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LOCALLY ALMOST NONEXPANSIVE MAPPINGS

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Abstract

Let X be a Banach space satisfying Opial's property, C a weakly compact convex subset of X, and T a locally almost nonexpansive self-mapping of C. We prove that $I - T$ is demiclosed on C, and if T is weakly asymptotically regular at $x \in C$ (i.e., $T^n x - T^{n+1}x \to 0$ weakly), then every weak cluster point of $(T^n x)$ is a fixed point of T. We also prove a fixed point theorem for multivalued locally almost nonexpansive mappings.

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1 Introduction

Let X be a Banach space and C a nonempty closed bounded convex subset of X. Recall that a mapping $T : C \to X$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. In 1965, F.E. Browder [1], D. Göhde [6] and W.A. Kirk [7] independently proved that if X is uniformly convex, then every nonexpansive self-mapping of C has a fixed point. In 1968, F.E. Browder [2] introduced a wider class of mappings called semicontractive. A mapping $T : C \to X$ is called *semicontractive* if there exists a mapping $V : C \times C \to X$ such that (a) $Tx = V(x, x)$ for $x \in C$; (b) for each fixed $y \in C$, $V(\cdot, y)$ is a

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nonexpansive map; and (c) given $z \in C$ and $\varepsilon > 0$, there is a weak neighborhood N_z of z in C such that for $x, y \in N_z$, $\|V(x,y) - V(x,z)\| < \varepsilon$. Later in 1972, R.D. Nussbaum [9] proposed the class of locally almost nonexpansive mappings. A mapping $T : C \to X$ is called *locally almost nonexpansive* (*LANE* for short) if for all $x \in C$ and $\varepsilon > 0$, there exists a weak neighborhood $N_x = N(x, \varepsilon)$ of x in C such that

$$\|Tu - Tv\| \le \|u - v\| + \varepsilon, \quad u, v \in N_x.$$

Nussbaum [9] observed that a semicontractive mapping is LANE. He also observed that if $T_1 : C \to X$ is nonexpansive, $T_2 : C \to X$ is completely continuous, and $T_3 : (T_1 + T_2)(C) \to X$ is nonexpansive, then $T_3(T_1 + T_2)$ is LANE.

One can use nets and sequences to characterize local almost nonexpansivity. The following lemma is not hard to prove. (Throughout this paper, '$\rightharpoonup$' stands for weak convergence and '$\to$' for strong convergence.)

Lemma 1.1. Let X be a Banach space and C a closed convex subset of X. Then $T : C \to X$ is LANE if and only if for all $x \in C$, and nets $(x_\alpha)_{\alpha \in \Lambda}$ and $(y_\alpha)_{\alpha \in \Lambda}$ in C with $x_\alpha \rightharpoonup x$ and $y_\alpha \rightharpoonup x$, it follows that

$$\limsup_{\alpha \in \Lambda} \|Tx_\alpha - Ty_\alpha\| \le \limsup_{\alpha \in \Lambda} \|x_\alpha - y_\alpha\|. \tag{1.1}$$

If C is separable, then T is LANE if and only if for all $x \in C$ and sequences (x_n) and (y_n) in C both weakly converging to x, it follows that

$$\limsup_{n \to \infty} \|Tx_n - Ty_n\| \le \limsup_{n \to \infty} \|x_n - y_n\|. \tag{1.2}$$

In particular, if T is LANE and $x_n \rightharpoonup x$, then

$$\limsup_{n \to \infty} \|Tx_n - Tx\| \le \limsup_{n \to \infty} \|x_n - x\|. \tag{1.3}$$

Remark 1.2. In both (1.1) and (1.2), 'lim sup' can be replaced with 'lim inf'.

As an immediate consequence of Lemma 1.1 we have

Corollary 1.3. Let X and C be as in Lemma 1.1.

1. T is LANE if it is weak-to-norm continuous on C (i.e., $x \in C$, $(x_\alpha) \subset C$, $x_\alpha \rightharpoonup x \Rightarrow T(x_\alpha) \to T(x)$).

2. If X is finite-dimensional, then T is LANE if anly only if T is continuous on C.

Corollary 1.3 points out a big difference between the class of nonexpansive mappings and the class of locally almost nonexpansive mappings. One thus concludes that many nice properties of nonexpansive mappings are not shared by the LANE mappings. We here include one which is pertinent to Theorem 3.1, one of the main results of this paper. It is easily seen that if $T : C \to C$ is nonexpansive and p is a fixed point, then $\lim \|T^n x - p\|$ exists for every $x \in C$. However, this is not true for LANE mappings. For example, let X be the real line, $C = [-1, 1]$ and $T : C \to C$ be given by $T(x) = -x$ if $-1 \le x \le -\frac{1}{2}$ or $\frac{1}{2} \le x \le 1$, $T(\frac{1}{4}) = \frac{1}{4}$ and T is linearly extended to the rest of C. Then $\frac{1}{4}$ is the only fixed point of T. Since $T^n(1) = (-1)^n$, $\lim |T^n(1) - \frac{1}{4}|$ does not exist.

Nussbaum [9] proved that if X is reflexive and $T : C \to X$ is LANE, then T is a 1-set contraction. He further proved that if X is uniformly convex, then $I - T$ (with I the identity) is demiclosed on C and hence T has a fixed point provided T is a self-mapping of C.

In the present paper we shall prove that if X satisfies Opial's property and $T : C \to X$ is LANE, then $I - T$ is demiclosed on C and hence T admits a fixed point when T maps C into itself; moreover, if T is weakly asymptotically regular at $x \in C$ (i.e., $w - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$), then every weak cluster point of $(T^n x)$ is a fixed point of T. We shall also prove a fixed point theorem for multivalued LANE mappings.

2 Demiclosedness Principle

Let X be a Banach space, let C be a nonempty closed bounded convex subset of X, and let $f : C \to X$ be a mapping. Recall that f is said to satisfy the *demiclosedness principle* or be *demiclosed on* C if for any sequence (x_n) in C, the conditions $x_n \rightharpoonup x$ and $f(x_n) \to y$ imply that $f(x) = y$. It is known [1] [10] that if X is a uniformly convex Banach space or satisfies Opial's property and $T : C \to X$ is nonexpansive, then $I - T$ is demiclosed on C. Recall that X is said to satisfy *Opial's property* [10] if given any sequence (x_n) in X with $x_n \rightharpoonup x_\infty$, we have

$$\limsup_{n\to\infty} \|x_n - x_\infty\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in X \setminus \{x_\infty\}.$$

Banach spaces having Opial's property include Hilbert spaces and the spaces l^p $(1 \le p < \infty)$. Further any separable Banach space can equivalently be renormed to have Opial's property [5].

Nussbaum [9], in a uniformly convex Banach space setting, extended the demiclosedness principle from nonexpansive mappings to locally almost nonexpansive mappings. Below we make a similar extension in the framework of Banach spaces with Opial's property.

Theorem 2.1. Assume that X is a Banach space satisfying Opial's property, that C is a nonempty closed convex subset of X, and that $T : C \to X$ is LANE. Then $I - T$ is demiclosed on C and hence $(I - T)(E)$ is closed for every weakly compact subset E of C.

Proof. Assume (x_n) is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$. If $(I - T)x \neq y$, then $x \neq Tx + y$. Hence Opial's property for X and (1.3) of Lemma 1.1 imply that

$$\begin{aligned}
\limsup_{n\to\infty} \|x_n - x\| &< \limsup_{n\to\infty} \|x_n - (Tx + y)\| \\
&= \limsup_{n\to\infty} \|x_n - Tx - (I - T)x_n\| \\
&= \limsup_{n\to\infty} \|Tx_n - Tx\| \\
&\le \limsup_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

This is a contradiction. So we must have $(I - T)x = y$ and $I - T$ is demiclosed. Finally if E is a weakly compact subset of C, then the closedness of $(I - T)(E)$ is an immediate consequence of the demiclosedness of $I - T$ and the weak compactness of C. ∎

Recall that the inward set to a closed convex set C at $x \in C$ is defined by

$$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, \ y \in C\}.$$

Let $\bar{I}_C(x) = \overline{I_C(x)}$, the closure of $I_C(x)$. A map $f : C \to X$ is said to satisfy the *weak inwardness condition* (or to be *weakly inward*) on C if $f(x) \in \bar{I}_C(x)$ for all $x \in C$. In case the interior of C is nonempty, we say that f satisfies the *Leray-Schauder condition* if there exists a point $z \in \operatorname{int} C$ such that

$$f(x) \neq z + \lambda(x - z) \quad \text{for all } \lambda > 1 \text{ and } x \in \partial C.$$

Note that in case $\operatorname{int} C \neq \emptyset$, the weak inwardness condition implies the Leray-Schauder condition.

Corollary 2.2. Let X be a Banach space satisfying Opial's property, C a weakly compact convex subset of X, and $T : C \to X$ LANE. If T is weakly inward on C, or if, in case $\operatorname{int} C \neq \emptyset$, T satisfies the Leray-Schauder condition, then T has a fixed point.

Proof. Fix an $x_0 \in C$ and define for each integer $n \geq 1$ a map $T_n : C \to X$ by

$$T_n x = \frac{1}{n} x_0 + (1 - \frac{1}{n}) T x, \quad x \in C.$$

Then [9, Lemma 1] imlpies that each T_n is a $(1 - \frac{1}{n}) - \gamma$—contraction and hence in either the weakly inward case or the Leray-Schauder case, T_n has a fixed point $x_n \in C$; see [3] [4]. It is easily seen that

$$\|(I - T)x_n\| \leq \frac{1}{n} \operatorname{diam}(C) \to 0 \ (n \to \infty).$$

Since C is weakly compact, by Theorem 2.1, we see that every weak cluster point of (x_n) is a fixed point of T. ∎

Remark 2.3. Nussbaum [9] proved that if X is a uniformly convex Banach space, if C is a closed bounded convex subset of X, and if $T : C \to X$ is LANE, then T has a fixed point in C provided T satisfies the Rothe boundary condition; namely, $T(\partial C) \subset C$. The above proof shows that Nussbaum's result is still valid if the Rothe boundary condition is relaxed to either the weak inwardness condition, or in case $\operatorname{int} C \neq \emptyset$, to the Leray-Schauder condition.

3 Weak Asymptotic Behavior

Let X be a Banach space and let C be a closed convex subset of X. Recall that a mapping $T : C \to C$ is *weakly asymptotically regular* at $x \in C$ if $w - \lim_{n \to \infty}(T^n x - T^{n+1} x) = 0$. Kuczumow [8] proved that if X satisfies Opial's property and $T : C \to C$ is nonexpansive, then the weak asymptotic regularity of T at $x \in C$ implies the weak convergence of the sequence $(T^n x)$ to a fixed point of T. We can not fully recover this result for a locally almost nonexpansive mapping T. The difficulties lie in that the local almost nonexpansiveness of T at x only works for a sequence (x_n) which is weakly convergent to x (see Lemma 1.1) and that for a fixed point p of T, the $\lim \|T^n x - p\|$ may fail to exist (see the example in Section one).

Theorem 3.1. Let X be a Banach space satisfying Opial's property, let C be a closed convex subset of X, and let $T : C \to C$ be LANE. Then, if T is weakly asymptotically regular at $x \in C$, we have that $\omega_w(x) \subset F(T)$, where $\omega_w(x)$ is the weak ω-limit set of T at x; i.e.,

$$\omega_w(x) = \left\{ z \in X : z = w - \lim_{i \to \infty} T^{n_i}x \text{ for some } n_i \to \infty \right\}.$$

If we assume, in addition, that the $\lim_{n \to \infty} \|T^n x - p\|$ exists for all fixed points p of T, then $(T^n x)$ weakly converges to a fixed point of T.

Proof. Let p be a point in $\omega_w(x)$. Then we have a subsequence (n_i), $n_i \to \infty$, such that $T^{n_i}x \rightharpoonup p$. By using the diagonal method and passing to a further subsequence if necessary we can assume for each integer $m \geq 0$ that $\lim_{i \to \infty} \|T^{n_i+m}x - p\| =: b_m$ exists. Note that by the weak asymptotic regularity, we have for all $m \geq 0$, $T^{n_i+m}x \rightharpoonup p$ as $i \to \infty$. It follows that

$$\begin{aligned} b_{m+1} &= \lim_{i \to \infty} \|T^{n_i+m+1}x - p\| \\ &\leq \limsup_{i \to \infty} \|T^{n_i+m+1}x - Tp\| \qquad \text{by Opial's property} \\ &\leq \lim_{i \to \infty} \|T^{n_i+m}x - p\| \qquad \text{by Lemma 1.1} \\ &= b_m. \end{aligned}$$

Hence (b_m) is a decreasing sequence. Put $b := \lim_{m \to \infty} b_m = \inf\{b_m : m \geq 0\}$. We now claim that p is a fixed point of T.

If $b = 0$, using Lemma 1.1 we get

$$\begin{aligned} \|Tp - p\| &\leq \limsup_{i \to \infty} \|Tp - T^{n_i+m+1}x\| + \limsup_{i \to \infty} \|T^{n_i+m+1} - p\| \\ &\leq b_m + b_{m+1} \to 0 \ (m \to \infty). \end{aligned}$$

Hence $Tp = p$. Assume next $b > 0$. Consider the countable set $G := \{T^{n_i+m}x : i \geq 1, m \geq 0\} \cup \{p\}$. Since the weak topology of X restricted to G satisfies the First Countable Axiom, we can find a countable weak neighborhoods (N_i) of p in G such that $\cap_{i=1}^{\infty} N_i = \{p\}$. Now for each $k \geq 1$ we select a subsequence $\left(m_i^{(k)}\right)_{i=1}^{\infty}$ of $\left(m_i^{(k-1)}\right)_{i=1}^{\infty}$ satisfying for $i \geq 1$,

$$\begin{aligned} \left\|T^{n_i^{(k)}+k}x - p\right\| &> b_k - \frac{1}{k}, \\ \left\|T^{n_i^{(k)}+k-1}x - p\right\| &< b_{k-1} + \frac{1}{k}, \\ T^{n_i^{(k)}+k-1}x &\in N_k. \end{aligned}$$

This is possible because for each $m \geq 0$, $T^{n_i+m}x \rightharpoonup p$ as $i \to \infty$ and $b_k = \lim_{i \to \infty} \|T^{n_i+k}x - p\| \leq \lim_{i \to \infty} \|T^{n_i+k-1}x - p\| = b_{k-1}$. Let $m_k = n_k^{(k)} + k - 1$. Then we have for $k \geq 1$,

$$\begin{aligned} T^{m_k}x &\rightharpoonup p, \\ \|T^{m_k}x - p\| &< b_{k-1} + \frac{1}{k}, \\ \|T^{m_k+1}x - p\| &> b_k - \frac{1}{k}. \end{aligned}$$

We therefore have if $Tp \neq p$,

$$\begin{aligned}
b &\leq \limsup_{k\to\infty} \|T^{m_k+1}x - p\| \\
&< \limsup_{k\to\infty} \|T^{m_k+1}x - Tp\| \qquad \text{by Opial's property} \\
&\leq \limsup_{k\to\infty} \|T^{m_k}x - p\| \qquad \text{by Lemma 1.1} \\
&\leq b.
\end{aligned}$$

This is a contradiction. So we must have $Tp = p$ and $\omega_w(x) \subset F(T)$. To finish the proof, assume in addition that $\lim_{n\to\infty} \|T^n x - y\|$ exists for all fixed points y of T. If $p, q \in \omega_w(x)$, then $T^{n_i}x \rightharpoonup p$ and $T^{m_j}x \rightharpoonup q$ for some $n_i \to \infty$ and $m_j \to \infty$. Since $p, q \in F(T)$, it follows that if $p \neq q$,

$$\begin{aligned}
\lim_{n\to\infty} \|T^n - p\| &= \lim_{i\to\infty} \|T^{n_i}x - p\| \\
&< \lim_{i\to\infty} \|T^{n_i}x - q\| \qquad \text{by Opial's property} \\
&= \lim_{j\to\infty} \|T^{m_j}x - q\| \\
&< \lim_{j\to\infty} \|T^{m_j}x - p\| \qquad \text{by Opial's property} \\
&= \lim_{n\to\infty} \|T^n x - p\|,
\end{aligned}$$

which is a contradiction. ■

4 Multivalued Extension

This section is devoted to a multivalued extension of the concept of locally almost nonexpansive mappings. For a nonempty closed convex subset C of a Banach space X, we denote by $K(C)$ (resp. $KC(C)$) the family of nonempty compact (resp. compact convex) subsets of C. Let H be the Hausdorff distance induced by the norm of X; thus we have for $A, B \in K(C)$,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(x, K) := \inf\{\|x - y\| : y \in K\}$ is the distance from a point $x \in X$ to a subset $K \subset X$.

Definition 4.1. A multivalued map $T : C \to K(X)$ is said to be *locally almost nonexpansive (LANE* for short) if for every $x \in C$ and $\varepsilon > 0$, there exists a weak neighborhood N_x in C of x such that

$$H(Tu, Tv) \leq \|u - v\| + \varepsilon, \quad u, v \in N_x.$$

Note that a LANE multivalued map T is continuous with respect to the Hausdorff distance and hence it is both upper and lower semicontinuous.

Recall that the Hausdorff measure of a bounded subset B of a Banach space X is defined by

$$\beta(B) := \inf\{r > 0 : \quad B \text{ can be covered by a finite family of balls each with radius less than } r\}.$$

A multivalued map $T : C \to K(X)$ is called *a 1-β-contraction* if $\beta(T(B)) \leq \beta(B)$ for all bounded subsets B of C. Here $T(B) = \cup\{Tx : x \in B\}$.

Lemma 4.2. Let C be a weakly compact convex subset of a Banach space X and let $T : C \to K(X)$ be LANE. Then T is a 1-β-contraction.

Proof. We need to prove

$$\beta(T(B)) \leq \beta(B) \quad \forall B \subset C.$$

For any $\varepsilon > 0$, by weak compactness, C can be covered by finitely many subsets of C, $N_1, N_2, \cdots, N_m$, such that

$$H(Tu, Tv) \leq \|u - v\| + \varepsilon, \quad u, v \in N_i, \ 1 \leq i \leq m.$$

Repeat the argument of Deimling [4, p. 113] to get $\beta(T(N_i)) \leq \beta(N_i) + \varepsilon$ for $1 \leq i \leq m$. Now for a subset B of C, we have $B = \cup_{i=1}^m B \cap N_i$, and so

$$\beta(T(B)) = \max_{1 \leq i \leq m} \beta(T(B \cap N_i)) \leq \max_{1 \leq i \leq m} \beta(B \cap N_i) + \varepsilon \leq \beta(B) + \varepsilon.$$

But, $\varepsilon > 0$ is arbitrary, we get $\beta(T(B)) \leq \beta(B)$. ■

Theorem 4.3. Let X be a Banach space satisfying Opial's property, let C be a weakly compact convex subset of X, and let $T : C \to KC(X)$ be LANE. Assume that

$$Tx \cap \bar{I}_C(x) \neq \emptyset \quad \text{for all } x \in C,$$

or in case $\operatorname{int} C \neq \emptyset$, that the Leray-Schauder condition holds; that is, there is some $z \in \operatorname{int} C$ for which

$$z + \lambda(x - z) \notin Tx \quad \text{for all } \lambda > 1 \text{ and } x \in \partial C.$$

Then T has a fixed point.

Proof. Take a fixed $x_0 \in C$ and define for each integer $n \geq 1$ a mapping $T_n : C \to KC(X)$ by

$$T_n x = \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx, \quad x \in C.$$

Then by Lemma 4.2, T_n is a $(1 - \frac{1}{n}) - \beta$-contraction. It is easily seen that T_n satisfies the same boundary condition as T does. Hence by Theorems 11.5 and 11.6 of Deimling [4], T_n has a fixed point $x_n \in C$; i.e., $x_n \in C$ is a solution to the inclusion

$$x_n \in \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx_n \tag{4.1}$$

Since C is weakly compact, we can assume that (x_n) is weakly convergent. Let $z = w - \lim x_n$. Take $y_n \in Tx_n$ such that $\|x_n - y_n\| = d(x_n, Tx_n) \leq \frac{1}{n} \operatorname{diam}(C)$ by (4.1). We also have $z_n \in Tz$ for which $\|y_n - z_n\| = d(y_n, Tz)$. By the compactness of Tz, we can assume that $z_n \to \bar{z} \in Tz$. Now since T is LANE, given any $\varepsilon > 0$, we have a weak neighborhood N_z of z such that

$$H(Tu, Tv) \leq \|u - v\| + \varepsilon, \quad u, v \in N_z.$$

As $x_n \rightharpoonup z$, we have for all large n,

$$H(Tx_n, Tz) \leq \|x_n - z\| + \varepsilon.$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\|x_n-\bar{z}\| &= \limsup_{n\to\infty}\|y_n-z_n\| \\ &= \limsup_{n\to\infty} d(y_n,Tz) \\ &\le \limsup_{n\to\infty} H(Tx_n,Tz) \\ &\le \limsup_{n\to\infty}\|x_n-z\|+\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\limsup_{n\to\infty}\|x_n-\bar{z}\| \le \limsup_{n\to\infty}\|x_n-z\|$. Opial's property of X then yields that $\bar{z} = z \in Tz$. ∎

References

[1] F.E. Browder, Nonexpansive nonlinear nonexpansive operators in a Banach space, *Proc. Nat. Acad. Sci. U.S.A., 54(1965),* 1041–1044.

[2] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.*, 74(1968), 660–668.

[3] K. Deimling, "Nonlinear Functional Analysis," Springer, 1985 .

[4] K. Deimling, "Multivalued Differential Equations," de Gruyter Series in Nonlinear Analysis and Applications, 1, alter de Gruyter, Berlin/New York, 1992.

[5] D. van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, *J. London Math. Soc.*, 25(1982), 139–144.

[6] D. Göhde, Zum Prinzip der kontraktiven Abbildung, *Math. Nach.*, 30(1965), 251–258.

[7] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*, 72(1965), 1004–1006.

[8] T. Kuczumow, Weak convergence theorems for nonexpansiive mappings and semigroups in Banach spaces with Opial's property, *Proc. Amer. math. Soc.*, 93(1985), 430–432.

[9] R.D. Nussbaum, Degree theory for local condensing maps, *J. Math. Anal. Appl.*, 37(1972), 741–766.

[10] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, 73(1967), 595–597.

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