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More on minimal invariant sets for nonexpansive mappings

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Abstract

Minimal invariant sets for nonexpansive mappings share some singular geometrical properties. Here we present some seemingly unknown ones.

Key words: nonexpansive mappings, fixed points, minimal invariant sets

1 Introduction

The basic technique for proving fixed point theorems for nonexpansive mappings is based on an analysis of whether the geometrical properties of the Banach space under consideration allow the existence of nontrivial *minimal invariant sets.* The classical results of F.E. Browder [4], D.Gőhde [10] and W.A. Kirk [13] and many further ones have been proved this way (see the books [1],[3],[9].). Since the work of D. Alspach [2] it is known that there are convex, minimal invariant, weakly compact sets of strictly positive diameter. Such sets display some 'bizarre' geometrical properties (see [7],[8],[12]).

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The aim of this paper is to add some seemingly new facts concerning these exotic objects.

Let C be a nonempty, convex, closed and bounded subset of a Banach space X, and let the mapping $T : C \to C$ be *nonexpansive,* in the sense that

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$.

The set C can contain many smaller nonempty, closed, convex, $T-$*invariant* subsets D. Here D is $T-$invariant means $T(D) \subseteq D$. The family $\mathcal{T}$ of all such T-invariant sets is partially ordered by inclusion. A set D in $\mathcal{T}$ is said to be *minimal invariant* if it is minimal with respect to this ordering, that is it does not have any elements of $\mathcal{T}$ as a proper subset. Obviously any set consisting of only one element x, necessarily a fixed point of T, is minimal. Without further assumptions on C, the existence of a minimal invariant subset for T is not assured. A natural condition to impose is that C be *weakly compact.* Then, the existence of a minimal invariant subset follows from a standard application of Zorn's Lemma. This ensures existence, but does not imply uniqueness. Indeed, we will see later that there can be more than one minimal invariant set for T in C. Further, the family $\mathcal{T}$ furnished with the Hausdorff metric is a complete metric space, as is the closed subfamily $\mathcal{T}_0$ consisting of all minimal invariant sets for T.

This allows us to use the following fact. If $D \subset C$ is closed and convex, thus weakly compact, then for any $z \in C$ there exists at least one point $x \in D$ such that $\|z - x\| = \operatorname{dist}(z, D)$. Moreover the set of all such points is closed and convex. This defines *the metric projection* of z onto D, which we will denote by $P_D(z)$. Obviously

$$P_D(z) = D \cap B[z, r],$$

where $r = \operatorname{dist}(z, D)$ and $B[z, r]$ denotes the closed ball centered at z and of radius r.

An answer to the question of whether there are weakly compact minimal invariant sets other than singleton ones is provided by the example of Dale Alspach [2].

Example 1. Let $X = L_1[0, 1]$ and let C be the order segment given by

$$C = \{f \in L_1 : 0 \leq f \leq 1\}.$$

The baker transformation

$$(Tf)(t) = \begin{cases} \min\{2f(2t), 1\} & \text{if } 0 \leq t \leq \frac{1}{2} \\ \max\{2f(2t-1) - 1, 0\} & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

isometrically transforms C into itself. Only the two constant functions, 0 and

1, are fixed points of T. The whole set $C = \bigcup_{a \in [0,1]} C_a$ where

$$C_a = \left\{ f \in C : \int_0^1 f = a \right\}$$

are $T-$invariant convex, closed *slices of* C by parallel hyperplanes. Since C, is an order segment and therefore weakly compact, so are all the slices C_a. In view of these observations, each of the sets C_a, for $0 < a < 1$, contains at least one nontrivial, that is of strictly positive diameter, minimal invariant subset. Further, since each C_a is $T-$invariant, these are the only minimal invariant sets for T. It is also known that except for $a = 0$ or 1 each of the C_a are not themselves minimal invariant and that the closed convex hull of an orbit need not be minimal invariant.

A variant of this example and the extension to a family of mappings can be found in [17].

It is interesting to note that when C is weakly compact the existence of a minimal invariant set is obtain through an application of the axiom of choice (Zorn's Lemma). So far no constructive examples of such sets are known. The only explicit examples of minimal invariant sets that we know of are not weakly compact. This is discussed further in the final section of the paper.

2 Known properties

Suppose $D \subset C$ is nonempty closed convex and $T-$invariant. We always have $\inf\{\|x - Tx\| : x \in D\} = 0$. Even more can be observed. Take any $s \in [0, 1)$, and for any given $x \in D$ consider the equation

$$y = (1 - s)\,x + sTy. \tag{1}$$

Since as a function of y the right hand side is a strict contraction of D into D, (2.1) has exactly one solution, say $y = x_s$ satisfying

$$x_s = (1 - s)\,x + sTx_s.$$

The mapping $x_s : [0, 1) \to D$ represents a continuous curve with $\lim_{s\to 1} \|x_s - Tx_s\| = 0$. All norm cluster points of this curve are fixed under T. Thus, if T is fixed point free the curve does not have cluster points and gives an embedding of the half open interval $[0, 1)$ into D as a closed set. This will always be the case if D is a nontrivial minimal invariant set for T. Sequences $\{y_n\}$ satisfying $\lim \|y_n - Ty_n\| = 0$ are said to be *approximate fixed point sequences* for T. Any sequence of the form $\{x_{s_n}\}$ with $s_n \to 1$ is such a sequence. A deeper result is the following. If we put $V = \frac{1}{2}(I + T)$ then for any $x_0 \in D$ the sequence of iterates $\{x_n = V^n x_0\}$ is an approximate fixed point sequence for V and T (see Ichikawa [11] and Edelstein-O'Brien [6]).

The first author, in two papers [7] and [8], discussed several properties of minimal invariant sets and families of them. Let $K \subset C$ be closed convex and minimal invariant for T. Then we have:

I: $K = \overline{co}T(K)$.

II: *For any* $z \in K$, $\sup_{x \in K} \|x - z\| = \operatorname{diam} K$.

The first is obvious since the right hand side of **I** is $T-$invariant and K is minimal. The second, saying that K is *a diametral set*, can be proved directly from **I** (see [13]). Indeed suppose there exists *a nondiametral point* $z \in K$ satisfying $\sup_{x \in K} \|x - z\| = r < \operatorname{diam}(K)$. Then the proper subset K_r of K consisting of all such points is nonempty closed and convex. Using the fact that in view of (I) each point of K can be approximated by a convex combination of points in $T(K)$ one can easily observe that K_r is also $T-$invariant. This contradicts the minimality of K.

III: *For any approximate fixed point sequence* $\{y_n\}$ *and any* $z \in K$, $\lim \|z - y_n\| = \operatorname{diam}(K)$.

To see this, observe that for the convex function $r(z) = \limsup \|z - y_n\|$, $z \in K$, all the level sets $K_a = \{z : r(z) \le a\}$, are closed convex and $T-$invariant. The minimality of K therefore implies that $r(z)$ has to be constant on K. Suppose that $r(z) = const = r < d = \operatorname{diam}(K)$. Take any finite collection of points $\{z_1, z_{2,\dots,}z_p\}$, and consider the finite collection of balls $B[z_i, \overline{r}]$, $i = 1, 2, .., p$, where $\overline{r} = \frac{1}{2}(r + d)$. The sequence $\{y_n\}$ eventually enters each one of these balls. It follows that the family of all balls centered at points of K and of radius $\frac{1}{2}(r + d)$ have the finite intersection property on K and thus by weak compactness a common point, say $\overline{z}$. Thus $\overline{z}$ is a nondiametral point and this contradicts **II**. Lastly since all the subsequences of an approximate fixed point sequence are also approximate fixed point sequences we can replace lim sup by lim.

Property **III** was independently discovered by the first author [7] and L. Karlovitz [12] and has proved to be one of the most useful tools for establishing the existence of fixed points for nonexpansive mappings, especially via ultraproduct method (see for example [1],[3],[9]).

As a consequence of **III** we get;

IV: *For any* $z \in K$ *and any* $x \in K$ *we have* $\lim \|x - V^n x\| = \operatorname{diam}(K)$,

V: *For any* $z \in K$ *and any* $x \in K$ *the curve* x_s *satisfies* $\lim_{s \to 1} \|z - x_s\| = \operatorname{diam}(K)$.

In [7], a notion was introduced which has not been thoroughly investigated. Let D be a closed convex set with diameter $d > 0$. Let us call a point $z \in D$ *almost nondiametral* if there exists $\varepsilon > 0$ such that all the path-connected components of the set $D \setminus B[z, d - \varepsilon]$ have diameters less then d. Even if D is diametral it can contain almost nondiametral points.

Example 2. In the space c_0 let the set D be the closed convex hull of the standard basis vectors

$$D = \left\{ (x_i) : x_i \geq 0, i = 1, 2, ..., \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

Then D is diametral with $\operatorname{diam}(D) = 1$, but for sufficiently small $\varepsilon > 0$ the set $D \backslash B\left[0, \frac{1}{2} + \varepsilon\right]$ consists of infinitely many disjoint path connected components of diameter $\frac{1}{2} - \varepsilon$. Thus, 0 is an almost nondiametral point.

Now **V** implies

VI: *K does not contain almost nondiametral points.*

Another consequence of **III** is that a minimal invariant set K can not be covered by a finite collection of sets with diameters smaller then $\operatorname{diam}(K)$. At least one of the sets would have to contain an approximate fixed point sequence.

Recall that for any set A the number

$$\alpha(A) = \inf\{ d : A \text{ can be covered by a finite number of sets with diameters not exceding } d\}$$

is called the *Kuratowski's measure of noncompactness* of A. Two basic properties of the measure of noncompactness α will be needed in a sequel. For any two bounded sets A_1, A_2 we have

$$\alpha(A_1 \cup A_2) = \max\{\alpha(A_1), \alpha(A_2)\} \text{ and } \alpha(A_1 + A_2) \leq \alpha(A_1) + \alpha(A_2).$$

Using Kuratowski's measure of noncompactness the last remark can be written as;

VII: *If K is a minimal invariant set then $\alpha(K) = \operatorname{diam}(K)$.*

Let us pass to properties which involve more than one invariant set. If D is invariant so are all its r–neihbourghoods in C, $B[D, r] = C \cap \bigcup \{B[x, r] : x \in D\}$. Since the intersections of invariant sets are invariant, by a minimality argument, we immediately get;

VIII: If *D is invariant and K is minimal invariant then* $\operatorname{dist}(x, D)$ *is constant for all $x \in K$.*

Consequently

IX: *If K_0, K_1 are minimal invariant then for any $x \in K_0, y \in K_1$,*

$$\operatorname{dist}(x, K_1) = \operatorname{dist}(y, K_2) = const = H(K_0, K_1) = \operatorname{dist}(K_0, K_1).$$

Here, H is the usual Hausdorff distance between sets. We leave the justification of the next observations to the reader.

X: *If K_0, K_1 are minimal invariant then for any $t \in [0,1]$ there exists a minimal invariant set K_t such that $H(K_0, K_t) = tH(K_0, K_1)$ and $H(K_t, K_1) = (1-t)H(K_0, K_1)$.*

Equivalently

XI: *The family of all minimal closed convex $T-$invariant subsets of C is metrically convex with respect to the Hausdorff metric.*

Since the family of minimal sets furnished with the Hausdorff metric is a complete space, in view of Menger's theorem [16], the sets K_t of **X** can be selected to form a continuous path joining K_0 and K_1 which is isometric to the interval $[0, H(K_0, K_1)]$.

3 New Findings

The first observations we are going to present here are connected with property **IX.** We will say that any two convex weakly compact sets satisfying the conclusion of **IX** are *metrically parallel.* There are some general facts connected with this notion which have an influence on the structure of the family $\mathcal{T}_0$ of all minimal invariant sets for T. Let sets K_0, K_1 be metrically parallel with $H(K_0, K_1) = d > 0$. Let $P_1 = P_{K_1} : K_0 \to K_1$ be the metric projection. Consider the set

$$M = M(K_0, K_1) = (I - P_1)(K_0) = \{x - y : x \in K_0,\ y \in P_1(x) \subset K_1\}.$$

Obviously for any $z \in M$ we have $\|z\| = d$. A straightforward calculation based on property **IX** shows that M is convex. It follows that $\frac{1}{d}M$ is a convex set contained in the unit sphere S of X. Also it is clear that reversing the roles of K_0 and K_1 we get $M(K_1, K_0) = -M = -M(K_0, K_1)$. Consequently $K_0 \subset K_1 + M$ and $K_1 \subset K_0 - M$. This means that given two metrically parallel, weakly compact, convex sets, each one is contained in a translate of the other by a convex subset of the sphere dS, where d is the distance between them.

For some spaces the above leads to interesting consequences. If the space X is strictly convex, then the metric projection is single valued and M consists of only one point. Consequently, as was observed in [7], we have

XII: *If X is a strictly convex space and K_0, K_1 are two minimal invariant subsets of C then K_1 is a translate of K_0 and in particular* $\operatorname{diam}(K_0) = \operatorname{diam}(K_1)$.

Further if $x_0 \in K_0$, $y_0 = P_1 x_0$ and $z = x_0 - y_0$ then for any $x \in K_0$ we have $x - P_1 x = z$ and, since $\|Tx - TP_1 x\| \leq \|x - P_1 x\| = \|z\| = d$, we have $TP_1 x = P_1 Tx = Tx - z$. Thus,

XIII: *If X is strictly convex and K_0, K_1 are two minimal invariant sets for T with $K_0 = K_1 + z$ then the mappings T and P_1 commute and for all*

$x \in K_0$ we have $TP_1x = P_1Tx = Tx - z$.

We now present a previously unpublished result of the above type due to T. Dalby and the second author showing that the last conclusion of **XII** holds for a much wider class of spaces.

Let us recall that the space X is said to have the Kadec-Klee property (X is a KK-space for short) if for any sequence (x_n) in X with $w - \lim x_n = x$ and $\lim \|x_n\| = \|x\|$ we have $\lim x_n = x$. In such spaces all convex, closed subsets lying on the unit sphere are norm compact.

Let X be a KK-space and let K_0, K_1 be two minimal invariant sets for $T : C \to C$ with $\operatorname{dist}(K_0, K_1) = d > 0$. Take $x \in K_0$ and $y \in P_1x \subset K_1$. Let V be the averaged map $V = \frac{1}{2}(I + T)$. In view of

$$d \leq \|V^n x - V^n y\| \leq \|x - y\| = d$$

we notice that $V^n y \in P_1 V^n x$. As we already mentioned both the sequences $(V^n x)$ and $(V^n y)$ are approximate fixed point sequences for T. Extracting subsequences so that $w - \lim V^{n_k} x = x_0$, and $w - \lim V^{n_k} y = y_0$ we have that

$$w - \lim (V^{n_k} x - V^{n_k} y) = x_0 - y_0.$$

But $x_0 - y_0$ and all the elements of the sequence $(V^{n_k} x - V^{n_k} y)$ are members of $M(K_0, K_1)$ which is norm compact. Thus the weak limit is actually a strong limit.

On the other hand in view of **III** we have

$$\lim \|V^{n_k} x - x_0\| = \operatorname{diam} K_0 \quad \text{and} \quad \lim \|V^{n_k} y - y_0\| = \operatorname{diam} K_1$$

implying

$$\begin{aligned} |\operatorname{diam}(K_0) - \operatorname{diam}(K_1)| &= \left| \lim_k \|V^{n_k} x - x\| - \lim_k \|V^{n_k} y - y\| \right| \\ &\leq \lim \|(V^{n_k} x - V^{n_k} y) - x_0 - y_0\| = 0. \end{aligned}$$

This establishes;

XIV: *If X is a KK-space then all the minimal invariant sets for $T : C \to C$ have the same diameter.*

Some further observations are the following. For any two bounded sets A_0, A_1 we have the obvious inequalities

$$|\operatorname{diam}(A_0) - \operatorname{diam}(A_1)| \leq 2H(A_0, A_1),$$

$$|\alpha(A_0) - \alpha(A_1)| \leq 2H(A_0, A_1).$$

For a given space X let $l(X)$ be the supremum of the diameters of all convex subsets of the unit sphere S. Similarly, let $k(X)$ be the supremum of the

Kuratowski measures of noncompactness of all such sets. Obviously $k(X) \leq l(X) \leq 2$. If K_0, K_1 are two metrically parallel convex sets then $K_0 \subset K_1 + M$ and $K_1 \subset K_0 - M$. Since $\operatorname{diam}(M) \leq l(X)\, H(K_0, K_1)$ and $\alpha(M) \leq k(X)\, H(K_0, K_1)$ we easily get

$$|\operatorname{diam}(K_0) - \operatorname{diam}(K_1)| \leq l(X)\, H(K_0, K_1),$$

$$|\alpha(K_0) - \alpha(K_1)| \leq k(X)\, H(K_0, K_1).$$

But for minimal invariant sets K_0, K_1 we have $\operatorname{diam}(K_i) = \alpha(K_i)$. Therefore

XV: *If K_0, K_1 are minimal invariant then*

$$|\operatorname{diam}(K_0) - \operatorname{diam}(K_1)| \leq k(X)\, H(K_0, K_1). \tag{2}$$

If X is strictly convex then $l(X) = 0$ while if X is a KK-space then $k(X) = 0$. The above shows that if $k(X) < 2$, then the diameter function on sets in $\mathcal{T}_0$ has a smaller Lipschitz constant than in general. It also gives an alternative proof of **XIV** and even more; it evaluates the span between smallest and largest diameters of minimal invariant sets. For spaces with $k(X) < 1$ we have

$$\sup \operatorname{diam}(K) - \inf \operatorname{diam}(K) \leq k(X) \operatorname{diam}(C) < \operatorname{diam}(C). \tag{3}$$

Here the supremum and infimum are taken over all $K \in \mathcal{T}_0$.

We now develop some additional properties of metrically parallel sets. Let K_0, K_1 be convex, weakly compact, metrically parallel sets with $H(K_0, K_1) = d > 0$. Without loss of generality assume that $0 \in K_0$. Then K_1 lies on the boundary of the convex body, $K_0 + dB$, more precisely $K_1 \subset K_0 + dS$. Let $\widetilde{K_1} \subset K_0 + dS$ be the maximal convex face containing K_1. Then there exists a linear functional $f \in X^*$, with $\|f\| = 1$ which supports $K_0 + dB$ at $\widetilde{K_1}$,. Thus, $f(x) = k$ for all $x \in \widetilde{K_1}$, where without loss of generality we may assume that $k \geq 0$, and then $f(x) \leq k$ for all $x \in K_0 + dB$.

Since $0 \in K_0$ then $dB \subset K_0 + dB$ and because the two sets K_0 and K_1 are metrically parallel there is a point $y \in K_1$ with $\|y\| = d$. Obviously we have $k = f(y) \leq \|y\| = d$. Strict inequality can not hold. Otherwise, since $\|f\| = 1$, if $k < d$ then there would be point $z \in dB$ with $f(z) > k$. Thus $k = d > 0$.

Finally, we show that the functional f is constant on K_0. Since $0 \in K_0$, f takes the value 0 in K_0. Suppose for some $x \in K_0$ we have $f(x) < 0$. then for any $y \in K_1$

$$\|y - x\| \geq f(y - x) = d - f(x) > d,$$

a contradiction. On the other hand if for $x \in K_0$,we have $f(x) = a > 0$ then choosing $z \in dB$ such that $f(z) \geq d - \frac{a}{2}$ we get $f(x + z) \geq a + d - \frac{a}{2} > d$ and again we arrive at a contradiction.

Thus for metrically parallel sets we have;

XVI: *If K_0, K_1 are weakly compact, convex metrically parallel sets then they lie in parallel hyperplanes that is, there exists a continuous linear functional which takes constant values on each set.*

An interesting consequence for minimal invariant sets is,

XVII: *Let K_0, K_1 be minimal invariant sets for $T : C \to C$. Suppose $K_t : [0,1] \to \mathcal{T}_0$ is a continuous path satisfying property* **X** *and finally let $f \in X^*$ be such that $f(K_0) = \{a\}$ and $f(K_1) = \{b\}$ then $f(A_t) = \{(1-t)\,a + tb\}$.*

The verification is left to the reader. This shows that the situation observed in Alspach's example is typical of the general case.

We now discuss properties connected to the 'size' of minimal invariant sets. Let $T : C \to C$ be a nonexpansive mapping with $FixT = \emptyset$. It is expected that some geometrical or topological properties of the set C will limit the size of minimal T−invariant subsets of C. The first observation follows immediately from the fact that $\operatorname{diam}(K) = \alpha(K)$,

XVIII: *If K is a convex minimal T−invariant subset of C then* $\operatorname{diam}(K) \le \alpha(C)$.

Following reasoning similar to that for property **VI** we can get a better evaluation. Take any closed convex subset $D \subset C$. Let $\mathcal{A}(D)$ be the, possibly uncountable, collection of all pathwise connected components of $C \setminus D$. Define

$$\eta(C) = \inf_{D \subset C} \max\left\{\alpha(D),\ \sup_{A \in \mathcal{A}(D)} \alpha(A)\right\}.$$

Obviously $\eta(C) \le \alpha(C)$.The example preceding property **VI** shows that strict inequality may hold. Now we get;

XIX: *If K is a minimal invariant subset of C then* $\operatorname{diam}(K) \le \eta(C)$.

To see this take any $x \in K$ and consider the curve x_s contained in K, defined by the equation (2.1). For any D the curve either enters one of the components in $\mathcal{A}(D)$ and stays there as $s \to 1$, or visits D infinitely many times. The conclusion then follows easily from **III.**

The technique proposed here for measuring the size of K has not been greatly exploited. More examples are needed especially because Example 2.1 cited in connection with property **VI** seems inadequate, since it is known (B. Maurey [15]) that all nonexpansive self mappings of nonempty weakly compact covex subsets of c_0 have fixed points. Nevertheless we hope that property **XIX** will lead the reader toward a new line of investigation.

A more subtle way of measuring the size of minimal invariant sets was proposed in [8]. For a closed bounded convex set C, with $\operatorname{diam}(C) > 0$, and a nonexpansive mapping $T : C \to C$, define the number

$$g(C,T) = \inf\{\operatorname{diam}(K) : K \subset C \text{ is a minimal invariant set for } T\}.$$

Further, let

$$g(C) = \sup\{g(C,T) : T \text{ is nonexpansive, } T : C \to C\}.$$

Obviously $g(C,T) = 0$ if T has a fixed point and $g(C,T) = \operatorname{diam}(C)$ if C itself is a minimal invariant set for T. Also $g(C) = 0$ if all nonexpansive self mappings of C have fixed points and $g(C) = \operatorname{diam}(C)$ if C is a minimal invariant set for at least one nonexpansive mapping $T : C \to C$.

Not much is known about these parameters. The main open questions concerning weakly compact convex sets C are: Does $g(C,T) = 0$ imply that the nonexpansive map T has a fixed point in C? Is the condition $g(C) = 0$ equivalent to every nonexpansive self mapping of C having a fixed point?

The answer to both of these questions is of course affirmative if the space is strictly convex or has the KK property.

In view of our observations, the situation $g(C,T) = 0$ but T is fixed point free could only occur in a space X with many large flats on its unit sphere.

Suppose $T : C \to C$ has $g(C,T) = 0$ with $FixT = \emptyset$. Then there exits a sequence of minimal T-invariant sets (K_i) with $\lim \operatorname{diam}(K_i) = 0$. Any sequence (x_i) with $x_i \in K_i$ is an approximate fixed point sequence for T and does not have cluster points. Fix any minimal invariant $K \subset C$. Passing to subsequences we can assume that $\lim_{i\to\infty} H(K,K_i) = r > 0$ and $\lim_{\substack{i,j\to\infty \\ i\neq j}} H(K_i,K_j) = d > 0$. Now consider the sequence of sets

$$U_i = \frac{1}{H(K,K_i)} M(K,K_i).$$

Obviously, the sets $U_i, i = 1,2,...$ are convex subsets of S. Also for each $i = 1,2,...$ select $x_i \in K_i$ and form the sets

$$V_i = \frac{1}{H(K,\{x_i\})}(K - x_i).$$

Each V_i being a normalized translate of K shares the properties of a minimal invariant set. Indeed, they are minimal invariant sets for a nonexpansive mapping defined on a homothet of C.

Putting $a = \frac{d}{r}$ we then have

XX: *If $T : C \to C$ has $g(C,T) = 0$ and $Fix(T) = \emptyset$, then for any $K \in \mathcal{T}_0$, there exist sequences of sets(U_i) and (V_i) such that, for $i = 1,2,...,$ the V_i are homothets of K contained in the unit ball, while the U_i are convex subsets of the unit sphere, and the following equalities hold;*

$$\lim_{i\to\infty} H(U_i,V_i) = 0,$$
$$\lim_{i\to\infty} H(\{0\},V_i) = \lim_{i\to\infty} \operatorname{dist}(0,V_i) = 1 = \operatorname{dist}(0,U_i) = H(\{0\},U_i),$$

$$\lim_{\substack{i,j\to\infty \\ i\neq j}} H(U_i, U_j) = \lim_{\substack{i,j\to\infty \\ i\neq j}} H(V_i, V_j) = a > 0.$$

One can use (3.1) and (3.2) to observe additionally that;

$$\lim_{i\to\infty} \operatorname{diam}(U_i) = \lim_{i\to\infty} \operatorname{diam}(V_i) \geq \frac{\operatorname{diam}(K)}{r}.$$

These exemplify the geometrically aberrant nature of the spaces under consideration, but unfortunately do not provide answers to the questions raised.

Further facts connected with the metrical parallelness of pairs of minimal invariant sets and the metric convexity of the family $\mathcal{T}_0$ are left as exercises to the reader.

Convexity properties of the unit ball are often measured by an appropriate *moduli of convexity*. Following this line let us propose a new modulus as follows.

In the unit ball B of X consider the family $\mathcal{D}$ of all convex diametral sets D. Let $m = \sup\{\operatorname{diam} D : D \in \mathcal{D}\}$. Define *the modulus of nondiametrality* as a function $\sigma : [0, m) \to [0, 1)$ by;

$$\sigma_X(\varepsilon) = \inf\{1 - \operatorname{dist}(0, D) : D \in \mathcal{D}, \operatorname{diam}(D) \geq \varepsilon\}.$$

Thus σ is an nondecreasing function and following a pattern used very often in the theory of nonexpansive mappings, in view of **XX** we can write

XXI: *IF $\sigma_X(\varepsilon) > 0$ for all $\varepsilon > 0$, then for each convex, weakly compact set $C \subset X$ and each nonexpansive mapping $T : C \to C$ with $g(C, T) = 0$ there exists a fixed point of T.*

Furthermore,

XXII: *If $\sigma_X(\varepsilon) > 0$ for all $\varepsilon > 0$, then each nonempty weakly compact, convex set C with $g(C) = 0$ has the fixed point property for nonexpansive mappings.*

These last two observations represent little more than a reformulation of the problem, as practically nothing is known about the function σ. Further investigation is called for.

4 Some Final Examples

So far our discussion has concerned objects, minimal invariant sets, whose existence is either proved in a nonconstructive way, or simply assumed. We conclude this note with some examples that help put our findings into context.

For the spaces c_0 and c denote the natural norms by $\|\|_\infty$ and for l_1 by $\|\|_1$. For any sequence x let x^+, x^- stand for the positive and negative parts of the sequence. Finally, for $n = 1, 2, ...$, let e^n be the standard basis vectors in l_1.

Recall that c_0 and c are both preduals that give rise to the same dual norm $\|\|_1$ in l_1, but induce different w^*−topologies.

It is known (see e.g.[9]) that when l_1 is taken as the dual of c_0 all $\|\|_1$ −nonexpansive self mappings of nonempty w^* compact convex subsets have fixed points. That is, $l_1 = c_0^*$ has the w^*−fpp.

Changing to an equivalent norm in c_0 does not change the w^*− topology in l_1, but induces a new equivalent dual norm with respect to which the class of nonexpansive mappings in l_1 may be different. In 1980 T.C. Lim [14] observed that this can lead to a loss of the w^*−fpp.

Example 3. Furnish c_0 with the equivalent norm: $\|x\| = \|x^-\|_\infty + \|x^+\|_\infty$. The dual norm in l_1 is $\|f\|^* = \max\{\|f^-\|_1, \|f^+\|_1\}$. The positive part of the unit ball, $B^+ = \{f \in l_1 : f_i \geq 0, \|f\|^* \leq 1\}$ is w^*−compact convex set and the mapping defined by

$$Tf = \left(1 - \sum_{i=1}^{\infty} f_i, f_1, f_2, ...\right)$$

is a $\|\|^*$− nonexpansive fixed point free mapping of B^+ into B^+. For each point $f \in B^+$ we see that, $w^* - \lim_{n\to\infty} T^n f = 0$. Thus 0 belongs to any T−invariant w^*−closed subset of B^+. Furthermore $\overline{co}\{T^n(0) : n = 1, 2, ...\} = \overline{co}\{e^n : n = 1, 2, ...\} = B^+$, showing that B^+ is itself a minimal invariant set for T.

This is one of the only known instances where a minimal invariant set for a nonexpansive mapping can be explicitly identified. However, it is in the w^*−setting. In l_1 weak compactness coincides with norm compactness and so B^+ is not weakly compact. One can also verify that T is not nonexpansive with respect to the natural norm $\|\|_1$. We only have

$$\|Tf - Tg\|_1 \leq 2\|f - g\|_1.$$

The next example is essentially due to C. Lennard [see 5]. This time we consider l_1 with the natural norm $\|\|_1$ but as the dual of c. So we retain the natural class of $\|\|_1$ −nonexpansive mappings but change the class of w^*−compact sets. Again, we see that this can lead to a failure of the w^*−fpp

Example 4. For conveniece, we take the dual action of $(f_n) \in l_1$ on $(x_n) \in c$ to be

$$(f_n)(x_n) = f_1 x_1 + f_2 \lim x_n + f_3 x_2 + ...).$$

Consider the sequence $x = (-1, 1, 1, ...) \in c$ as a continuous linear functional over l_1. Then

$$\ker x = \left\{f \in l_1 : f_1 = \sum_{i=2}^{\infty} f_i\right\}$$

is a w^*-closed hyperplane and consequently the set

$$C=\left\{f:f_i\geq 0,f_1=\sum_{i=2}^{\infty}f_i\leq 1,\right\}$$

being the intersection of $\ker x$ and w^*-closed halfspaces is itself convex and w^*-closed. Obviously $C\subset 2B^+$. So C is w^*-compact. Choose $\delta\in(0,1]$ and a sequence $(\varepsilon_k)\subset[0,1)$ such that $\sum_{k=1}^{\infty}\varepsilon_k<\infty$ (consequently $\prod_{k=1}^{\infty}(1-\varepsilon_k)>0$) and define a mapping by

$$T(f)=\left(\delta(1-f_1)+\sum_{k=1}^{\infty}(1-\varepsilon_k)f_{k+1},\delta(1-f_1),(1-\varepsilon_1)f_2,(1-\varepsilon_2)f_3,...\right).$$

There is some technicality involved in proving that T is a self mapping of C and that it is $\|\cdot\|_1$ $-$nonexpansive. The summability condition imposed on the sequence (ε_n) implies that T is fixed point free.

A few further remarks about the above construction were made by M. A. Smyth [18]. In particular it was observed that if all $\varepsilon_k>0$ and $\delta<1$, then T is *contractive*, in the sense that

$$f\neq g\Rightarrow\|Tf-Tg\|<\|f-g\|.$$

It was observed in [8] and it easily follows from the metric parallelness that contractive mappings have only one minimal invariant set.

We do not know if C is the minimal invariant set for T in these cases. However, when $\varepsilon_k=0$, for $k=1,2,...$, we observe that this is not the case. There is a smaller w^*-closed convex $T-$ invariant set; namely,

$$C'=\{f\in C:f_1=1\}$$

and a slightly more subtle variant of the argument used in example 3 shows that C' is in fact the unique minimal invariant set for the nonexpansive map $T:C\to C$.

We present our last example thanks to the courtesy of S. Prus who kindly communicated to us his unpublished result.

Let $C\subset L_1$ be the order interval used in Alspach's example and let T be the Alspach map. Moreover let X be the c_0 product of infinitely many copies of L_1,

$$X=\left\{f=(f_k):f_k\in L_1,\lim_{k\to\infty}\|f_k\|=0\right\}$$

furnished with the norm

$$\|f\|_X=\max\{\|f_k\|:k=1,2,...\}.$$

Example 5. For $n = 1, 2, ...$, define, $s_n \in c_0$ by $s_n = (1, 1, ...1, \frac{1}{n}, 0, 0, ...)$ with $\frac{1}{n}$ in $n-$th place. Let $A = \overline{co}\{s_n\}$. Then $0 \notin A$ and A does not contain any sequence with entries of only 0 or 1. Now define the set $\widetilde{C} \subset X$, by

$$\widetilde{C} = \left\{ f \in X : f_n \in C, \left(\int f_n \right) \in A \right\}$$

and a mapping $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ by

$$\widetilde{T}f = \widetilde{T}(f_n) = (Tf_n).$$

The set $\widetilde{C}$ is closed convex but not weakly compact. The mapping $\widetilde{T}$ is nonexpansive, and fixed point free. But each of the sets $D_m \subset \widetilde{C}$ given by

$$D_m = \left\{ (f_n) \in C : \left(\int f_n \right) = s_m \right\},$$

for $, m = 1, 2, ...$ are in view of Alspach's example, weakly compact and $T-$invariant. Thus each D_m contains a minimal invariant subset K_m with $\operatorname{diam}(K_m) \leq \operatorname{diam}(D_m) \leq \frac{2}{m}$.

Thus there are closed, bounded convex sets and fixed point free nonexpansive mappings with $g(C, T) = 0$. But, we know of no example where C is a weakly compact convex set.

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