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Nonlinear Isometries in Superreflexive Spaces*

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We extend Maurey's theorem on the existence of a fixed point for an isometry of a nonempty closed bounded convex subset of a superreflexive space to obtain the existence of common fixed points for countable families of commuting isometries.
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Using ideas developed in [7], B. Maurey proved in 1981 that any nonlinear isometry which maps a bounded closed convex subset of a superreflexive Banach space into itself has a nonempty fixed point set. (For an explicit proof of this fact, see [4] or [1].) In this paper we use a retraction theorem due to R. E. Bruck [2] and an iteration process of Ishikawa to prove the following extension of Maurey's result.

THEOREM 1. *Let X be a superreflexive Banach space and let K be a nonempty bounded closed convex subset of X. Then any countable family of commuting nonlinear isometries of K into K has a nonempty common fixed point set.*

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Our proof is based on several facts, the first of which is a refinement of a theorem of Bruck [2]. To formulate this result we need the following definition. Let C be a closed convex subset of a Banach space X. A family $\mathcal{F}$ of mappings of C into C is said to satisfy the *conditional fixed point property* (CFP) if the common fixed point set $\operatorname{fix}(\mathcal{F})$ of $\mathcal{F}$ is either empty or if $\operatorname{fix}(\mathcal{F})$ intersects every nonempty bounded closed convex set which is left invariant by each mapping $T \in \mathcal{F}$.

THEOREM 2. *If C is locally weakly compact, if $\mathcal{F}$ is a family of nonexpansive mappings each of which maps C into C, and if $\mathcal{F}$ satisfies* (CFP), *then $F(\mathcal{F}) := \operatorname{fix}(\mathcal{F})$ is a nonexpansive retract of C.*

Proof. A detailed proof may be found in [6]. However, for completeness we note here that the only modifications needed are the following changes to the proof of Theorem 2 of Bruck [2]. Replace T by $\mathcal{F}$ in the statement of that theorem and throughout its proof. The proof is then identical except for the concluding three sentences which should now read as follows. Since $T \circ f \in N(F(\mathcal{F}))$ whenever $f \in N(F(\mathcal{F}))$ and T is in $\mathcal{F}$, we also have $T(K) \subseteq K$ for each $T \in \mathcal{F}$. But $\mathcal{F}$ satisfies (CFP), has a nonempty common fixed point set, and leaves K invariant; therefore $\mathcal{F}$ has a common fixed point in K. That is, there exists $h \in N(F(\mathcal{F}))$ with $h(z)$ in $F(\mathcal{F})$. Since this is so for each z in C the conclusion now follows from Theorem 1 of [2]. ∎

We also need the following result of S. Ishikawa [5].

LEMMA 3. *Let D be a bounded convex subset of a Banach space and let R be a nonexpansive retraction of D into a subset of D which is left invariant under a nonexpansive mapping $G: D \to D$. Let y_0 be any point in D and let $\alpha \in (0,1)$. Then the sequences $(y_n - Gy_n)$ and $(y_n - Ry_n)$ respectively converge to* 0, *where (y_n) is defined by*

$$y_n = (1 - \alpha) z_n + \alpha G z_n, \qquad z_n = R y_{n-1}.$$

Note that as an immediate consequence of the above one may conclude that there is a sequence (x_n) in $R(D)$ such that $(x_n - Gx_n)$ converges to 0. We shall call such a sequence an *approximate fixed point sequence* (a.f.p.s.) for G. It is the existence of such sequences in the above setting that we use in Step 2 of the proof for Theorem 1 given below.

A crucial key to the proof of Theorem 1 is the following fact which we extract from the proof of the Theorem F given in [4]. This requires some preliminary explanation. In the proof of Theorem F a function $\phi: K \to \mathbf{R}^+$ is constructed as follows: Let $\tilde{X}$ be the Banach space ultrapower of X with respect to some nontrivial ultrafilter U over $\mathbf{N}$, and let $\tilde{K}$ be the subset of $\tilde{X}$ defined by

$$\tilde{K} := \{(k_n)_U : k_n \in K, \text{for } n = 1,2,\dots\}.$$

Given $f \in \tilde{K}$, ϕ is the supremum of the "girths" of "configurations" in $\tilde{K}$ built between points $y \in K$, identified with their natural embedding into $\tilde{K}$, and f. Such a function ϕ always satisfies (i) of the lemma below, and it satisfies (ii) for any isometry $\tilde{T}$ that fixes f.

LEMMA 4. *Let K be a bounded closed convex subset of a superreflexive Banach space X and let $T: K \to K$ be an isometry. Then corresponding to each approximate fixed point sequence of T there exists a bounded function $\phi: K \to \mathbf{R}^+$ such that*

(i) $\phi(\frac{1}{2}(x+y)) \geq \frac{1}{2}(\phi(x) + \phi(y)) + \|\frac{1}{2}(x-y)\|^2$,
(ii) $\phi(Tx) \geq \phi(x)$.

Note the passage to a minimal invariant set K in the proof of Elton *et al.* in [4] is not required for the construction of ϕ. Further, Maurey's result also may be derived from the above lemma without passing to a minimal set by using the following lemma.

LEMMA 5. *Let K be a nonempty closed bounded convex set in a Banach space X and let $T: K \to K$ be a continuous map for which there exists a bounded function $\phi: K \to \mathbf{R}^+$ satisfying* (i) *and* (ii) *of Lemma* 4. *Then T has a fixed point in K.*

Proof. Let $M = \sup \phi(K)$ and for each $n \in \mathbf{N}$ define $K_n = \{x \in K : \phi(x) \geq M - 1/n\}$. Then each K_n is nonempty and, by (ii) of Lemma 4, $T(K_n) \subseteq K_n$. Hence $\overline{K}_n$ is also invariant under T. Further, by (i) of Lemma 4, for $x, y \in K_n$

$$M \geq \phi\left(\tfrac{1}{2}(x+y)\right) \geq M - \frac{1}{n} + \|\tfrac{1}{2}(x-y)\|^2$$

and so $\operatorname{diam}(\overline{K}_n) \leq 2/\sqrt{n}$.

Thus, by Cantor's intersection theorem, there exists $x_0 \in K$ with

$$\bigcap_{n=1}^{\infty} \overline{K}_n = \{x_0\},$$

but then $Tx_0 \in \overline{K}_n$, for all n, and so $Tx_0 = x_0$. ∎

We now proceed to the proof of our main theorem.

Proof of Theorem 1.

STEP 1. If $T: K \to K$ is an isometry then $\mathrm{fix}(T)$ is a nonexpansive retract of K.

Proof. Note that by Maurey's Theorem T satisfies (CFP) on K. Step 1 is then immediate from Theorem 2 for the case $\mathscr{F}$ consists of a single mapping.

STEP 2. Now suppose T, G are two commuting isometries of K into K. Then $\mathrm{fix}(T) \cap \mathrm{fix}(G) \neq \emptyset$.

Proof. By Step 1 there exists a retraction $R: K \to \mathrm{fix}(T)$ and, since T and G commute, $G: \mathrm{fix}(T) \to \mathrm{fix}(T)$. Thus by Lemma 3, G has an a.f.p.s. in $\mathrm{fix}(T)$ which is also (trivially) an a.f.p.s. for T. Now let ϕ be the function of Lemma 4 corresponding to this a.f.p.s. Then by Lemma 5 this ϕ yields a fixed point of both T and G.

STEP 3. Under the assumptions of Step 2, $\mathrm{fix}(T) \cap \mathrm{fix}(G)$ is a nonexpansive retract of K.

Proof. If H is a bounded closed convex subset of K which is invariant under both T and G, then by Step 2 (applied to H) we have $\mathrm{fix}(T) \cap \mathrm{fix}(G) \cap H \neq \emptyset$. Thus the family $\mathscr{F} := \{T, G\}$ satisfies (CFP) on K, so Step 3 also follows from Theorem 2.

STEP 4. Theorem 1 holds for finite families.

Proof. This follows from Step 3 and a routine induction argument.

Proof of Theorem 1 *Completed.* Now let $\mathscr{F} = \{T_1, T_2, \ldots\}$ be a countable family of commuting isometries each of which maps K into K. By Step 4 each of the sets

$$F_n = \bigcap_{i=1}^{n} \mathrm{fix}(T_i)$$

is nonempty. Select $x_n \in F_n$ for $n = 1, 2, \ldots$. Note that (x_n) is an a.f.p.s. for each T_i, $i = 1, 2, \ldots$. Let ϕ be the function of Lemma 4 corresponding to this (x_n). By Lemma 5 this ϕ yields a unique point which is fixed under each of the mappings T_i, $i = 1, 2, \ldots$. This completes the proof. ∎

Remark. It remains unknown whether nonempty closed bounded convex subsets of superreflexive spaces have the fixed point property for nonexpansive self mappings. Of course should this happen to be true Theorem 1 (even for uncountable families $\mathscr{F}$) would follow from Bruck's result of [3].

Note Added in Proof. M. A. Khamsi has observed (personal communication, 1995) that, in fact, Theorem 1 holds for arbitrary families. In particular, he notes that if $\{T_\alpha: \alpha \in \Gamma\}$ is an arbitrary family of nonlinear isometries of K into K, then one may replace the ultrapower $\tilde{X}$ of X over $\mathbf{N}$ with an ultrapower $(X)_{\mathscr{U}}$ of X over $\mathscr{U}$, where $\mathscr{U}$ is an ultrafilter containing the filter generated by $I := \{i \in \Gamma : I \text{ is finite}\}$.

REFERENCES

1. A. G. Aksoy and M. A. Khamsi, "Nonstandard Methods in Fixed Point Theory," Universitext, Springer-Verlag, New York/Berlin, 1990.
2. R. E. Bruck, Properties of fixed—point sets of nonexpansive mappings in Banach spaces, *Trans. Amer. Math. Soc.* **179** (1973), 251–262.
3. R. E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappings, *Pacific J. Math.* **53** (1974), 59–71.
4. J. Elton, Pei-Kee Lin, E. Odell, and S. Szarek, Remarks on the fixed point problem for nonexpansive maps, *in* "Contemporary Math.," Vol. 18, pp. 87–120, Amer. Math. Soc., Princeton, 1983.
5. S. Ishikawa, Common fixed points and iteration of commuting nonexpansive mappings, *Pacific. J. Math.* **80** (1979), 493–501.
6. W. A. Kirk, Common fixed points and nonexpansive retracts, *in* "Optimization and Nonlinear Analysis" (A. Ioffe, M. Marcus, and S. Reich, Eds.), pp. 155–168, Longman, Essex, England, 1992.
7. B. Maurey, Points fixe des contractions sur un convexe forme de L^1, *in* "Seminaire d'Analyse Fonctionnelle," Vol. 80–81, Exposé No. VIII, Ecole Normale Polytechnique, Palaiseau, 1981.