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NONEXPANSIVE MAPPINGS ON BANACH LATTICES

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1. Introduction.

A mapping T defined on a weakly compact convex subset C of a Banach space X is said to be nonexpansive if $||T(x)-T(y)|| \leq ||x-y||$ for all x and y in C; and X is said to have the (weak) fixed point property (FPP) if every such mapping has a fixed point. Classical results ([4],[7]) show that every uniformly convex Banach space and those with normal structure have the fixed point property. Until recently other positive results remained fragmentary. Moreover, it was only in 1981 that Alspach [1] showed that $L_1[0,1]$ does not have the FPP (see also [11],[12]).

Also, Maurey [10] showed, using ultrapower techniques, that $c_0(\mathbb{N})$ and reflexive subspaces of $L_1[0,1]$ have the FPP. In this paper we refine Maurey's ideas on c_0 and remove the dependence on ultrapowers. We are then able to show that many Banach spaces satisfy simple and verifiable lattice-theoretic criteria and so have the FPP. In particular we: (1) characterize order-complete M spaces with the FPP; (2) show that $c_0(S)$,$c(S)$, and Day's norm on $c_0(S)$ have the FPP; (3) recover-much strengthened-examples due to Karlovitz [6] and others; (4) exhibit spaces without asymptotic normal structure which have the FPP.

Full details of these and other results are forthcoming in [3].

2. Basic results.

A sequence (x_n) in C is approximately fixed for T if $||x_n-T(x_n)||$ tends to 0 . It is a simple consequence of the Contraction Principle that such sequences exist. The following basic construction allows us to replace approximate fixed points by fixed points of a related mapping. Let $l_\infty(X)$ and $c_0(X)$ denote the substitution spaces of X into $l_\infty(\mathbb{N})$ and

$c_0(\mathbb{N})$, with elements $[x]:=(x_n)$. Define $[X]$ to be the quotient $l_\infty(X)/c_0(X)$ with $\|[X]\|:=\limsup_{n\to\infty}\|x_n\|$, and define $[C]:=(\prod_{n=1}^{\infty} C)/c_0(X)$. Then $[C]$ is closed bounded and convex and $[T]([x]):=[T(x_n)]$ defines a nonexpansive mapping on $[C]$. It is clear that $[x]$ is fixed for $[T]$ exactly when (x_n) is approximately fixed for T.

In any Banach space, the quasi-midpoint set $Q(y,z):=\{w\in C:\|y-w\|=\|z-w\|=(1/2)\|y-z\|\}$ is a nonempty closed convex subset of C. If T is nonexpansive and C is closed convex and T-invariant then $Q(y,z)$ is also T-invariant whenever y and z are fixed. The first result follows from the Contraction Principle and a diagonal arguement applied to $[T]$ on $Q([x],[y])$. Maurey gives a similar (slightly weaker) result using ultrapowers.

Proposition 1. ([3]) *Suppose that C is a minimal T-invariant weakly compact convex subset of X containing 0. Suppose also that (x_n) and (y_n) are approximately fixed for T in C and that $\lim_{n\to\infty}\|x_n-y_n\| = \operatorname{diam}(C)$. Then there exists a sequence (z_n) of approximately fixed points in C with*

$$\lim_{n\to\infty}\|x_n-z_n\| = \lim_{n\to\infty}\|y_n-z_n\| = (1/2)\lim_{n\to\infty}\|z_n\| = (1/2)\operatorname{diam}(C) \quad (1)$$

The existence of diametral approximately fixed points can always be guaranteed ([3],[6],[10]). We also need two lattice-theoretic concepts. A subset C of a Banach lattice will be called weakly orthogonal if

$$\liminf_{n\to\infty}\liminf_{m\to\infty}\| |x_n-\bar{x}|\wedge|x_m-\bar{x}| \| = 0 \quad (2)$$

whenever (x_n) converges weakly to $\bar{x}$ in C. A lattice Y is said to be weakly orthogonal whenever all its weakly compact subsets are. It is relatively easy to show that $c(S)$, $c_0(S)$, $l_p(S)$ $(1\le p<\infty)$ are weakly orthogonal while $l_\infty(\mathbb{N})$ and non-atomic L_p spaces are not. Obviously, any norm-compact set is weakly orthogonal. It is also easy to see that every Orlicz sequence space with the "delta-two" condition [9] is weakly orthogonal. Finally, we define the Riesz angle of a Banach lattice by

(3) $$a(X):=\sup\{\|\,|x|\vee|y|\,\|:\|x\|\le 1,\|y\|\le 1\}.$$

Proposition 2. ([3]) (a) For any Banach lattice $X, 1 \le a(X) \le 2$ with $a(X)=1$ if and only if X is an M space. (b) If X is an abstract L_p space $(1 \le p \le \infty)$ then $a(X) = 2^{1/p}$. (c) Let X be a full substitution space on a index set I and $(X_i : i\in I)$ is a family of Banach lattices. Then for the substitution space $P := P(X_i, X)$

(4) $$a(P)\le a(X)\sup_{i\in I} a(X_i) \quad .$$

The fundamental inequality involving the Riesz angle is:

(5) $$\|z\| \le a(X)(\|x-z\|\vee\|x-y\|) + \|\,|x|\wedge|y|\,\|.$$

and is easily established.

3. Fixed point theorems.

Recall that the Mazur distance between two Banach spaces X and Y is given by $d(X,Y) := \inf\{\|U\|\|U^{-1}\| : U$ is an isomorphism of X onto $Y\}$.

Theorem 1. ([3]) A Banach space X has the FPF if there exists a weakly orthogonal Banach lattice Y such that

(6) $$d(X,Y)\, a(Y) < 2 \quad .$$

Proof. We suppose that T is nonexpansive on a minimal invariant weakly compact subset C. Select an approximately fixed sequence (a_n). On extracting a subsequence and translating we may assume that (a_n) is weakly null and that 0 lies in C. Now use (6) to pick an isomorphism U of X onto Y with (7) $\|U\|\,\|U^{-1}\|a(Y) < 2$. Since Y is weakly orthogonal, we can find subsequences (x_n) and (y_n) of (a_n) with (8) $\| |Ux_n| \wedge |Uy_n| \|$ tending to zero in norm. Since any approximately fixed sequence is diametrizing [6] we can also assume that (9) $\|x_n - y_n\|$ tends to diam(C). Proposition 1 produces a third approximately fixed sequence (z_n) satisfying (1). Now (5) shows that

$$(10) \quad \limsup_{n\to\infty} \|Uz_n\| \leq a(Y) \limsup_{n\to\infty} (\|Ux_n - Uz_n\|) \vee (\|Uy_n - Uz_n\|) + \limsup_{n\to\infty} \| |Ux_n| \wedge |Uy_n| \|$$

Now (1), (8) and (10) combine to show that 2 diam(C) $\leq \|U\| \, \|U^{-1}\| a(Y)$diam(C). Since (7) holds C must be singleton.

Corollary 1. (a) Every weakly orthogonal lattice with Riesz angle less than two has the FPP.

(b) A Banach space X such that $d(X, l_p) < 2^{1-1/p}$ has the FPP (for $1<p<\infty$) .

(c) A Banach space X such that $d(X, c(S)) < 2$ or $d(X, c_0(S)) < 2$ has the FPP.

Corollary 2. An abstract L_p space $(1 \leq p \leq \infty)$ or an abstract order-complete M space has the FPP if and only if it contains no isometric copy of $L_1[0,1]$.

Proof. For $p>1$, L_p spaces are uniformly convex and the result is trivial. For $p = 1$, it is a consequence of Alspach's example, the fact that atomic L_1 spaces have the FPP, and the fact that a non-atomic L_1 space contains a copy of $L_1[0,1]$. Finally, any order-complete M space which contains no copy of $l_\infty(\mathbb{N})$ is isomorphic and isometric to $c_0(S)$ on some index set S [8].

Example 1. ([6]) (a) Let $1 \leq p \leq r < \infty$ and let $t > 0$. Consider $X := X(p,r,t)$ as $l_p(S)$ renormed by $\|x\| := \|x\|_r \vee (t\|x\|_p)$. For $p = 2$,

$r = \infty$ these norms were studied in [2], [6]. It is immediate that X is weakly orthogonal and Proposition 2 (c) shows that $a(X) \le 2^{1/p} < 2$. Thus X has the FPP. Baillon and Schönberg [2] showed that $X(2,\infty,t)$ has normal structure only for $t > 1/\sqrt{2}$, and asymptotic normal structure only for $t > 1/2$. Their results, therefore, only apply for $t > 1/2$.

(b) It is almost immediate that Day's norm on $c_0(S)$ [5] has d′ $_0(S)) \le \sqrt{3}/2 < 2$. Corollary 1 (c) shows that Day's norm has the FPP. Note that $d(c(S),c_0(S)) = 3$ and the two parts of Corollary 1(c) are thus distinct. Note also that $c_0(S)$ is locally uniformly convex, but is not uniformly convex in every direction [5].

Remark 1. Our results can be rephrased so that they apply to weakly orthogonal sets in arbitrary lattices. In particular, every weakly orthogonal subset of $l_\infty(S)$ has the FPP. This is interesting because every Banach space is isometric to a subspace of some $l_\infty(S)$. One can therefore show that a class of spaces has the FPP by showing that their isometric images are weakly orthogonal in the l_∞ lattice structure. Conversely, it follows that any isometric image of the $L_1[0,1]$ unit interval must fail to be weakly orthogonal since such sets admit nonexpansive mappings without fixed points [12]. Is it possible to use these ideas to show that (super-) reflexive spaces have the FPP?

5. eferences.

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