_{i=i_n+1}^{\infty} v_n(i) e_i \right\| - \left\| \sum_{i=1}^{i_{m-1}} v_m(i) e_i \right\| - \left\| \sum_{i=i_m+1}^{\infty} u_m(i) e_i \right\| \\ &\geq \|z_n - z_m\| - 4\varepsilon. \end{aligned} \tag{10.4}$$

This means that $A(\{x_n\}) = A(\{v_n\}) \geq A(\{z_n\}) - 4\varepsilon$. Put $u_n = z_n / \|z_n\|$ for $n = 2, 3, \dots$. Then

$$u_n \in S(\mathrm{ces}_p); \tag{10.5}$$

$$A(\{x_n\}) \geq 1 - \varepsilon A(\{u_n\}) - 4\varepsilon. \tag{10.6}$$

On the other hand

$$\|v_n - v_m\| \leq \|z_n - z_m\| + 4\varepsilon \leq \|u_n - u_m\| + 4\varepsilon$$

for every $m, n \in \mathbf{N},\ m \neq n$. Therefore

$$A(\{u_n\}) \geq A(\{x_n\}) - 4\varepsilon. \tag{10.7}$$

By the arbitrariness of $\varepsilon > 0$, we have from (10.5), (10.6) and (10.7) that

$$WCS(\mathrm{ces}_p) = \inf\Big\{A(\{u_n\}) \colon u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)e_i \in S(\mathrm{ces}_p),$$
$$0 = i_0 < i_1 < i_2 < \dots,\ u_n \xrightarrow{w} 0\Big\}.$$

Using Lemma 2 in [Z 92], we have

$$WCS(\mathrm{ces}_p) = \inf\Big\{A(\{u_n\}) \colon u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)e_i \in S(\mathrm{ces}_p),\ 0 = i_0 < i_1 < \cdots,$$
$$u_n \xrightarrow{w} 0 \text{ and } \{u_n\} \text{ is asymptotic equidistant}\Big\}.$$

Take $m \in \mathbf{N}$ large enough such that

$$\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^p < \varepsilon,$$

where $b := \sum_{i=i_{n-1}+1}^{i_n} |u_n(i)|$. We have for $n < m$

$$\begin{aligned}
\|u_n - u_m\|^p &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p \\
&\geq \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p \\
&= \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\right)^p - \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^p + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p \\
&> 1 - \varepsilon + 1 = 2 - \varepsilon, \quad \text{that is, } A_1(\{u_1\}) \geq (2-\varepsilon)^{\frac{1}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}|u_m(i)|\right)\right)^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k} + \frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p\right]^{\frac{1}{p}} \\
&\leq \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^p\right]^{\frac{1}{p}} + \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p\right]^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}} + 1.
\end{aligned}$$

Therefore

$$\begin{aligned}\|u_n - u_m\|^p &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b+\sum_{i=1}^{k}|u_m(i)|\right)\right)^p \\ &\le \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b+\sum_{i=1}^{k}|u_m(i)|\right)\right)^p \\ &\le 1 + (1+\varepsilon^{\frac{1}{p}})^p\end{aligned}$$

for any $n, m \in \mathbf{N}$, $m \neq n$. This yields $A(\{u_n\}) \le [1 + (1+\varepsilon^{\frac{1}{p}})^p]^{\frac{1}{p}}$ and, by the arbitrariness of $\varepsilon > 0$, we obtain $WCS(\mathrm{ces}_p) = 2^{\frac{1}{p}}$. ■

Corollary 10.5 *For $1 < p < \infty$, ces_p has the weak uniform normal structure and normal structure.*

Corollary 10.6 *For any $1 < p < \infty$, we have $\gamma(\mathrm{ces}_p) = 2^{(p-1)/p}$.*

Proof. By [Ay-D 93], if X is reflexive Banach space with the uniform Opial property, then $\gamma(X) = 2/WCS(X)$. Since, by Theorem 10.1, ces_p is NUC for $1 < p < \infty$ and property NUC implies reflexivity, Theorem 10.2 yields $\gamma(\mathrm{ces}_p) = 2/2^{1/p} = 2^{(p-1)p}$. ■

Note. Banach-Saks and weak Banach-Saks properties in Cesaro sequence spaces has been characterized in [Cu-H 99b].

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