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NON-EXPANSIVE MAPPINGS ON BANACH LATTICES AND RELATED TOPICS

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ABSTRACT. We give a new lattice theoretic criterion for a non-expansive mapping defined on a weakly compact convex subset of a Banach space to have a fixed point. Our condition allows us to show that a wide variety of Banach sequence spaces, including $c_0(\Gamma)$ and $c(\Gamma)$, have the fixed point property.

§1. **Introduction.** A self-mapping T of a closed convex subset C in a Banach space X is said to be *non-expansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all x,y in C.

We say X has the (*weak*) *fixed point property* if every non-expansive mapping defined on a non-empty weakly compact convex subset of X has a fixed point.

Classical results of Browder [3], Kirk [11] and others [7, 9, 15] established that every uniformly convex space and every space with "normal structure" has the fixed point property.

Until recently further positive results have been fragmentary. It remained open as to whether or not every Banach space possessed the fixed point property until Alspach [1] gave an example of a fixed point free non-expansive mapping on a weakly compact convex subset of $\mathcal{L}_1[0,1]$ (see also [18], [20]). Shortly afterwards Maurey [14], using ultrafilter methods, succeeded in showing that $c_0(N)$ and reflexive subspaces of $\mathcal{L}_1[0,1]$ do have the fixed point property.

In this paper we simultaneously refine some of Maurey's ideas and remove the dependence on ultrafilters. In consequence we are able to show that a large variety of Banach spaces have the fixed point property. Our techniques are lattice theoretic in spirit and allow us to give surprisingly general and simple criteria for a Banach space to have the fixed point property. In particular we are able to:

(i) characterize order complete M-spaces with the fixed point property;

(ii) recover substantially strengthened versions of examples used by Karlovitz [10] and others;

(iii) show that $c_0(\Gamma)$, $c(\Gamma)$ and $c_0(\Gamma)$ with Day's l.u.c. norm [16] have the fixed point property.

It is worth observing that a large number of the examples and counter-examples in non-expansive fixed point theory have had a lattice theoretic underpinning, but to the best of our knowledge, before Maurey's paper, this has remained largely implicit.

§2. **Some basic constructions for non-expansive mappings.** Let C be a non-empty weakly compact convex subset of a Banach space X and let $T: C \to C$ be a non-expansive mapping. A standard application of Zorn's lemma ensures the existence of a *minimal invariant subset* D from the class of non-empty weakly compact convex subsets of C which are invariant under T.

Geometric properties of the space such as UCED or the Opial condition have been used to rule out the existence of weakly compact convex diametral sets containing more than one point (normal structure). That such spaces have the fixed point property then follows from Proposition 2.2 below.

LEMMA 2.1. *If* $\psi: D \to \mathbb{R}$ *is a weak-lower semi-continuous mapping with* $\psi(Tx) \leqslant \psi(x)$ *for all* $x \in D$, *then* ψ *is constant on* D.

PROOF. Let $x_0 \in D$ be such that $\psi(x_0) = \inf \psi(D)$ and let $E = \{x \in D: \psi(x) = \psi(x_0)\}$ then E is a non-empty weakly compact convex set which is invariant under T and so by minimality E = D.

PROPOSITION 2.2. [Kirk, 1965] D *is diametral.*

PROOF.

$$\psi(x) := \operatorname{Sup}\{\|x - y\|: y \in D\}$$

$$= \operatorname{Sup}\{\|x - Ty\|: y \in D\}$$

$$(\text{as } \overline{\text{co}}T(D) = D \text{ by minimality})$$

satisfies the conditions of Lemma 2.1. Thus ψ is constant on D with value $\operatorname*{Sup}_{x \in D} \operatorname*{Sup}_{y \in D} \|x - y\| = \operatorname{diam}(D)$ and so D is diametral.

Applying the Banach contraction mapping principle to the strict contraction $(1 - \frac{1}{n})T + \frac{1}{n}I$ yields a *sequence* (x_n) *of approximate fixed points* for T in C: $\|Tx_n - x_n\| \to 0$. (Note, this does not require C to be weakly compact, only closed and convex.)

PROPOSITION 2.3. [Karlovitz, 1976] *If* (x_n) *is a sequence of approximate* *fixed points for* T *in the minimal invariant set* D, *then*

$$\lim_n \|x - x_n\| = \mathrm{diam}(D), \textit{ for all } x \textit{ in } D.$$

PROOF. Let (y_n) be any sequence of approximate fixed points for T in D and let $\psi(x) = \overline{\lim_n} \|x - y_n\|$. Then ψ satisfies the assumptions of Lemma 1 and so ψ is constant on D with value K say. Let (y_{n_k}) be a subsequence with $y_{n_k} \overset{w}{\to} y_0$, then

$$K \geqslant \overline{\lim_k} \|x - y_{n_k}\| \geqslant \underline{\lim_k} \|x - y_{n_k}\| \geqslant \|x - y_0\|.$$

Thus $K \geqslant \sup_{x \in D} \|x - y_0\| = \mathrm{diam}(D)$ (by Proposition 2.2).

Now taking (y_n) to be any subsequence (x_{n_k}) of (x_n) we therefore have $\overline{\lim_k} \|x - x_{n_k}\| = \mathrm{diam}(D)$ for all x in D and so

$$\lim_n \|x - x_n\| = \mathrm{diam}(D)$$

If (x_n) is the orbit of a point x_0 under T: $x_n = T^n x_0$, then replacing "lim sup" by a Banach limit ϕ in the above argument and using the translational invariance of ϕ to establish $\psi(Tx) \leqslant \psi(x)$ where $\psi(x) = \phi(\|x - T^n x_0\|)$ we may conclude that $\phi(\|x - T^n x_0\|) = \mathrm{diam}(D)$. Indeed replacing T by ½(T + I) and using the asymptotic regularity of the latter operator [8] we may, without loss of generality, simultaneously assume that (x_n) is both an orbit and a sequence of approximate fixed points.

We now develop a basic construction which in part is motivated by the desire to replace sequences of approximate fixed points by fixed points in a related space. Maurey constructed such a space using an ultraproduct. We realize it as a quotient of appropriate substitution spaces. An analogy with the role of the Calkin algebra in operator theory may also be noted.

Denote by $\ell_\infty(X)$ and $c_0(X)$ the spaces obtained by substitution of the Banach space X into $\ell_\infty(N)$ and $c_0(N)$.

Define

$$\overline{\lim}(X) := \ell_\infty(X)/c_0(X)$$

and denote by $[x_n]$ or $[\tilde{x}]$ the equivalence class $\tilde{x} + c_0(X)$, where $\tilde{x} = (x_n) \in \ell_\infty(X)$. The quotient norm is given by

$$\|[x_n]\| = \overline{\lim_n} \|x_n\|.$$

and with this norm $\overline{\lim}(X)$ is an order complete Banach lattice provided X is an order complete Banach lattice.

Denote by J the cannonical embedding of X into $\overline{\lim}(X)$; $J(x) := [x_n]$, where $x_n = x$ for all n.

If C is a closed bounded convex subset of X and $T: C \to C$ is a non-expansive mapping then it follows that

$$[C] := \Pi_{n=1}^{\infty} C / c_0(X)$$

is a closed bounded convex subset of $\overline{\lim}(X)$ and that

$$[T][\tilde{x}] := [Tx_n] \text{ where } (x_n) \in [\tilde{x}] \text{ and } x_n \in C,$$

is a well defined non-expansive mapping on [C].

Of basic importance is the observation that (x_n) is a sequence of approximate fixed points for T in C if and only if $(Tx_n - x_n) \in c_0(X)$, and so if and only if $[x_n]$ is fixed by [T]. In particular then [T] always has fixed points in [C].

Before proceeding to the main result of the section let us recall that the set of *quasi-midpoints* for two points x and y in a Banach space is

$$Q(x,y) = \{z: \|x - z\| = \|y - z\| = \tfrac{1}{2}\|x - y\|\}.$$

Q(x,y) is a non-empty ($\tfrac{1}{2}(x + y) \in Q(x,y)$), closed and convex set. Further if x and y are two fixed points of the non-expansive mapping T on the closed convex subset C, then $Q(x,y) \cap C$ is invariant under T.

LEMMA 2.4. *Let* T *be a non-expansive mapping of the non-empty* ω*-compact convex set* C *into itself. Assume that* C *is a minimal invariant set for* T *with* $0 \in C$.

Suppose that (x_n) *and* (y_n) *are sequences of approximate fixed points for* T *in* C *with:*

$$\lim_n \|x_n - y_n\| = \operatorname{diam}(C).$$

Then, there exists a sequence (z_n) *of approximate fixed points for* T *in* C *with:*

$$\lim_n \|x_n - z_n\| = \lim_n \|y_n - z_n\| = \tfrac{1}{2}\lim_n \|z_n\| = \tfrac{1}{2}\operatorname{diam}(C).$$

PROOF. Since $Q := [C] \cap Q([\tilde{x}],[\tilde{y}])$ is a non-empty closed convex subset which is invariant under the non-expansive mapping [T] we can construct a sequence $[\tilde{t}^m]$ of approximate fixed points for [T] in Q with $\|[T][\tilde{t}^m] - [\tilde{t}^m]\| \leqslant 2^{-(m+1)}$. Let

$(t_n^m) \in [\widetilde{t}^m]$ with t_n^m in C, then $\overline{\lim_n}\|Tt_n^m - t_n^m\| \leqslant 2^{-(m+1)}$ and so for sufficiently large n we have

$$\|Tt_n^m - t_n^m\| < 2^{-m}.$$

Let d = diam(C) = diam[C], then from

$$\|[\widetilde{x}] - [\widetilde{t}^m]\| = \|[\widetilde{y}] - [\widetilde{t}^m]\| = \tfrac{1}{2}\|[\widetilde{x}] - [\widetilde{y}]\| = \tfrac{1}{2}d$$

we have that

$$\overline{\lim_n}\|x_n - t_n^m\| = \overline{\lim_n}\|y_n - t_n^m\| = \tfrac{1}{2}d.$$

but then

$$d = \lim_n\|x_n - y_n\| \leqslant \underline{\lim_n}\|x_n - t_n^m\| + \overline{\lim_n}\|y_n - t_n^m\|$$

$$= \underline{\lim_n}\|x_n - t_n^m\| + \tfrac{1}{2}d$$

whence

$$\tfrac{1}{2}d \leqslant \underline{\lim_n}\|x_n - t_n^m\| \leqslant \overline{\lim_n}\|x_n - t_n^m\| = \tfrac{1}{2}d$$

and so

$$\lim_n\|x_n - t_n^m\| = \tfrac{1}{2}d.$$

A symmetric argument yields

$$\lim_n\|y_n - t_n^m\| = \tfrac{1}{2}d.$$

Thus there exists an increasing sequence N(m) such that, for $n \geqslant N(m)$ we have

$$|\,\|x_n - t_n^m\| - \tfrac{1}{2}d| \leqslant 2^{-m}$$

$$|\,\|y_n - t_n^m\| - \tfrac{1}{2}d| \leqslant 2^{-m}$$

$$\|Tt_n^m - t_n^m\| \leqslant 2^{-m}.$$

Taking $z_n = t_n^m$ for $N(m) \leqslant n \leqslant N(m+1)$ we conclude that (z_n) is a sequence of approximate fixed points for T in C with

$$\lim_n\|x_n - z_n\| = \lim_n\|y_n - z_n\| = \tfrac{1}{2}d.$$

To complete the proof it suffices to note that since $0 \in C$, by Proposition 2.3 we have $\lim_n\|z_n\| = \lim_n\|z_n - 0\| = d$.

In the following proof we indicate how it is always possible to construct a pair of sequences (x_n) and (y_n) satisfying the hypotheses of Lemma 2.4.

COROLLARY 2.5. *For* T *and* C *as in Lemma 2.4, there exists fixed points* $[\tilde{x}]$, $[\tilde{y}]$ *and* $[\tilde{z}]$ *of* $[T]$ *in* $[C]$ *with* $[\tilde{z}] \in Q([\tilde{x}],[\tilde{y}])$ *and* $\|[\tilde{x}]\| = \|[\tilde{y}]\| = \|[\tilde{z}]\| = \|[\tilde{x}] - [\tilde{y}]\| = \mathrm{diam}(C)$.

PROOF. Let (x_n) be any sequence of approximate fixed points for T in C. By Proposition 2.3, for each n,

$$\lim_m \|x_n - x_m\| = \mathrm{diam}(C),$$

so we may extract a subsequence (x_{n_k}) such that

$$\lim_k \|x_k - x_{n_k}\| = \mathrm{diam}(C).$$

The result now follows by taking $\tilde{x} = (x_n)$, $\tilde{y} = (x_{n_k})$ and applying Lemma 2.4.

§3. **Weak orthogonality in Banach lattices.** Let X be a Banach lattice. Given a sequence (x_n) weakly convergent to x_0 we will say that (x_n) is *weakly orthogonal* if

$$\varliminf_n \varliminf_m \|\, |x_n - x_0| \wedge |x_m - x_0| \,\| = 0.$$

Every weakly convergent monotone sequence is weakly orthogonal.

A subset C of X is a *weakly orthogonal set* if every weakly convergent sequence of points of C is weakly orthogonal.

Obviously, every compact subset of a Banach lattice is weakly orthogonal.

We say X *is weakly orthogonal* if every weakly compact convex subset of X is weakly orthogonal.

To obtain an easily verified sufficient condition for a space to be weakly orthogonal we introduce the following.

A Banach lattice X has the *Riesz Approximation Property* (R.A.P.) if there exists a family $\mathcal{P}$ of linear projections with $P|x| = |Px|$, for all $P \in \mathcal{P}$, which satisfy:

(i) P(X), the range of P, is a finite dimensional ideal:

(ii) for each $x \in X$, $\inf_{p \in \mathcal{P}} \|Px - x\| = 0$.

THEOREM 3.1. *Let* X *be a Banach lattice with the* R.A.P., *then* X *is weakly orthogonal.*

PROOF. It suffices to prove the stronger result: For any x in X the mapping $y \mapsto |x| \wedge |y|$ is weak to norm continuous at 0.

Given $x \in X$ and $\epsilon > 0$, select $P \in \mathcal{P}$ such that $\| \, |x| - P|x| \, \| < \epsilon/2$. Then, if (y_α) is any net weakly convergent to zero, there is an α_0 s.t. for $\alpha \geqslant \alpha_0$

$$\|P|y_\alpha| \, \| = \| \, |Py_\alpha| \, \| = \|Py_\alpha\| < \epsilon/2.$$

That this is possible follows because P is weakly continuous with a finite dimensional range. Now,

$$|x| \wedge |y_\alpha| \leqslant P|x| \wedge P|y_\alpha| + |\,|x| - P|x|\,| \wedge |y_\alpha| + P|x| \wedge |\,|y_\alpha| - P|y_\alpha|\,|.$$

Putting $z = P|x| \wedge |\,|y_\alpha| - P|y_\alpha|\,|$ we see that z lies in the ideal P(X) and $z \geqslant 0$.

Thus, $0 \leqslant z = Pz = P|x| \wedge |\,P|y_\alpha| - P|y_\alpha|\,| = 0$, and so $z = 0$.

Hence

$$|x| \wedge |y_\alpha| \leqslant P|x| \wedge P|y_\alpha| + |\,|x| - P|x|\,| \wedge |y_\alpha|$$

$$\leqslant P|x| \wedge P|y_\alpha| + |\,|x| - P|x|\,|$$

and so, for $\alpha > \alpha_0$,

$$\| \, |x| \wedge |y_\alpha| \, \| \leqslant \|P|y_\alpha| \, \| + \| \, |x| - P|x| \, \| < \epsilon.$$

Taking $\mathcal{P}$ to be the standard bases projections we have for any set Γ that the spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$ $(1 \leqslant p < \infty)$ have the R.A.P. as does any separable Orlicz space ℓ_M (that is to say, M satisfies the Δ_2 condition) together with a variety of Lorentz spaces. We also observe that R.A.P. is preserved under Riesz isomorphisms.

The spaces $\ell_\infty(\Gamma)$, $c(\Gamma)$ and $\mathcal{L}_p[0,1]$ $(1 \leqslant p \leqslant \infty)$ fail to have the R.A.P. and with the exception of $c(\Gamma)$ also fail to be weakly orthogonal. (In the case of $\mathcal{L}_p$ spaces consider the sequences of Rademacher functions.) To see that $c(\Gamma)$ is weakly orthogonal, let $x^n \overset{w}{\to} 0$ in $c(\Gamma)$ then $y^n = x^n - \mathrm{Lim}(x^n)e$ defines a weak null sequence in $c_0(\Gamma)$ where Lim denotes the norm, and hence weakly, continuous limit functional on $c(\Gamma)$. Thus, for each n, $\lim\limits_{m \to \infty} |y^n| \wedge |y^m| = 0$, but then, since $|x^n| \leqslant |y^n| + |\mathrm{Lim}(x^n)|e$, we have

$$|x^n| \wedge |x^m| \leqslant |y^n| \wedge |y^m| + |y^n| \wedge |\mathrm{Lim}(x^m)|e + |\mathrm{Lim}(x^n)|e \wedge |x^m|$$

and so $\overline{\lim\limits_{n \to \infty}} \; \overline{\lim\limits_{m \to \infty}} |x^n| \wedge |x^m| = 0$ as $\mathrm{Lim}(x^n) \to 0$.

§4. **The Riesz angle of a Banach lattice.** We define the *Riesz angle* of a Banach lattice X to be

$$\alpha(X) = \mathrm{Sup}\{ \| \, |x| \vee |y| \, \| : \|x\| \leqslant 1, \|y\| \leqslant 1 \}.$$

The *Banach-Mazur distance* of two Banach spaces X and Y is

$$d(X,Y) = \inf \|U\| \, \|U^{-1}\|$$

where the infimum is taken over all linear isomorphisms U of X onto Y.

We similarly define the *Riesz distance* $d_R(X,Y)$ by restricting the infimum to only Riesz isomorphisms.

Observe that if $\|\cdot\|_1$, and $\|\cdot\|_2$ are two equivalent Riesz norms for the Banach lattice X satisfying $m\|x\|_1 \leqslant \|x\|_2 \leqslant M\|x\|_1$, then

$$d(X_1,X_2) \leqslant M/m$$

where X_i denotes X with $\|\cdot\|_i$ $(i = 1,2)$.

PROPOSITION 4.1. (a) *For any Banach lattice* X *we have* $1 \leqslant \alpha(X) \leqslant 2$.

(b) $\alpha(X) = 1$ *if and only if* X *is an* M-*space.*

(c) *If* X *is an abstract* L_p *space* $(1 \leqslant p \leqslant \infty)$, *then* $\alpha(X) = 2^{1/p}$.

(d) *If* X *is a full substitutionspace on an index set* I *(and hence a Banach lattice with respect to pointwise order) and* (X_i) $(i \in I)$ *is a family of Banach lattices, then for the substitution space* $P_X(X_i)$ *we have*

$$\alpha(P_X(X_i)) \leqslant \alpha(X) \sup_{i \in I} \alpha(X_i).$$

(e) *For any pair of Banach lattices* X *and* Y, $\alpha(Y) \leqslant d_R(X,Y)\alpha(X)$.

PROOF. (a) follows immediately from the inequality

$$\|x\| \leqslant \| \, |x| \vee |y| \, \| \leqslant \| \, |x| + |y| \, \| \leqslant \|x\| + \|y\|.$$

(b) is immediate from the definition of an M-space.

(c) By Bonhenblust's theorem [12, Theorem 15.3] we may assume that X is $L_p(\mu)$ for some measure μ, from which it follows that for $0 \leqslant x,y$ we have

$$\|x \vee y\|^p \leqslant \|x\|^p + \|y\|^p \leqslant 2(\|x\|^p \vee \|y\|^p.$$

Thus

$$\|x \vee y\| \leqslant 2^{1/p}(\|x\| \vee \|y\|)$$

and we have

$$\alpha(x) \leqslant 2^{1/p}.$$

Taking $\|x\| = \|y\| = 1$ and $x \wedge y = 0$ we have $\|x \vee y\|^p = 2$ so $\alpha(x) \geqslant 2^{1/p}$.

(d) Let $(x_i),(y_i) \in P_X(X_i)$ and let $\alpha_i = \alpha(X_i)$, then for each i:

$$\|x_i \vee y_i\|_i \leqslant \alpha_i(\|x_i\| \vee \|y_i\|),$$

so

$$\|(x_i) \vee (y_i)\|_X \leqslant \alpha_i\|(x_i) \vee (y_i)\|_X$$

$$\leqslant \alpha(X)[\operatorname{Sup} \alpha_i]\,\|(x_i)\|_X \vee \|y_i\|_X.$$

(e) Let U: X → Y be any Riesz isomorphism, then

$$\|\,|x| \vee |y|\,\|_Y \leqslant \|U\|\,\|\,|U^{-1}x| \vee |U^{-1}y|\,\|_X$$

$$\leqslant \|U\|\alpha(X)\|U^{-1}x\|_X \vee \|U^{-1}y\|_X$$

$$\leqslant \|U\|\,\|U^{-1}\|\alpha(X)\|x\|_Y \vee \|y\|_Y,$$

whence $\alpha(Y) \leqslant \|U\|\,\|U^{-1}\|\alpha(X)$ for all Riesz isomorphisms U of X onto Y and so $\alpha(Y) \leqslant d_R(X,Y)\alpha(X)$.

We also note that for an Orlicz sequence lattice ℓ_M where $M(1) = 1$ we have $\alpha(\ell_M) \leqslant 1/g^{-1}(½)$ where $g(t) = \sup_{0<z\leqslant 1} M(tz)/M(z)$. In particular $\alpha(\ell_M) < 2$ if $2M(z) < M(2z)$ for $0 < z \leqslant ½$.

To see this, let $\alpha_0 = g^{-1}(½)$, then for x,y with $\|x\|_M, \|y\|_M \leqslant 1$ we have

$$\Sigma\, M(\alpha_0(|x_i| \vee |y_i|)) \leqslant g(\alpha_0)\Sigma\, M(|x_i| \vee |y_i|)$$

$$\leqslant ½(\Sigma\, M(|x_i|) + \Sigma\, M(|y_i|)$$

$$\leqslant 1$$

whence $\|\,|x| \vee |y|\,\|_M \leqslant 1/\alpha_0$.

The final conclusion now follows by observing that $g(t) \leqslant t$ so $\alpha_0 \geqslant ½$.

The fundamental inequality involving the Riesz angle is:

THEOREM 4.2. *Suppose that* X *is a Banach lattice with Riesz angle* α(X). *Then for* x, y, z *in* X *we have*

$$\|z\| \leqslant \alpha(X)(\|x - z\| \vee \|y - z\|) + \|\,|x| \wedge |y|\,\|.$$

PROOF.

$$|z| \leqslant |x| + |x - z| \leqslant |x - z| \vee |y - z| + |x|.$$

Similarly

$$|z| \leqslant |x - z| \vee |y - z| + |y|.$$

Thus,

$$|z| \leqslant |x - z| \vee |y - z| + |y| \wedge |x|.$$

By Riesz properties

$$\|z\| \leqslant \| |x - z| \vee |y - z| \| + \| |y| \wedge |x| \|$$

and so, by definition of $\alpha(X)$,

$$\|z\| \leqslant \alpha(X)(\|x - z\| \vee \|y - z\|) + \| |y| \wedge |x| \|.$$

There seems to be only a tenuous connection between the geometry of a space and its Riesz angle. Every uniformly convex Banach lattice X has a Riesz angle $\alpha(X) < 2$. If this were not the case, for each n there would exist x_n and y_n with $\|x_n\|,\|y_n\| \leqslant 1$ and $2 - \frac{1}{n} \leqslant \| |x_n| \vee |y_n| \|$, but then

$$2 - \frac{1}{n} \leqslant \| |x_n| \vee |y_n| \| \leqslant \| |x_n| + |y_n| \| \leqslant \|x_n\| + \|y_n\| \leqslant 2$$

and so by the uniform convexity $\| |x_n| - |y_n| \| \to 0$. In which case, since $\| |x_n| \vee |y_n| \| \leqslant \|x_n\| + 2\| |x_n| - |y_n| \|$, we have $\overline{\lim_n} \| |x_n| \vee |y_n| \| \leqslant 1$, a contradiction.

On the other hand, Davis, Ghoussoub and Lindenstrauss [6] have constructed an equivalent locally uniform convex Riesz norm for $\mathcal{L}_1[0,1]$. Equipped with this norm $\mathcal{L}_1[0,1]$ retains a Riesz angle of 2. Indeed we know of no way of equivalently renorming a space to effect a reduction in the Riesz angle.

§5. (Weak) fixed point results.

THEOREM 5.1. *A Banach space* X *has the fixed point property if there exists a weakly orthogonal Banach lattice* Y *such that*

(5.1) $d(X,Y)\alpha(Y) < 2.$

PROOF. Let T be a non-expansive mapping on a non-empty weakly compact convex set C and let D be a minimal invariant subset for T. Select a sequence (x_n) of approximate fixed points for T in D. By the extraction of a subsequence and a translation we may assume that $x_n \overset{w}{\to} 0$ (in particular, then $0 \in D$).

Now let U be a linear isomorphism from X onto Y with $\|U\|\,\|U^{-1}\|\alpha(Y) < 2$, the existence of such a U is ensured by (5.1).

By the weak orthogonality of Y, for each k in N there exists n_k in N such that

$$\underline{\lim}_m \| |Ux_{n_k}| \wedge |Ux_m| \| \leqslant \frac{1}{k}.$$

Since by Proposition 2.3, $\lim_m \|x_{n_k} - x_m\| = \mathrm{diam}(D)$ we may find $m_k \geqslant n_k$ such that

$$\|x_{n_k} - x_{m_k}\| \geqslant \mathrm{diam}(D) - \frac{1}{k}$$

and

$$(5.2)\quad \| |x_{n_k}| \wedge |x_{m_k}| \| \leqslant \frac{1}{k}.$$

Applying Lemma 2.4 we obtain a further sequence of approximate fixed points (z_k) such that

$$\lim_k \|z_k\| = \mathrm{diam}(D)$$

and

$$(5.3)\quad \lim_k \|x_{n_k} - z_k\| = \lim_k \|x_{m_k} - z_k\| = \tfrac{1}{2}\mathrm{diam}(D).$$

Theorem 4.2 shows that

$$(5.4)\quad \overline{\lim}_k \|Uz_k\| \leqslant \alpha(Y)\, \overline{\lim}_k (\|Ux_{n_k} - Uz_k\| \vee \|Ux_{m_k} - Uz_k\|)$$

$$+ \overline{\lim}_k \| |Ux_{n_k}| \vee |Ux_{m_k}| \|.$$

Then (5.2), (5.3) and (5.4) combine to show that

$$2\,\mathrm{diam}(D) = 2\,\overline{\lim}_k \|z_k\| \leqslant 2\|U^{-1}\|\, \|U\|\alpha(Y)\, \overline{\lim}_k (\|x_{n_k} - z_k\| \vee \|x_{m_k} - z_k\|)$$

$$\leqslant \|U^{-1}\|\, \|U\|\alpha(Y)\, \mathrm{diam}(D).$$

Since $2 > \|U^{-1}\|\, \|U\|\alpha(Y)$ it follows that $\mathrm{diam}(D) = 0$ and D is the singleton $\{0\}$. Thus T has a fixed point in C.

COROLLARY 5.2. *Let* X *be a weakly orthogonal lattice such that* $\alpha(X) < 2$. *Then* X *has the weak fixed point property.*

Before listing 2 further immediate corollaries we remark that the above arguments establish the fixed point property for a Banach lattice X provided $\alpha(X)$ is less than 2 and any weakly convergent sequence which is diameterizing for a weakly compact convex set (in particular, any weakly convergent sequence of approximate fixed points in a minimal invariant set for a non-expansive map) is weakly orthogonal. Indeed for such a sequence we only require that $\underline{\lim}_n\, \underline{\lim}_m \| |x_n - x_0| \wedge |x_m - x_0| \|$ is sufficiently small.

COROLLARY 5.3. *A Banach space* X *has the fixed point property if for some* Γ *and* $1 < p < \infty$ *we have*

$$d(X,\ell_p(\Gamma)) < 2^{1/q} \quad (\frac{1}{p}+\frac{1}{q}=1).$$

COROLLARY 5.4. *A Banach space* X *has the weak fixed point property if either*

(i) $d(X,c(\Gamma)) < 2$,

(ii) $d(X,c_0(\Gamma)) < 2$.

EXAMPLES 5.5. (Generalized Karlovitz norms) (a) Let $1 < p < r \leqslant \infty$ and let $\lambda > 0$. Consider $X_\lambda^{p,r}$ as $\ell_p(\Gamma)$ renormed by $\|\ \|_\lambda = (\lambda\|\ \|_p) \vee \|\ \|_r$ For $p = 2$, $r = \infty$ these norms were studied in [10, 2 and 4]. It is immediate that $X_\lambda^{p,r}$ is a weakly orthogonal lattice. Moreover, by Proposition 4.1

$$\alpha(X_\lambda^{p,r}) \leqslant \max \{\alpha(\lambda\|\cdot\|_p), \alpha(\|\cdot\|_r)\} = 2^{1/p} < 2.$$

Thus Corollary 5.2 applies and $X_\lambda^{p,r}$ has the weak fixed point property. Now for $p = 2$, $r = \infty$ Baillon and Schönberg [2] showed that $X_\lambda^{2,\infty}$ has *normal structure* if and only if $\lambda > 1/\sqrt{2}$ while *asymptotic normal structure* [4] obtains for $\lambda > 1/2$. Their paper thus establishes the existence of fixed points only for $\lambda > 1/2$, while our lattice arguments work equally easily for all $\lambda > 0$ and for general p and r.

(b) Consider

$$N: = \{X | d(X,X_\lambda^{2,\infty}) < 2^{-\frac{1}{2}}\}.$$

It follows from Theorem 5.1 and $\alpha(X_\lambda^{2,\infty}) < 2^{\frac{1}{2}}$ that all Banach spaces in N have the weak fixed point property. In particular N contains spaces which are uniformly convex, spaces with normal structure and without uniform convexity, spaces with only asymptotic normal structure and spaces without asymptotic normal structure. This emphasizes the fact that our results allow one to move entirely away from the classical geometric conditions by studying both lattice and isomorphic conditions.

(c) Let $c_0^d(\Gamma)$ denote $c_0(\Gamma)$ with Day's equivalent (l.u.c.) norm [16]. It follows easily that $d(c_0(\Gamma),c_0^d(\Gamma)) \leqslant \sqrt{3}/2 < 2$ and Corollary 5.4(ii) implies that $c_0^d(\Gamma)$ has the fixed point property. In fact with a little effort it can be seen that $c_0^d(\Gamma)$ is a Banach [illegible] with R.A.P. and so Corollary 5.2 also applies (in conjunction with Proposition [illegible]). We note in passing that $\alpha(c_0^d(\Gamma)) > 1$.

(d) It is known that $d(c_0(\Gamma), c(\Gamma)) = 3$ [5]. It follows that one can not deduce Corollary 5.4(i) from Corollary 5.4(ii) nor conversely.

(e) In [6] it is shown that order continuous Banach lattices are exactly those possessing equivalent (l.u.c.) lattice renorms. It follows from this and standard renorming techniques that a wide variety of spaces with the weak fixed point property may be given an equivalent (l.u.c.) renorm with the weak fixed point property.

(f) Since R.A.P. is preserved under substitution into a space with R.A.P. it follows that one can establish the fixed point property for a profusion of spaces. In particular regarding the space $X = (\ell_2 \in \ell_3 \oplus \cdots)_2$ as the substitution of countably many copies of ℓ_∞^1 together with the spaces $\ell_3, \ell_4, \ldots$ into $\ell_2(N)$ it follows from Proposition 4.1 that $\alpha(X) \leqslant 2^{1/2} \cdot 2^{1/3} < 2$ and so X has the weak fixed point property. Similarly $\ell_\infty(\Gamma) \times \ell_2(\Gamma)$ has the fixed point property in the maximum norm $\|(x,y)\| := \|x\|_\infty \vee \|y\|_2$.

(g) Bynum [4] has shown that the fixed point property is inherited by Banach spaces whose distance from a uniformly convex space is not too large. In particular, he shows that if $d(\ell_p(\Gamma), X) \leqslant 2^{1/p}$ $(1 < p < \infty)$ then X has the weak fixed point property. For $p > 2$ this is weaker than our Corollary 5.3, but for $p \leqslant 2$ Bynum's result is stronger. He also points out that the space $\ell_{p,\infty}(\Gamma)$ (which is $\ell_p(\Gamma)$ renormed by $\|x\|_{p,\omega} := \|x^+\|_p \vee \|x^-\|_p$) does not have asymptotically normal structure but has the fixed point property. Our results as given do not recapture this for $p \leqslant 2$. Moreover, $\ell_{p,\infty}(\Gamma)$ is not a Banach lattice. This presents no obstacle for $p > 2$ as the following results shows.

COROLLARY 5.6. *Let* X *be a weakly orthogonal Banach lattice and let* Y *be* X *in an equivalent monotone lattice norm. Suppose that* $\alpha(Y) < 2$. *Then* Y *has the weak fixed point property.*

PROOF. Since X is weakly orthogonal it follows that $\lim_{n\to\infty} |y_n| \wedge |y_m| = 0$ whenever $y_n \to 0$ weakly in Y. The result now follows from Theorem 5.1 because only monotonicity of the norm is required in the proof of Theorem 4.2.

To recapture Bynum's result for $p > 2$ it remains to verify that $\alpha(\ell_{p,\infty}(\Gamma)) \leqslant$

[illegible] < 2 whenever $p > 2$.

§6. Fixed point theorems in M-spaces.

THEOREM 6.1. *Let* X *be a countably order complete* M-*space. The following are equivalent.*

(i) X *has the fixed point property.*

(ii) X *is isometric and lattice isomorphic to* $c_0(\Gamma)$ *for some index set* Γ.

(iii) X *is order continuous.*

(iv) X *has weakly compact order intervals.*

(v) X *contains no (lattice, or norm) copy of* $\ell_\infty(N)$. *(See* [17] *for other equivalences.)*

(vi) X *contains no isometric copy of* $\mathcal{L}_1[0,1]$.

PROOF. (iv) and (iii) coincide for any countably order complete Banach lattice. By Ando's Theorem [12, Theorem 16.2] it follows that (iii) and (ii) coincide and Theorem 5.1 now shows that X has the weak fixed point property. Alspach's example shows that (i) implies (vi). Since (vi) implies X contains no norm copy of $\ell_\infty(N)$ it can certainly not contain a lattice copy. Since this more restrictive form of (v) is equivalent to (iv), the equivalences are established.

EXAMPLE 6.2. (a) The theorem is manifoldly false without the hypothesis of order completeness as is best seen by considering $c(\Gamma)$. Indeed $c(\Gamma)$ satisfies only (i). (v) and (vi).

(b) The space $\overline{\lim}\,\mathbf{R} = \ell_\infty(N)/c_0(N) = C_\infty(N^*)$ with the induced lattice structure is an order complete M-space. Since $[e]$ is an order unit for $\overline{\lim}\,\mathbf{R}$ it can not satisfy (ii) and hence fails to have weak fixed point property.

COROLLARY 6.3. *An abstract* L_p *space* $(1 \leqslant p \leqslant \infty)$ X *has the fixed point property if and only if* X *contains no isometric copy of* $\mathcal{L}_1[0,1]$.

PROOF. For $p = \infty$ this is covered by Theorem 6.1. For $1 < p < \infty$ L_p is uniformly convex and so has the fixed point property.

For $p = 1$ we use the fact that any abstract L_1 space either (i) contains a copy of $L_1[0,1]$ or (ii) is purely atomic [12, page 136]. In the later case X is $\ell(\Gamma)$ on some index set and has the weak fixed point property. This is an immediate consequence of the Opial condition [9] and the Shur property.

These last two results and the scope of our main theorems suggest the conjecture t Corollary 6.3 holds for arbitrary Banach spaces. This is further reinforced by urey's result [14] that reflexive subspaces of $\mathcal{L}_1[0,1]$ have the weak fixed point perty.

This result of Maurey's may be combined with Alspach's example to show the lowing:

EXAMPLE 6.4. The closed convex hull of an orbit of a nonexpansive mapping a weakly compact convex set need not be invariant. Specifically consider T as in spach's paper and $x_0 := 1$. It is easily calculated that $r_n := T^n(1)$ is the n'th ademacher function. Now define D to be $\overline{\text{co}}\{T^n(1): n \in N\}$. Since the closed linear an of the Rademacher functions is isomorphic to $\mathcal{L}_2[0,1]$ [13, page 133], D can t be invariant. Indeed, were it invariant Maurey's result would imply that T ssessed a fixed point on D and, a fortiori, on C.

We next turn to a theorem on weakly compact weakly orthogonal subsets of a nach lattice.

THEOREM 6.5. *Let* C *be a weakly compact convex weakly orthogonal subset of Banach lattice* X *such that* $\alpha(X) < 2$. *Then every non-expansive mapping leaving* C *ariant has a fixed point.*

PROOF. We merely observe that the proof of Theorem 5.1, when Y = X, needs to be weakly orthogonal.

COROLLARY 6.6. *Let* C *be a weakly compact convex subset of a Banach space Suppose that the isometric image of* C *in any* $\ell_\infty(\Gamma)$ *is weakly orthogonal then ry non-expansive mapping leaving* C *invariant has a fixed point.*

PROOF. We use the isometry to lift the problem from X to an M-space. The ult now follows from Theorem 6.5.

Every Banach space X isometrically embeds in some $\ell_\infty(\Gamma)$. Thus one sees that e may establish the (weak) fixed point property for X reflexive, respectively perreflexive, by showing that all weakly compact subsets of $\ell_\infty(\Gamma)$ whose span is lexive, respectively superreflexive, are weakly orthogonal in $\ell_\infty(\Gamma)$. Is this true? ess ambitiously, is it true for separable reflexive subspaces of $\ell_\infty(N)$. This would low one to recapture Maurey's result.

EXAMPLE 6.6. Alspach's example and Corollary 6.5 show that the isometric image of the unit interval in $\mathcal{L}_1(0,1)$ is not weakly orthogonal.

§7. **Miscellaneous.** A complementary question to those considered so far is whether or not every non-expansive mapping on a weak compact order interval has a fixed point. By a slight modification of Alspach's example, Robert Sine [20] obtains an example of a fixed point free non-expansive map on the order interval $0 \leqslant f \leqslant 2$ in $\mathcal{L}_1[0,1]$. In the positive direction we show that a non-expansive map on an order complete interval of any abstract M-space has a fixed point. This may be seen as a mild extension of an earlier result of Sine's [19] that a non-expansive map on an order interval in $\mathcal{L}_\infty(\mu)$ has a fixed point.

THEOREM 7.1. *Let* I *be an order complete order interval in an* M-*space* X *with a unit* e, *and let* T: I → I *be non-expansive, then* T *has a fixed point in* I.

PROOF. We first establish the existence of a minimal invariant order interval in I. This will follows from Zorn's lemma provided we show that for any decreasing chain $\{I_\alpha\}$ of non-empty intervals in I the intersection $I_\infty = \cap_\alpha I_\alpha$ is itself a non-empty interval. To see this note that $a_\infty = \sup_\alpha \inf(I_\alpha)$ and $b_\infty = \inf_\alpha \sup(I_\alpha)$ exist in I by order completeness and that therefore $[a_\infty, b_\infty] = I_\infty$.

Now, let I_0 be a minimal invariant interval in I and set $a_0 = \inf I_0$, $b_0 = \sup I_0$ and $m = \frac{1}{2}(b_0 - a_0)$. Define $N = \{x \in I_0 \colon \sup_{y \in I_0} \|x - y\| \leqslant \|m\|\}$, clearly $N \subseteq Q(a_0, b_0)$. For $x \in Q(a_0, b_0) \cap I$ we also have

$$-a_0 - \frac{d}{2}e \leqslant x \leqslant a_0 + \frac{d}{2}$$

and

$$-b_0 - \frac{d}{2}e \leqslant x \leqslant b_0 + \frac{d}{2}$$

where $d = \operatorname{diam}(I_0) = 2\|m\|$.

Consequently for $a_0 \leqslant y \leqslant b_0$ we have

$$-\frac{d}{2}e \leqslant a_0 - x \leqslant y - x \leqslant b_0 - x \leqslant \frac{d}{2}e$$

and so $\|y - x\| \leqslant \|m\|$ or $x \in N$.

Thus $N = Q(a_0, b_0)$ and so N is invariant under T. The proof is completed by showing that N is in fact an order interval and therefore equal to I_0. This is an sibility unless I_0 is a singleton.

From the definition of M we know that

$$- \|m\|e + y \leqslant x \leqslant \|m\|e + y$$

so $y \in I_0$ if and only if $x \in N$. As a result for every $y \in I_0$ we have

$$- \|m\|e + y \leqslant \inf N \leqslant \operatorname{Sup} N \leqslant \|m\|e + y$$

[inf N, Sup N] = N as required.

While there are several geometric conditions known to imply the (weak) fixed point property, many are little removed from the notion of diametrality and with the exception of uniform convexity most are difficult to verify.

That uniformly convex spaces have the (weak) fixed point property is a simple consequence of Corollary 2.5 translated "approximately back" into the space X.

Some further results in this direction are discussed below, however we omit the [illegible]

We note that if, for any Banach space X we define, $\beta_X(t) = 1 - \liminf_{s \to t} \delta_X(s)$, where $\delta_X(s)$ is the modulus of convexity for X, and let $\lambda_X = \operatorname{Sup}\{t \in [0,2): \delta_X(t) = 0\}$ then it follows from Lemma 2.4 that X has the (weak) fixed point property if $(1 - \lambda_X)^2 < \beta_X(½)$. That the inequality holds locally among Banach spaces follows from the inequality

$$\beta_Y(t) \leqslant d(X,Y)\beta_X(t/d(X,Y)).$$

It should also be observed that by the Day-Norlander theorem $\beta_X(½) \geqslant \sqrt{15}/4$ and so the inequality $(1 - \lambda_X)^2 < \beta_X(½)$ is only feasible for $\lambda_X < 1 - ½\sqrt[4]{15}$, a small and uninteresting number.

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