ON THE CONVERGENCE OF ITERATION PROCESSES FOR SEMIGROUPS OF NONLINEAR MAPPINGS IN BANACH SPACES

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In tribute to Jonathan Borwein on his 60th birthday

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ABSTRACT. *Let C be a bounded, closed, convex subset of a uniformly convex Banach space X. We investigate the convergence of the generalized Krasnosel'skii-Mann and Ishikawa iteration processes to common fixed points of pointwise Lipschitzian semigroups of nonlinear mappings $T_t : C \to C$. Each of T_t is assumed to be pointwise Lipschitzian, that is, there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \le \alpha_t(x)\|x - y\|$ for $x, y \in C$.*

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1. Introduction

Let C be a bounded, closed, convex subset of a Banach space X. Let us consider a pointwise Lipschitzian semigroup of nonlinear mappings, that is, a family of mappings $T_t : C \to C$ satisfying the following conditions: $T_0(x) = x$, $T_{s+t}(x) = T_s(T_t(x))$, $t \mapsto T_t(x)$ is strong continuous for each $x \in C$, and each T_t is pointwise Lipschitzian. The latter means that there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \le \alpha_t(x)\|x - y\|$ for $x, y \in C$ (see Definitions 2.1, 2.2, and 2.3 for more details). Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the Banach space X would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. Common fixed *points of such a semigroup can be interpreted as points that are fixed during the state space transformation* T_t *at any given point of time* t*. Our results cater for both the continuous and the discrete time cases. In the setting of this paper, the state space may be an infinite dimensional Banach space. Therefore, it is natural to apply these result not only to deterministic dynamical systems but also to stochastic dynamical systems.*

The existence of common fixed points for families of contractions and nonexpansive mappings have been investigated since the early 1960s, see e.g. DeMarr [8]*, Browder* [4]*, Belluce and Kirk* [2, 3]*, Lim* [22]*, Bruck* [5, 6]*. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has been also investigated for some time, see e.g. Tan and Xu* [33]*. It is worthwhile mentioning the recent studies on the special case, when the parameter set for the semigroup is equal to* $\{0, 1, 2, 3, ...\}$ *and* $T_n = T^n$*, the* n*-th iterate of an asymptotic pointwise nonexpansive mapping, i.e. such a* $T : C \to C$ *that there exists a sequence of functions* $\alpha_n : C \to [0, \infty)$ *with* $\|T^n(x) - T^n(y)\| \le \alpha_n(x)\|x - y\|$ *and* $\limsup_{n\to\infty} \alpha_n(x) \le 1$*. Kirk and Xu* [17] *proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi extended this result to metric spaces* [11]*, and Khamsi and Kozlowski to modular function spaces* [14], [15]*. Recently, Kozlowski proved existence of common fixed points for semigroups of nonlinear contractions and nonexpansive mappings in modular functions spaces,* [20]*.*

Several authors studied the generalizations of known iterative fixed point construction processes like the Mann process (see e.g. [23, 10]*) or the Ishikawa process (see e.g.* [12]*) to the case of asymptotic and pointwise asymptotic nonexpansive mappings. There exists an extensive literature on the subject of iterative fixed point construction processes for asymptotically nonexpansive mappings in Hilbert, Banach and metric spaces, see e.g.* [1, 29, 27, 9, 30, 31, 32, 36, 37, 33, 34, 7, 35, 28, 25, 13, 11, 24] *and the works referred there. Schu* [31] *proved the weak convergence of the modified Mann iteration process to a fixed point of asymptotic nonexpansive mappings in uniformly convex Banach spaces with the Opial property, and the strong convergence for compact asymptotic nonexpansive mappings in uniformly convex Banach spaces. Tan and Xu* [35] *proved the weak* *convergence of the modified Mann and modified Ishikawa iteration processes for asymptotic nonexpansive mappings in uniformly convex Banach spaces satisfying the Opial condition or possessing Fréchet differentiable norm. Kozlowski* [18] *proved that - under some reasonable assumptions - the generalized Mann and Ishikawa processes converge weakly to a fixed point of an asymptotic pointwise nonexpansive mapping $T : C \to C$, where C is a bounded, closed and convex subset of a uniformly convex Banach space X which satisfies the Opial condition.*

Let us note that the existence of common fixed points for asymptotic pointwise nonexpansive semigroups has been recently proved by Kozlowski in [19]*. However, the proof of this result does not provide a constructive method of finding such common fixed points. The aim of the current paper is to fill this gap. We prove that - under some reasonable assumptions - the generalized Krasnosel'skii-Mann and Ishikawa processes converge weakly, and - under additional assumptions - strongly, to a common fixed point of the asymptotic pointwise nonexpansive semigroups.*

The paper is organized as follows:

- *(a) Section 2 provides necessary preliminary material.*
- *(b) Section 3 presents some technical results on approximate fixed point sequences.*
- *(c) Section 4 is devoted to proving the Demiclosedness Principle in a version relevant for this paper.*
- *(d) Section 5 deals with the weak convergence of generalized Krasnosel'skii-Mann iteration processes to common fixed points of asymptotic pointwise nonexpansive semigroups.*
- *(e) Section 6 deals with the weak convergence of generalized Ishikawa iteration processes to common fixed points of asymptotic pointwise nonexpansive semigroups.*
- *(f) Section 7 presents the strong convergence result for both Krasnosel'skii-Mann and Ishikawa processes.*

2. Preliminaries

Throughout this paper X will denote a Banach space, C a nonempty, bounded, closed and convex subset of X, and J will be a fixed parameter semigroup of nonnegative numbers, i.e. a subsemigroup of $[0, \infty)$ with normal addition. We assume that $0 \in J$ and that there exists $t > 0$ such that $t \in J$. The latter *assumption implies immediately that $+\infty$ is a cluster point of J in the sense of the natural topology inherited by J from $[0,\infty)$. Typical examples are: $J = [0,\infty)$ and ideals of the form $J = \{n\alpha : n = 0, 1, 2, 3, ...\}$ for a given $\alpha > 0$. The notation $t \to \infty$ will mean that t tends to infinity over J.*

Let us start with more formal definitions of pointwise Lipschitzian mappings and pointwise Lipschitzian semigroups of mappings, and associated notational conventions.

Definition 2.1. *We say that $T : C \to C$ is a pointwise Lipschitzian mapping if there exists a function $\alpha : C \to [0,\infty)$ such that*

$$\|T(x) - T(y)\| \le \alpha(x)\|x - y\| \text{ for all } x, y \in C. \tag{2.1}$$

If the function $\alpha(x) < 1$ for every $x \in C$, then we say that T is a pointwise contraction. Similarly, if $\alpha(x) \le 1$ for every $x \in C$, then T is said to be a pointwise nonexpansive mapping.

Definition 2.2. *A one-parameter family $\mathcal{F} = \{T_t; t \in J\}$ of mappings from C into itself is said to be a pointwise Lipschitzian semigroup on C if $\mathcal{F}$ satisfies the following conditions:*

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$;

(iii) for each $t \in J$, T_t is a pointwise Lipschitzian mapping, i.e. there exists a function $\alpha_t : C \to [0,\infty)$ such that

$$\|T_t(x) - T_t(y)\| \le \alpha_t(x)\|x - y\| \text{ for all } x, y \in C. \tag{2.2}$$

(iv) for each $x \in C$, the mapping $t \mapsto T_t(x)$ is strong continuous.

For each $t \in J$ let $F(T_t)$ denote the set of its fixed points. Note that if $x \in F(T_t)$ then x is a periodic point (with period t) for the semigroup $\mathcal{F}$, i.e. $T_{kt}(x) = x$ for every natural k. Define then the set of all common set points for mappings from $\mathcal{F}$ as the following intersection

$$F(\mathcal{F}) = \bigcap_{t \in J} F(T_t).$$

The common fixed points are frequently interpreted as the stationary points of the semigroup $\mathcal{F}$.

Definition 2.3. *Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup. $\mathcal{F}$ is said to be asymptotic pointwise nonexpansive if* $\limsup_{t\to\infty} \alpha_t(x) \le 1$ *for every $x \in C$.*

Denoting $a_0 \equiv 1$ and $a_t(x) = \max(\alpha_t(x), 1)$ for $t > 0$, we note that without loss of generality we can assume that $\mathcal{F}$ is asymptotically nonexpansive if

$$\|T_t(x) - T_t(y)\| \le a_t(x)\|x - y\| \text{ for all } x, y \in C,\ t \in J, \tag{2.3}$$

$$\lim_{t\to\infty} a_t(x) = 1,\ a_t(x) \ge 1 \text{ for all } x \in C, \text{ and } t \in J. \tag{2.4}$$

Define $b_t(x) = a_t(x) - 1$. In view of (2.4), we have

$$\lim_{t\to\infty} b_t(x) = 0. \tag{2.5}$$

The above notation will be consistently used throughout this paper.

Definition 2.4. *By $\mathcal{S}(C)$ we will denote the class of all asymptotic pointwise nonexpansive semigroups on C such that*

$$M_t = \sup\{a_t(x) : x \in C\} < \infty,\ \text{for every } t \in J, \tag{2.6}$$

$$\limsup_{t\to\infty} M_t = 1. \tag{2.7}$$

Note that we do not assume that all functions a_t are bounded by a common constant. Therefore, we do not assume that $\mathcal{F}$ is uniformly Lipschitzian.

Definition 2.5. *We will say that a semigroup $\mathcal{F} \in \mathcal{S}(C)$ is equicontinuous if the family of mappings $\{t \mapsto T_t(x) : x \in C\}$ is equicontinuous at $t = 0$.*

The following result of Kozlowski will be used in this paper to ensure existence of common fixed points.

Theorem 2.1. [19] *Assume X is uniformly convex. Let $\mathcal{F}$ be an asymptotically nonexpansive pointwise Lipschitzian semigroup on C. Then $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is closed and convex.*

The following elementary, easy to prove, lemma will be used in this paper.

Lemma 2.1. [7] *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly-index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \le 0,\ \text{and } r_{k+n} \le r_k + d_{k,n}$$

for each $k, n \geq 1$. *Then* $\{r_k\}$ *converges to an* $r \in \mathbb{R}$.

The notion of bounded away sequences of real numbers will be used extensively throughout this paper.

Definition 2.6. *A sequence* $\{c_n\} \subset (0,1)$ *is called bounded away from* 0 *if there exists* $0 < a < 1$ *such that* $c_n > a$ *for every* $n \in \mathbb{N}$. *Similarly,* $\{c_n\} \subset (0,1)$ *is called bounded away from* 1 *if there exists* $0 < b < 1$ *such that* $c_n < b$ *for every* $n \in \mathbb{N}$.

The following property of uniformly convex Banach spaces will play an important role in this paper.

Lemma 2.2. [31, 38] *Let* X *be a uniformly convex Banach space. Let* $\{c_n\} \subset (0,1)$ *be bounded away from* 0 *and* 1, *and* $\{u_n\}, \{v_n\} \subset X$ *be such that*

$$\limsup_{n\to\infty} \|u_n\| \leq a, \ \limsup_{n\to\infty} \|v_n\| \leq a, \ \lim_{n\to\infty} \|c_n u_n + (1-c_n)v_n\| = a.$$

Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Using Kirk's result [16] *(Proposition 2.1), Kozlowski* [19] *proved the following proposition.*

Proposition 2.1. *Let* $\mathcal{F}$ *be a semigroup on* C. *Assume that all mappings* $T_t \in \mathcal{F}$ *are continuously Fréchet differentiable on an open convex set* A *containing* C. *Then* $\mathcal{F}$ *is asymptotic pointwise nonexpansive on* C *if and only if for each* $x \in C$

$$\limsup_{t\to\infty} \|(T_t)'_x\| \leq 1. \tag{2.8}$$

This result, combined with Theorem 2.1, *produces the following fixed point theorem.*

Theorem 2.2. [19] *(Theorem 3.5) Assume* X *is uniformly convex. Let* $\mathcal{F}$ *be a pointwise Lipschitzian semigroup on* C. *Assume that all mappings* $T_t \in \mathcal{F}$ *are continuously Fréchet differentiable on an open convex set* A *containing* C *and for each* $x \in C$

$$\limsup_{t\to\infty} \|(T_t)'_x\| \leq 1. \tag{2.9}$$

Then $\mathcal{F}$ *has a common fixed point and the set* $F(\mathcal{F})$ *of common fixed points is closed convex.*

Because of the above, all the results of this paper can be applied to the semigroups of nonlinear mappings satisfying condition (2.9)*. This approach may be very useful for applications provided the Fréchet derivatives can be estimated.*

3. Approximate fixed point sequences

The technique of approximate fixed point sequences will play a critical role in proving fixed convergence to common fixed points for semigroups of mappings. Let us recall that given $T : C \to C$, a sequence $\{x_k\} \subset C$ is called an approximate fixed point sequence for T if $\|T(x_k) - x_k\| \to 0$ as $k \to \infty$. We will also use extensively the following notion of a generating set.

Definition 3.1. *A set $A \subset J$ is called a generating set for the parameter semigroup J if for every $0 < u \in J$ there exist $m \in \mathbb{N}$, $s \in A$, $t \in A$ such that $u = ms + t$.*

Lemma 3.1. *Let C be a nonempty, bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. If $\|T_s(x_n) - x_n\| \to 0$ for an $s \in J$ as $n \to \infty$ then for any $m \in \mathbb{N}$, $\|T_{ms}(x_n) - x_n\| \to 0$ as $n \to \infty$*

Proof. It follows from the fact that every a_t is a bounded function that there exists a finite constant $M > 0$ such that

$$\sum_{j=1}^{m-1} sup\{a_{js}(x); x \in C\} \;\leq\; M. \tag{3.1}$$

It follows from

$$\begin{aligned} \|T_{ms}(x_n) - x_n\| &\leq \sum_{j=1}^{m-1} \|T_{(j+1)s}(x_n) - T_{js}(x_n)\| + \|T_s(x_n) - x_n\| \\ &\leq \|T_s(x_n) - x_n\|\Big(\sum_{j=1}^{m-1} a_{js}(x_n) + 1\Big) \leq (M+1)\|T_s(x_n) - x_n\| \end{aligned} \tag{3.2}$$

that

$$\lim_{n\to\infty} \|T_{ms}(x_n) - x_n\| = 0, \tag{3.3}$$

which completes the proof. □

Lemma 3.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. If $\{x_k\} \subset C$ is a approximate fixed point sequence for $T_s \in \mathcal{F}$ for any $s \in A$ where A is a generating set for J then $\{x_k\}$ is a approximate fixed point sequence for T_s for any $s \in J$.*

Proof. Let $s, t \in A$ and $m \in \mathbb{N}$. We need to show that $\|T_{ms+t}(x_n) - x_n\| \to 0$ as $n \to \infty$. Indeed,

$$\|T_{ms+t}(x_n) - x_n\| \leq \|T_{ms+t}(x_n) - T_{ms}(x_n)\| + \|T_{ms}(x_n) - x_n\|$$
$$\leq a_{ms}(x_n)\|T_t(x_n) - x_n\| + \|T_{ms}(x_n) - x_n\|,$$

which tends to zero by boundedness of the function a_{ms} and by Lemma 3.1. □

Lemma 3.3. *Let $\mathcal{F} \in \mathcal{S}(C)$ be equicontinuous and $\overline{B} = A \subset J$. If $\{x_k\} \subset C$ is an approximate fixed point sequence for T_t for every $t \in B$ then $\{x_k\}$ is an approximate fixed point sequence for T_t for every $t \in A$.*

Proof. Let $s \in A$, then there exists a sequence $\{s_n\} \subset B$ such that $s_n \to s$. Note that

$$\begin{aligned} \|T_s(x_k) - x_k\| &\leq \|T_s(x_k) - T_{s_n}(x_k)\| + \|T_{s_n}(x_k) - x_k\| \\ &\leq \sup_{x \in C} a_{min(s,s_n)}(x) \sup_{x \in C} \|T_{|s-s_n|}(x) - x\| + \|T_{s_n}(x_k) - x_k\|. \end{aligned} \tag{3.4}$$

Fix $\varepsilon > 0$. By equicontinuity of $\mathcal{F}$ and by (2.6) there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x \in C} a_{min(s,s_{n_0})}(x) \sup_{x \in C} \|T_{|s-s_{n_0}|}(x) - x\| < \frac{\varepsilon}{2}. \tag{3.5}$$

Since $\{x_k\}$ is an approximate fixed point for $T_{s_{n_0}}$ we can find $k_0 \in \mathbb{N}$ such that for every natural $k \geq k_0$

$$\|T_{s_{n_0}}(x_k) - x_k\| < \frac{\varepsilon}{2}. \tag{3.6}$$

By substituting (3.5) and (3.6) into (3.4) we get $\|T_s(x_k) - x_k\| < \varepsilon$ for large k. Hence $\{x_k\}$ is an approximate fixed point for T_s as claimed. □

4. The Demiclosedness Principle

The following version of the Demiclosedness Principle will be used in the proof of our main convergence theorems. There exist several versions of the Demiclosedness Principle for the case of asymptotic nonexpansive mappings, see e.g. Li and Sims [21]*, Gornicki* [9] *or Xu* [37]*. Recently, Kozlowski* [18] *proved a version of the Demiclosedness Principle for the asymptotic pointwise nonexpansive mappings, using the "parallelogram inequality" valid in the uniformly convex Banach spaces (Theorem 2 in* [36]*). For the completeness sake, we provide the proof for asymptotic pointwise nonexpansive semigroups.*

Let us recall the definition of the Opial property which will play an essential role in this paper.

Definition 4.1. [26] *A Banach space X is said to have the Opial property if for each sequence $\{x_n\} \subset X$ weakly converging to a point $x \in X$ (denoted as $x_n \rightharpoonup x$) and for any $y \in X$ such that $y \neq x$ there holds*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{4.1}$$

or equivalently

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|. \tag{4.2}$$

Theorem 4.1. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a nonempty, bounded, closed and convex subset of X, and let $\mathcal{F} \in \mathcal{S}(C)$. Assume that there exists $w \in X$ and $\{x_n\} \subset C$ such that $x_n \rightharpoonup w$. Assume that there exists an $s \in J$ such that $\|T_s(x_n) - x_n\| \to 0$ as $n \to \infty$. Then $w \in F(T_{ks})$ for any natural k.*

Proof. Define a type $\varphi(x) = \limsup_{n\to\infty} \|x_n - x\|$ for $x \in C$. Let us fix $m \in \mathbb{N}$, $m > 2$ and observe that

$$\begin{aligned}\|T_{ms}(x_n) - x\| &\leq \sum_{i=1}^{m} \|T_{is}(x_n) - T_{(i-1)s}(x_n)\| + \|x_n - x\| \\ &\leq \|T_s(x_n) - x_n\| \Big(\sum_{i=2}^{m} a_{(i-1)s}(x_n) + 1 \Big) + \|x_n - x\|.\end{aligned}$$

Since all functions a_i are bounded and $\|T_s(x_n) - x_n\| \to 0$, it follows that

$$\limsup_{n\to\infty} \|T_{ms}(x_n) - x\| \le \limsup_{n\to\infty} \|x_n - x\| = \varphi(x).$$

On the other hand, by Lemma 3.1, we have

$$\varphi(x) \le \limsup_{n\to\infty} \|x_n - T_{ms}(x_n)\| + \limsup_{n\to\infty} \|T_{ms}(x_n) - x\| = \limsup_{n\to\infty} \|T_{ms}(x_n) - x\|.$$

Hence,

$$\varphi(x) = \limsup_{n\to\infty} \|T_{ms}(x_n) - x\|. \tag{4.3}$$

Because $\mathcal{F}$ is asymptotic pointwise nonexpansive, it follows that $\varphi\Big(T_{ms}(x)\Big) \le a_{ms}(x)\varphi(x)$ for every $x \in C$. Applying this to w and passing with $m \to \infty$, we obtain

$$\lim_{m\to\infty} \varphi\Big(T_{ms}(w)\Big) \le \varphi(w). \tag{4.4}$$

Since $x_n \rightharpoonup w$, by the Opial property of X we have that for any $x \ne w$

$$\varphi(w) = \limsup_{n\to\infty} \|x_n - w\| < \limsup_{n\to\infty} \|x_n - x\| = \varphi(x),$$

which implies that $\varphi(w) = inf\{\varphi(x); x \in C\}$. This together with (4.4) gives us

$$\lim_{m\to\infty} \varphi\Big(T_{ms}(w)\Big) = \varphi(w). \tag{4.5}$$

By Proposition 3.4 in [17] (see also Theorem 2 in [36]) for each $d > 0$ there exists a continuous function $\lambda : [0,\infty) \to [0,\infty)$ such that $\lambda(t) = 0 \Leftrightarrow t = 0$, and

$$\|\alpha x + (1-\alpha)y\|^2 \le \ \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\lambda(\|x-y\|), \tag{4.6}$$

for any $\alpha \in [0,1]$ and all $x, y \in X$ such that $\|x\| \le d$ and $\|y\| \le d$. Applying (4.6) to $x = x_n - w$, $y = x_n - T_{ms}(w)$ and $\alpha = \frac{1}{2}$ we obtain the following inequality

$$\|x_n - \frac{1}{2}(w + T_{ms}(w))\|^2 \le \frac{1}{2}\|x_n - w\|^2 + \frac{1}{2}\|x_n - T_{ms}(w)\|^2 - \frac{1}{4}\lambda\Big(\|T_{ms}(w) - w\|\Big).$$

Applying to both side $\limsup_{n\to\infty}$ and remembering that $\varphi(w) = inf\{\varphi(x); x \in C\}$ we have

$$\varphi(w)^2 \le \frac{1}{2}\varphi(w)^2 + \frac{1}{2}\varphi\Big(T_{ms}(w)\Big)^2 - \frac{1}{4}\lambda\Big(\|T_{ms}(w) - w\|\Big),$$

which implies

$$\lambda\Big(\|T_{ms}(w) - w\|\Big) \le 2\varphi\Big(T_{ms}(w)\Big)^2 - 2\varphi(w)^2.$$

Letting $m \to \infty$ and applying (4.5) we conclude that

$$\lim_{m\to\infty} \lambda\Big(\|T_{ms}(w) - w\|\Big) = 0.$$

By the properties of λ, we have $T_{ms}(w) \to w$. Fix any natural number k. Observe that, using the same argument, we conclude that $T_{(m+k)s}(w) \to w$. Note that

$$T_{ks}(T_{ms}(w)) = T_{(m+k)s}(w) \to w$$

By the continuity of T_{ks},

$$T_{ks}(T_{ms}(w)) \to T_{ks}(w)$$

and finally $T_{ks}(w) = w$ as claimed. □

5. Weak convergence of generalized Krasnosel'skii-Mann iteration processes

Let us start with the precise definition of the generalized Krasnosel'skii-Mann iteration process for semigroups of nonlinear mappings.

Definition 5.1. *Let $\mathcal{F} \in \mathcal{S}(\mathcal{C})$, $\{t_k\} \subset J$ and $\{c_k\} \subset (0,1)$. The generalized Krasnosel'skii-Mann iteration process gKM($\mathcal{F}, \{c_k\}, \{t_k\}$) generated by the semigroup $\mathcal{F}$, the sequences $\{c_k\}$ and $\{t_k\}$, is defined by the following iterative formula:*

(5.1) $$x_{k+1} = c_k T_{t_k}(x_k) + (1 - c_k)x_k, \text{ where } x_1 \in C \text{ is chosen arbitrarily},$$

and

(1) *$\{c_k\}$ is bounded away from 0 and 1,*
(2) *$\lim_{k\to\infty} t_k = \infty$,*
(3) *$\sum_{n=1}^{\infty} b_{t_n}(x) < \infty$ for every $x \in C$.*

Definition 5.2. *We say that a generalized Krasnosel'skii-Mann iteration process gKM($\mathcal{F}, \{c_k\}, \{t_k\}$) is well defined if*

(5.2) $$\limsup_{k\to\infty} a_{t_k}(x_k) = 1.$$

We will prove a series of lemmas necessary for the proof of the generalized Krasnosel'skii-Mann process convergence theorems.

Lemma 5.1. *Let C be a bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$, $w \in F(\mathcal{F})$, and let $\{x_k\}$ be a sequence generated by a generalized Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$. Then there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.*

Proof. Let $w \in F(\mathcal{F})$. Since

$$\begin{aligned}
\|x_{k+1} - w\| &\le c_k \|T_{t_k}(x_k) - w\| + (1 - c_k)\|x_k - w\| \\
&= c_k \|T_{t_k}(x_k) - T_{t_k}(w)\| + (1 - c_k)\|x_k - w\| \\
&\le c_k(1 + b_{t_k}(w))\|x_k - w\| + (1 - c_k)\|x_k - w\| \\
&\le c_k b_{t_k}(w)\|x_k - w\| + \|x_k - w\| \\
&\le b_{t_k}(w) diam(C) + \|x_k - w\|,
\end{aligned}$$

it follows that for every $n \in \mathbb{N}$,

$$\|x_{k+n} - w\| \le \|x_k - w\| + diam(C) \sum_{i=k}^{k+n-1} b_{t_i}(w). \tag{5.3}$$

Denote $r_p = \|x_p - w\|$ for every $p \in \mathbb{N}$ and $d_{k,n} = diam(C) \sum_{i=k}^{k+n-1} b_{t_i}(w)$. Observe that $\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} = 0$. By Lemma 2.1 then, there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. □

Lemma 5.2. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $\{x_k\}$ be a sequence generated by a well defined generalized Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$. Then*

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0 \tag{5.4}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{5.5}$$

Proof. By Theorem 2.1, $F(\mathcal{F}) \neq \emptyset$. Let us fix $w \in F(\mathcal{F})$. By Lemma 5.1 there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. Because $w \in F(\mathcal{F})$, and the process is well defined, then there holds

$$\begin{aligned}\limsup_{k\to\infty} \|T_{t_k}(x_k) - w\| &= \limsup_{k\to\infty} \|T_{t_k}(x_k) - T_{t_k}(w)\| \\ &\leq \limsup_{k\to\infty} a_{t_k}(x_k)\|x_k - w\| = r.\end{aligned}$$

Observe that

$$\lim_{k\to\infty} \|c_k(T_{t_k}(x_k) - w) + (1 - c_k)(x_k - w)\| = \lim_{k\to\infty} \|x_{k+1} - w\| = r.$$

By Lemma 2.2 applied to $u_k = x_k - w$, $v_k = T_{t_k}(x_k) - w$,

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0, \tag{5.6}$$

which by the construction of the sequence $\{x_k\}$ is equivalent to

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{5.7}$$

□

Let us prove an important technical result which demonstrates that under suitable assumption the sequence $\{x_k\}$ generated by the generalized Krasnosel'skii-Mann iteration process becomes an approximate fixed point sequence, which will provide a crucial step for proving the process convergence.

Lemma 5.3. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(\mathcal{C})$. Let the generalized Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ be well defined. Let $A \subset J$ be such that to every $s \in A$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

(a) $\|x_k - x_{j_k}\| \to 0$ *as* $k \to \infty$,

(b) $\lim_{k\to\infty} \|T_{d_k}(x_{j_k}) - x_{j_k}\| = 0$, *where* $d_k = |t_{j_{k+1}} - t_{j_k} - s|$.

Then $\{x_k\}$ is an approximate fixed point sequence for all mappings $\{T_{ms}\}$ where $s \in A$ and $m \in \mathbb{N}$, that is

$$\lim_{k\to\infty} \|T_{ms}(x_k) - x_k\| = 0 \tag{5.8}$$

for every $s \in A$ and $m \in \mathbb{N}$. If, in addition, A is a generating set for J then

$$\lim_{k\to\infty} \|T_t(x_k) - x_k\| = 0 \tag{5.9}$$

for any $t \in J$.

Proof. In view of Lemma 3.1, it is enough to prove (5.8) for $m = 1$. To this end, let us fix $s \in A$. Note that

$$\|x_{j_k} - x_{j_{k+1}}\| \to 0 \text{ as } k \to \infty. \tag{5.10}$$

Indeed,

$$\|x_{j_k} - x_{j_{k+1}}\| \le \|x_{j_k} - x_k\| + \|x_k - x_{k+1}\| + \|x_{k+1} - x_{j_{k+1}}\| \to 0, \tag{5.11}$$

in view of the above assumption (a) and of (5.5) in Lemma 5.2.

Observe that

$$\|x_{j_k} - T_s(x_{j_k})\| \to 0 \text{ as } k \to \infty. \tag{5.12}$$

Indeed,

$$\begin{aligned}\|x_{j_k} - T_s(x_{j_k})\| &\le \|x_{j_k} - x_{j_{k+1}}\| + \|x_{j_{k+1}} - T_{t_{j_{k+1}}}(x_{j_{k+1}})\| + \|T_{t_{j_{k+1}}}(x_{j_{k+1}}) - T_{t_{j_{k+1}}}(x_{j_k})\| \\ &\quad + \|T_{t_{j_{k+1}}}(x_{j_k}) - T_{s+t_{j_k}}(x_{j_k})\| + \|T_{s+t_{j_k}}(x_{j_k}) - T_s(x_{j_k})\| \\ &\le \|x_{j_k} - x_{j_{k+1}}\| + \|x_{j_{k+1}} - T_{t_{j_{k+1}}}(x_{j_{k+1}})\| + a_{t_{j_{k+1}}}(x_{j_{k+1}})\|x_{j_{k+1}} - x_{j_k}\| \\ &\quad + a_{s+t_{j_k}}(x_{j_k})\|T_{d_k}(x_{j_k}) - x_{j_k}\| + \sup_{x\in C} a_s(x)\|T_{t_{j_k}}(x_{j_k}) - x_{j_k}\|\end{aligned}$$

which tends to the zero as $k \to \infty$ because of (5.10), Lemma 5.2, the fact that the process is well defined, assumptions (b) and (2.7), and the boundedness of each function a_s.

On the other hand,

$$\begin{aligned}\|x_k - T_s(x_k)\| &\le \|x_k - x_{j_k}\| + \|x_{j_k} - T_{t_{j_k}}(x_{j_k})\| + \|T_{t_{j_k}}(x_{j_k}) - T_{s+t_{j_k}}(x_{j_k})\| \\ &\quad + \|T_{s+t_{j_k}}(x_{j_k}) - T_s(x_{j_k})\| + \|T_s(x_{j_k}) - T_s(x_k)\| \\ &\le \|x_k - x_{j_k}\| + \|x_{j_k} - T_{t_{j_k}}(x_{j_k})\| + a_{t_{j_k}}(x_{j_k})\|x_{j_k} - T_s(x_{j_k})\| \\ &\quad + a_s(x_{j_k})\|T_{t_{j_k}}(x_{j_k}) - x_{j_k}\| + a_s(x_k)\|x_{j_k} - x_k\|\end{aligned}$$

which tends to the zero as $k \to \infty$ because of assumption (a), Lemma 5.2, the fact that the process is well defined, and the fact that the semigroup is asymptotic pointwise nonexpansive. If A is a generating set for J then by Lemma 3.2, $\{x_k\}$

is an approximate fixed point sequence for any T_s. This completes the proof of the Lemma. □

We will prove next a generic version of the weak convergence theorem for the sequences $\{x_k\}$ which are generated by the Krasnosel'skii-Mann iteration process and are at the same time approximate fixed point sequences.

Theorem 5.1. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a bounded, closed and convex subset of a X. Let $\mathcal{F} \in \mathcal{S}(C)$. Assume that $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ is a well defined Krasnosel'skii-Mann iteration process. If the sequence $\{x_k\}$ generated by $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ is an approximate fixed point sequence for every $s \in A \subset J$ where A is a generating set for J, then $\{x_k\}$ converges weakly to a common fixed fixed point $w \in F(\mathcal{F})$.*

Proof. Consider $y, z \in C$, two weak cluster points of the sequence $\{x_k\}$. Then there exist two subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$. Fix any $s \in A$. Since $\{x_k\}$ is an approximate fixed point sequence for s it follows that

$$\lim_{k\to\infty} \|T_s(x_k) - x_k\| = 0. \tag{5.13}$$

It follows from the Demiclosedness Principle (Theorem 4.1) that $T_s(y) = y$ and $T_s(z) = z$. By Lemma 5.1 the following limits exist

$$r_1 = \lim_{k\to\infty} \|x_k - y\|, \ r_2 = \lim_{k\to\infty} \|x_k - z\|. \tag{5.14}$$

We claim that $y = z$. Indeed, assume to the contrary that $y \neq z$. By the Opial property we have

$$\begin{aligned} r_1 &= \liminf_{k\to\infty} \|y_k - y\| < \liminf_{k\to\infty} \|y_k - z\| = r_2 \\ &= \liminf_{k\to\infty} \|z_k - z\| < \liminf_{k\to\infty} \|z_k - y\| = r_1. \end{aligned} \tag{5.15}$$

The contradiction implies $y = z$ which means that the sequence $\{x_k\}$ has at most one weak cluster point. Since C is weakly sequentially compact, we deduce that the sequence $\{x_k\}$ has exactly one weak cluster point $w \in C$, which means that $x_k \rightharpoonup w$. Applying the Demiclosedness Principle again, we get $T_s(w) = w$. Since $s \in A$ was chosen arbitrarily and the construction of w did not depend on the selection of s, and A is a generating set for J, we conclude that $T_t(w) = w$ for any $t \in J$, as claimed. □

Let us apply the above result to some more specific situations. Let us start with a discrete case. First, we need to recall the following notions.

Definition 5.3. *A strictly increasing sequence $\{n_i\} \subset \mathbb{N}$ is called quasi-periodic if the sequence $\{n_{i+1} - n_i\}$ is bounded, or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of p consecutive natural numbers must contain a term of the sequence $\{n_i\}$. The smallest of such numbers p will be called a quasi-period of $\{n_i\}$.*

Theorem 5.2. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a bounded, closed and convex subset of a X. Let $\mathcal{F} \in \mathcal{S}(C)$ be a semigroup with a discrete generating set $A = \{\alpha_1, \alpha_2, \alpha_3 ...\}$. Assume that $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ is a well defined Krasnosel'skii-Mann iteration process. Assume that for every $m \in \mathbb{N}$ with $m \leq card(A)$, there exists a strictly increasing, quasi-periodic sequence of natural numbers $\{j_k(m)\}$, with a quasi-period p_m, such that for every $k \in \mathbb{N}$, $t_{j_{k+1}(m)} = \alpha_m + t_{j_k(m)}$. Then the sequence $\{x_k\}$ generated by $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ converges weakly to a common fixed point $w \in F(\mathcal{F})$.*

Proof. We will apply Lemma 5.3. Note that the assumption (b) of Lemma 5.3 is trivially satisfied since $t_{j_{k+1}(m)} - t_{j_k(m)} - \alpha_m = 0$. To prove (a), observe that by the quasi-periodicity of $\{j_k(m)\}$, to every positive integer k there exists $j_k(m)$ such that $|k - j_k(m)| \leq p_m$. Assume that $k - p_m \leq j_k(m) \leq k$ (the proof for the other case is identical). Fix $\varepsilon > 0$. Note that by Lemma 5.2, $\|x_{k+1} - x_k\| < \dfrac{\varepsilon}{p_m}$ for k sufficiently large. Hence for k sufficiently large there holds

$$\|x_k - x_{j_k}\| \leq \|x_k - x_{k-1}\| + ... + \|x_{j_k(m)+1} - x_{j_k(m)}\| \leq p_m \frac{\varepsilon}{p_m} = \varepsilon. \tag{5.16}$$

This proves that (a) is also satisfied. Therefore, by Lemma 5.3 $\{x_k\}$ is an approximate fixed point sequence for every T_s where $s \in \mathcal{J}$. By Theorem 5.1, $\{x_k\}$ converges weakly to a common fixed fixed point $w \in F(\mathcal{F})$. □

Remark 5.1. *Note that Theorem 4.1 in* [18] *is actually a special case of Theorem* 5.2 *with* $A = \{1\}$.

Remark 5.2. *It is easy to see that we can always construct a sequence* $\{t_k\}$ *with the properties specified in the assumptions of Theorem* 5.2*. When constructing concrete implementations of this algorithm, the difficulty will be to ensure that the constructed sequence* $\{t_k\}$ *is not "too sparse" in the sense that the Krasnosel'skii-Mann process* $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ *remains well defined (see Definition* 5.2*).*

The following theorem is an immediate consequence of Theorem 5.1 *and Lemmas* 3.2*,* 5.3*, and* 3.3*.*

Theorem 5.3. *Let* X *be a uniformly convex Banach space* X *with the Opial property. Let* C *be a bounded, closed and convex subset of* X*. Let* $\mathcal{F} \in \mathcal{S}(C)$ *be equicontinuous and* $B \subset \overline{B} = A \subset J$ *where* A *is a generating set for* J*. Let* $\{x_k\}$ *be generated by a well defined Krasnosel'skii-Mann iteration process* $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$*. If to every* $s \in B$ *there exists a strictly increasing sequence of natural numbers* $\{j_k\}$ *satisfying the following conditions:*

(a) $t_{j_{k+1}} - t_{j_k} \to s$ *as* $k \to \infty$*,*

(b) $\|x_k - x_{j_k}\| \to 0$ *as* $k \to \infty$*,*

then the sequence $\{x_k\}$ *converges weakly to a common fixed point* $w \in F(\mathcal{F})$*.*

Remark 5.3. *Observe that the set* B *in Theorem* 5.3 *can be made countable. Hence by Remark* 5.2 *a sequence* $\{t_k\}$ *satisfying assumptions of Theorem* 5.3 *can be always constructed. Again, the main difficulty is in ensuring that the corresponding process* $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ *is well defined.*

6. Weak convergence of generalized Ishikawa iteration processes

The two-step Ishikawa iteration process is a generalization of the one-step Krasnosel'skii-Mann process. The Ishikawa iteration process provides more flexibility in defining the algorithm parameters which is important from the numerical implementation perspective.

Definition 6.1. *Let* $\mathcal{F} \in \mathcal{S}(\mathcal{C})$*,* $\{t_k\} \subset J$*. Let* $\{c_k\} \subset (0,1)$*, and* $\{d_k\} \subset (0,1)$*. The generalized Ishikawa iteration process* $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ *generated by the semigroup* $\mathcal{F}$*, the sequences* $\{c_k\}$*,* $\{d_k\}$ *and* $\{t_k\}$*, is defined by the following* *iterative formula:*

$$x_{k+1} = c_k T_{t_k}(d_k T_{t_k}(x_k) + (1-d_k)x_k) + (1-c_k)x_k, \tag{6.1}$$

where $x_1 \in C$ is chosen arbitrarily, and

(1) $\{c_k\}$, $\{d_k\}$ *are bounded away from* 0 *and* 1,
(2) $\lim_{k\to\infty} t_k = \infty$,
(3) $\sum_{n=1}^{\infty} b_{t_n}(x) < \infty$ *for every* $x \in C$.

Definition 6.2. *We say that a generalized Ishikawa iteration process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ is well defined if*

$$\limsup_{k\to\infty} a_{t_k}(x_k) = 1. \tag{6.2}$$

Lemma 6.1. *Let C be a bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$, $w \in F(\mathcal{F})$, and let $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ be a generalized Ishikawa process. Then there exists an $r \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} \|x_k - w\| = r$.*

Proof. Define $G_k : C \to C$ by

$$G_k(x) = c_k T_{t_k}\Big(d_k T_{t_k}(x) + (1-d_k)x\Big) + (1-c_k)x, \ x \in C. \tag{6.3}$$

It is easy to see that $x_{k+1} = G_k(x_k)$ and that $F(\mathcal{F}) \subset F(G_k)$ for every $k \geq 1$. Moreover, a straight calculation shows that each G_k satisfies

$$\|G_k(x) - G_k(y)\| \leq A_k(x)\|x - y\|, \tag{6.4}$$

where

$$A_k(x) = 1 + c_k a_{t_k}\Big(d_k T_{t_k}(x) + (1-d_k)x\Big)(1 + d_k a_{t_k}(x) - d_k) - c_k. \tag{6.5}$$

Note that $A_k(x) \geq 1$ which follows directly from the fact that $a_{t_k}(z) \geq 1$ for any $z \in C$. Using (6.5) and remembering that $w \in F(\mathcal{F})$ we have

$$B_k(w) = A_k(w) - 1 = c_k(1 + d_k a_{t_k}(w))(a_{t_k}(w) - 1) \leq (1 + a_{t_k}(w))b_{t_k}(w). \tag{6.6}$$

Fix any $M > 1$. Since $\lim\limits_{k\to\infty} a_{t_k}(w) = 1$, it follows that there exists a $k_0 \geq 1$ such that for $k > k_0$, $a_{t_k}(w) \leq M$. Therefore, using the same argument as in the proof of Lemma 5.1, we deduce that for $k > k_0$ and $n > 1$

$$\begin{aligned}\|x_{k+n} - w\| &\leq \|x_k - w\| + diam(C) \sum_{i=k}^{k+n-1} B_{t_i}(w) \\ &\leq \|x_k - w\| + diam(C)(1+M) \sum_{i=k}^{k+n-1} b_{t_i}(w).\end{aligned} \tag{6.7}$$

Arguing like in the proof of Lemma 5.1, we conclude that there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. □

Lemma 6.2. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ be a well defined generalized Ishikawa iteration process. Then*

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0 \tag{6.8}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{6.9}$$

Proof. By Theorem 2.1, $F(\mathcal{F}) \neq \emptyset$. Let us fix $w \in F(\mathcal{F})$. By Lemma 6.1, $\lim_{k\to\infty} \|x_k - w\|$ exists. Let us denote it by r. Let us define

$$y_k = d_k T_{t_k}(x_k) + (1 - d_k)x_k. \tag{6.10}$$

Since $w \in F(\mathcal{F})$, $\mathcal{F} \in \mathcal{S}(C)$, and $\lim_{k\to\infty} \|x_k - w\| = r$, we have the following

$$\begin{aligned}&\limsup_{k\to\infty} \|T_{t_k}(y_k) - w\| = \limsup_{k\to\infty} \|T_{t_k}(y_k) - T_{t_k}(w)\| \\ &\leq \limsup_{k\to\infty} a_{t_k}(w)\|y_k - w\| = \limsup_{k\to\infty} a_{t_k}(w)\|d_k T_{t_k}(x_k) + (1-d_k)x_k - w\| \\ &\leq \limsup_{k\to\infty} \Big(d_k a_{t_k}(w)\|T_{t_k}(x_k) - w\| + (1-d_k)a_{t_k}(w)\|x_k - w\|\Big) \\ &\leq \lim_{k\to\infty} \Big(d_k a_{t_k}^2(w)\|x_k - w\| + (1-d_k)a_{t_k}(w)\|x_k - w\|\Big) \leq r.\end{aligned} \tag{6.11}$$

Note that

$$\begin{aligned}&\lim_{k\to\infty} \|d_k(T_{t_k}(y_k) - w) + (1-d_k)(x_k - w)\| \\ &= \lim_{k\to\infty} \|d_k T_{t_k}(y_k) + (1-d_k)x_k - w\| = \lim_{k\to\infty} \|x_{k+1} - w\| = r.\end{aligned} \tag{6.12}$$

Applying Lemma 2.2 with $u_k = T_{t_k}(y_k) - w$ and $v_k = x_k - w$, we obtain the equality $\lim_{k\to\infty} \|T_{t_k}(y_k) - x_k\| = 0$. This fact, combined with the construction formulas for x_{k+1} and y_k, proves (6.9).

Since

$$\begin{aligned} \|T_{t_k}(x_k) - x_k\| &\leq \|T_{t_k}(x_k) - T_{t_k}(y_k)\| + \|T_{t_k}(y_k) - x_k\| \\ &\leq a_{t_k}(x_k)\|x_k - y_k\| + \|T_{t_k}(y_k) - x_k\| \\ &= d_k a_{t_k}(x_k)\|T_{t_k}(x_k) - x_k\| + \|T_{t_k}(y_k) - x_k\|, \end{aligned} \tag{6.13}$$

it follows that

$$\|T_{t_k}(x_k) - x_k\| \leq (1 - d_k a_{t_k}(x_k))^{-1}\|T_{t_k}(y_k) - x_k\|. \tag{6.14}$$

The right-hand side of this inequality tends to zero because $\|T_{t_k}(y_k) - x_k\| \to 0$, $\limsup_{k\to\infty} a_{t_k}(x_k) = 1$ by the fact that the Ishikawa process is well defined, and $\{d_k\} \subset (0,1)$ is bounded away from 1. □

We need the following technical result being the Ishikawa version of Lemma 5.3.

Lemma 6.3. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(\mathcal{C})$. Let the generalized Ishikawa process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ be well defined. Let $A \subset J$ be such that to every $s \in A$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

(a) $\|x_k - x_{j_k}\| \to 0$ as $k \to \infty$,

(b) $\lim_{k\to\infty} \|T_{e_k}(x_{j_k}) - x_{j_k}\| = 0$, where $e_k = |t_{j_{k+1}} - t_{j_k} - s|$.

Then $\{x_k\}$ is an approximate fixed point sequence for all mappings $\{T_{ms}\}$ where $s \in A$ and $m \in \mathbb{N}$, that is

$$\lim_{k\to\infty} \|T_{ms}(x_k) - x_k\| = 0 \tag{6.15}$$

for every $s \in A$ and $m \in \mathbb{N}$. If, in addition, A is a generating set for J then

$$\lim_{k\to\infty} \|T_t(x_k) - x_k\| = 0 \tag{6.16}$$

for any $t \in J$.

Proof. The proof is analogous to that of Lemma 5.3 with Lemma 5.1 replaced by Lemma 6.1, and Lemma 5.2 replaced by Lemma 6.2. □

We are now ready to provide the weak convergence results for the Ishikawa iteration processes.

Theorem 6.1. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a bounded, closed and convex subset of a X. Let $\mathcal{F} \in \mathcal{S}(C)$. Assume that $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ is a well defined Ishikawa iteration process. If the sequence $\{x_k\}$ generated by $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ is an approximate fixed point sequence for every $s \in A \subset J$ where A is a generating set for J, then $\{x_k\}$ converges weakly to a common fixed fixed point $w \in F(\mathcal{F})$.*

Proof. The proof is analogous to that of Theorem 5.1 with Lemma 5.3 replaced by Lemma 6.3, and Lemma 5.1 replaced by Lemma 6.1. □

Similarly, it is easy to modify the proof of Theorems 5.2 *and* 5.3 *to obtain the next two results.*

Theorem 6.2. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a bounded, closed and convex subset of a X. Let $\mathcal{F} \in \mathcal{S}(C)$ be a semigroup with a discrete generating set $A = \{\alpha_1, \alpha_2, \alpha_3 ...\}$. Assume that $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ is a well defined Ishikawa iteration process. Assume that for every $m \in \mathbb{N}$ there exists a strictly increasing, quasi-periodic sequence of natural numbers $\{j_k(m)\}$, with a quasi-period p_m, such that for every $k \in \mathbb{N}$, $t_{j_{k+1}(m)} = \alpha_m + t_{j_k(m)}$. Then the sequence $\{x_k\}$ generated by $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ converges weakly to a common fixed point $w \in F(\mathcal{F})$.*

Theorem 6.3. *Let X be a uniformly convex Banach space X with the Opial property. Let C be a bounded, closed and convex subset of X. Let $\mathcal{F} \in \mathcal{S}(C)$ be equicontinuous and $B \subset \overline{B} = A \subset J$ where A is a generating set for J. Let $\{x_k\}$ be generated by a well defined Ishikawa iteration process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$. If to every $s \in B$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

(a) $t_{j_{k+1}} - t_{j_k} \to s$ *as* $k \to \infty$,

(b) $\|x_k - x_{j_k}\| \to 0$ *as* $k \to \infty$,

then the sequence $\{x_k\}$ converges weakly to a common fixed point $w \in F(\mathcal{F})$.

7. Strong convergence of generalized Krasnosel'skii-Mann and Ishikawa iteration processes

Lemma 7.1. *Let C be a compact subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$ and $\{s_n\} \subset J$. If $s_n \to 0$ as $n \to \infty$ then $\mathcal{F}$ is equicontinuous, that is*

$$\lim_{n\to\infty} \sup_{x\in C} \|T_{s_n}(x) - x\| = 0. \tag{7.1}$$

Proof. Assume to the contrary that (7.1) does not hold. Then there exist $\{w_k\}$ a subsequence of $\{s_n\}$, a sequence $\{y_k\} \subset C$ and $\eta > 0$ such that for every $k \in \mathbb{N}$ there holds

$$\|T_{w_k}(y_k) - y_k\| > \eta > 0. \tag{7.2}$$

Using compactness of C and passing to a subsequence of $\{y_k\}$ if necessary we can assume that there exists $w \in C$ such that $\|y_k - w\| \to 0$ as $k \to \infty$.

$$\begin{aligned} 0 < \eta &\leq \limsup_{k\to\infty} \|T_{w_k}(y_k) - y_k\| \\ &\leq \limsup_{k\to\infty} (\|T_{w_k}(y_k) - T_{w_k}(w)\| + \|T_{w_k}(w) - w\| + \|w - y_k\|) \\ &\leq \limsup_{k\to\infty} (a_{w_k}(w)\|y_k - w\| + \|T_{w_k}(w) - w\| + \|w - y_k\|) = 0 \end{aligned} \tag{7.3}$$

since $\limsup_{k\to\infty} a_{w_k}(w) \leq 1$ and $t \mapsto T_t(w)$ is continuous. Contradiction. □

Theorem 7.1. *Let C be a compact, convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$ and $B \subset \overline{B} = A \subset J$ where A is a generating set for J. Let $\{x_k\}$ be generated by a well defined Krasnosel'skii-Mann iteration process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ (resp. generalized Ishikawa process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$). If to every $s \in B$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

(a) $t_{j_{k+1}} - t_{j_k} \to s$ as $k \to \infty$,

(b) $\|x_k - x_{j_k}\| \to 0$ as $k \to \infty$,

then the sequence $\{x_k\}$ converges strongly to a common fixed point $x \in F(\mathcal{F})$.

Proof. We apply Lemma 5.3 (resp. Lemma 6.3) for the parameter set B. Note that condition (a) of Lemma 5.3 (resp. Lemma 6.3) is assumed. By Lemma 7.1 the semigroup $\mathcal{F}$ is equicontinuous and hence the assumption (b) of Lemma 5.3 (resp. Lemma 6.3) is satisfied. By Lemma 5.3 (resp. Lemma 6.3) then $\{x_k\}$ is an approximate fixed point sequence for any T_t where $t \in B$. By Lemma 3.3 $\{x_k\}$ is an approximate fixed point sequence for any T_t where $t \in A$. Since A is a generating set for J, it follows that $\{x_k\}$ is an approximate fixed point sequence for any T_t where $t \in J$ (again, by Lemma 5.3 or respectively Lemma 6.3 for the Ishikawa case). Hence for every $t \in J$

$$\|T_t(x_k) - x_k\| \to 0 \text{ as } k \to \infty. \tag{7.4}$$

Since C is compact there exist a subsequence $\{x_{p_k}\}$ of $\{x_k\}$, and $x \in C$ such that

$$\|T_t(x_{p_k}) - x\| \to 0 \text{ as } k \to \infty. \tag{7.5}$$

Observe that

$$\|x_{p_k} - x\| \leq \|x_{p_k} - T_t(x_{p_k})\| + \|T_t(x_{p_k}) - x\|, \tag{7.6}$$

which tends to zero as $k \to \infty$ by (7.4) and (7.5). Hence

$$\lim_{k\to\infty} \|x_{p_k} - x\| = 0. \tag{7.7}$$

Finally

$$\begin{aligned} \|T_t(x) - x\| &\leq \|T_t(x) - T_t(x_{p_k})\| + \|T_t(x_{p_k}) - x_{p_k}\| + \|x_{p_k} - x\| \\ &\leq a_t(x)\|x_{p_k} - x\| + \|T_t(x_{p_k}) - x_{p_k}\| + \|x_{p_k} - x\|, \end{aligned} \tag{7.8}$$

which tends to zero as $k \to \infty$ by boundedness of the function a_t, by (7.7), and (7.4). Therefore, $T_t(x) = x$ for every $t \in J$, that is x is a common fixed point for the semigroup $\mathcal{F}$. By Lemma 5.1 (resp. Lemma 6.1), $\lim_{k\to\infty} \|x_k - x\|$ exists which, via (7.7) implies that

$$\lim_{k\to\infty} \|x_k - x\| = 0 \tag{7.9}$$

completing the proof of the theorem. □

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