PROPERTY (A_2^{ε}) IN ORLICZ SEQUENCE SPACES

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Abstract. *In this paper, we introduce a new geometric property* $(A_2^{\varepsilon})^*$ *and we show that if a separable Banach space has property* $(A_2^{\varepsilon})^*$ *then both* X *and its dual* X^* *have the weak fixed point property. Criteria for Orlicz spaces to have the properties* (A_2^{ε}), $(A_2^{\varepsilon})^*$ *and* (NUS^*) *are given.*

Keywords and Phrases. Orlicz space; Property (A_2^{ε}); Fixed point property, The weak Banach-saks property.

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§ 1. INTRODUCTIONS

Let X be a *Banach space* and let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively.

Given any element $x \in S(X)$ and any positive number δ, we define

$$S^*(x,\delta) = \{x^* \in B(X^*) : x^*(x) \geq 1 - \delta\}.$$

Let A be a bounded subset of X. Its Kuratowski measure of noncompactness $\alpha(A)$ is defined as the infimum of all numbers $d > 0$ such that A may be covered by a finite family of sets of diameters smaller than d.

A Banach space X is said to be NUS^* provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S(X)$, then $\alpha(S^*(x,\delta)) \leq \varepsilon$.

A *Banach* space X is said to have the weak Banach-Saks property whenever given any weakly null sequence $\{x_n\}$ in X there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that the sequence $\{\frac{1}{k}(z_1 + z_2 + \cdots + z_k)\}$ converges to zero strongly.

A Banach space X is said to have property (A_2) if there exists a number $\Theta \in (0,2)$ such that for each weakly null sequence $\{x_n\}$ in $S(X)$, there are $n_1, n_2 \in \mathcal{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$. It is well known that if X has property (A_2) then X has the weak Banach-Saks property (see [3]).

A Banach space X is said to have property (A_2^{ε}) if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for any $t \in (0,\delta)$ and each weakly null sequence $\{x_n\}$ in $S(X)$, there is $k \in \mathcal{N}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ (see [10]).

Now, we introduce the notions of (UA_2^{ε}) and $(A_2^{\varepsilon})^*$-properties.

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A Banach space X is said to have property (UA_2^ε) if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each weakly null sequence $\{x_n\}$ in $S(X)$, there is $k \in \mathcal{N}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ for all $t \in (0, \delta)$.

The dual space X^* of a Banach space X is said to have property $(A_2^\varepsilon)^*$ if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $0 < t < \delta$ and each weak star null sequence $\{x_n^*\}$ of $S(X^*)$, there is $k \in \mathcal{N}$ satisfying $\|x_1^* + tx_k^*\| < 1 + t\varepsilon$.

Notice that for reflexive Banach spaces the properties (A_2^ε) and $(A_2^\varepsilon)^*$ coincide.

Prus (see [9]) has proved that X is NUS^* if and only if X has property (A_2^ε) and X contains no copy of l_1. He also proved that if X is NUS^*, then X has the weak Banach-Saks property.

A natural generalization of this notion is property (WA_2^ε).

A Banach space X has property (WA_2^ε) whenever it satisfies the condition from the definition of property (A_2^ε) with "for every $\varepsilon > 0$" replaced by "for some $\varepsilon \in (0, 1)$".

Let C be a nonempty bounded closed convex subset of X. A mapping $T : C \to C$ is said to be nonexpansive whenever the inequality $\|Tx - Ty\| \leq \|x - y\|$ holds for every $x, y \in C$.

We will say that X has the weak fixed point property (**WFPP** for short) if every nonexpansive mapping $T : K \to K$ from a nonempty weakly compact convex subset K of X into itself has a fixed point.

R. Browder, D. Gohde, W. A. Kirk (see [5]) and other authors have established that conditions of a geometric nature on the norm of X, guarantee the **WFPP**. Uniform convexity and normal structure are examples of such conditions.

To obtain the weak fixed point property in Banach spaces, García-Falset [3] introduced the coefficient $R(X)$ as follows:

$$R(X) = \sup \left\{ \liminf_{n\to\infty} \|x_n + x\| : \{x_n\} \subset B(X), x_n \xrightarrow{w} 0, x \in B(X) \right\}.$$

He proved that a Banach space X with $R(X) < 2$ has the weak fixed point property (see [4]).

It is clear that a Banach space X with property (WA_2^ε) has $R(X) < 2$. Therefore, a Banach space X with property (WA_2^ε) has the fixed point property.

Let $\|\cdot\|$ be a norm in X. We say that $\|\cdot\|$ is a uniformly *Frechet* differentiable norm (**UF**-norm for short) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly over x, $y \in S(X)$.

Denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of natural and real numbers, respectively. Let (G, Σ, μ) be a measure space with a finite and non-atomic measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G. Let l^0 stand for the space of all real sequences.

A map $\Phi : \mathcal{R} \to [0,\infty)$ is said to be an *Orlicz function* if it is even, convex, vanishes at 0, but not identically 0.

An Orlicz function is called an *N-function* if

$$\lim_{u\to 0}\frac{\Phi(u)}{u}=\infty.$$

By the *Orlicz function space* L_Φ we mean

$$L_\Phi=\left\{x\in L^0 : I_\Phi(cx)=\int_G \Phi\left(cx(t)\right)d\mu<\infty \quad for \;\; some \;\; c>0\right\}.$$

Analogously, we define the *Orlicz sequence space*

$$l_\Phi=\left\{x\in l^0 : I_\Phi(cx)=\sum_{i=1}^{\infty}\Phi(cx(i))<\infty \quad for \;\; some \;\; c>0\right\}.$$

The spaces L_Φ and l_Φ are equipped with the so-called *Luxemburg norm*

$$\|x\|=\inf\{\varepsilon>0 : I_\Phi(\frac{x}{\varepsilon})\le 1\}$$

or with the equivalent one

$$\|x\|_0=\inf_{k>0}\frac{1}{k}\left(1+I_\Phi(kx)\right),$$

called the *Orlicz* or the *Amemiya norm.* It is well known that if Φ is an N-function, then for any $x\neq 0$ there exists a number k such that

$$\|x\|_0=\frac{1}{k}\left(1+I_\Phi(kx)\right).$$

(see [1]).

To simplify notations, we put $L_\Phi=(L_\Phi,\|\cdot\|)\,,\; l_\Phi=(l_\Phi,\|\cdot\|)\,,\; L^0_\Phi=(L_\Phi,\|\cdot\|_0)$ and $l^0_\Phi=(l^0_\Phi,\|\cdot\|_0)$.

For any Orlicz function Φ we define its *complementary function* $\Psi:\mathcal{R}\longrightarrow[0,\infty)$ by the formula

$$\Psi\left(v\right)=\sup_{u>0}\left\{u\left|v\right|-\Phi\left(u\right)\right\}$$

for every $v\in\mathcal{R}$. The complementary function Ψ is also a convex function vanishing at zero.

We say an Orlicz function Φ *satisfies the* Δ_2*-condition* (δ_2*-condition*) if there exist constants $k\ge 2$ and $u_0>0$ such that $\Phi(u_0)>0$ and

$$\Phi\left(2u\right)\le k\Phi\left(u\right)$$

for every $|u|\ge u_0$ (for every $|u|\le u_0$), respectively (see [1], [7] an [10]).

We say an Orlicz function Φ *satisfies the* ∇_2-condition ($\overline{\delta}_2$-condition) if its complementary function Ψ satisfies the Δ_2-condition (δ_2-condition), respectively.

An Orlicz function Φ is said to be *uniformly convex* on $[0, u_0]$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{\Phi(u)+\Phi(v)}{2}$$

for all $u, v \in [0, u_0]$ satisfying $|u - v| \geq \epsilon \max\{u, v\}$.

We say an Orlicz function Φ is *strictly convex* if for any $u \neq v$ and $\alpha \in (0, 1)$ we have

$$\Phi(\alpha u + (1-\alpha)v) < \alpha\Phi(u) + (1-\alpha)\Phi(v).$$

For more details on Orlicz functions and Orlicz spaces we refer to [1], [8] and [11].

§2. RESULTS

Theorem 1. If a norm $\|\cdot\|$ in a Banach space X is a **UF**-norm, then X has property (UA_2^{ε}).

Proof: Since $\|\cdot\|$ is a **UF**-norm in X, we get that the Banach space X is Gateaux differentiable, i.e., X is smooth. Let $f_x \in S(X^*)$ be the unique supporting functional at $x \in S(X)$. It is well know that the norm $\|\cdot\|$ on a Banach space X is **UF** if and only if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} = f_x(y)$$

exists uniformly for $x, y \in S(X)$.

Now, for any $\varepsilon > 0$ and each weakly null sequence $\{x_n\}$ in $S(X)$, there exists $n_0 \in \mathcal{N}$ such that

$$|f_x(x_n)| < \frac{\varepsilon}{2},$$

for all $n \geq n_0$. Since the norm $\|\cdot\|$ on a Banach space X is **UF**, there exists a $\delta > 0$ such that

$$\left|\frac{\|x+tx_{n_0}\|-\|x\|}{t} - f_x(x_{n_0})\right| < \frac{\varepsilon}{2}$$

whenever $|t| < \delta$, whence

$$\|x + tx_{n_0}\| - \|x\| < \frac{t\varepsilon}{2} + |f_x(x_{n_0})|\, t < t\varepsilon$$

uniformly with respect $x \in S(X)$. This means that X has property (A_2^{ε}).

Theorem 2. Suppose that a Banach space X has property (WA_2^{ε}). Then X has the weak Banach-Saks property and the weak fixed point property.

Proof: Since X has property (WA_2^{ε}), there exist $\varepsilon \in (0, 1)$ and $\delta > 0$ such that for $t \in [0, \delta]$ and weak null sequence $\{x_n\} \in B(X)$ there exists $k \in N, k > 1$ such that $\|x_1 + tx_k\| < 1 + \varepsilon\delta$. Hence

$$\|x_1 + x_k\| = \|x_1 + \delta x_k + (1-\delta)x_k\|$$

$$\leq \|x_1 + \delta x_k\| + (1-\delta) \leq 1 + \varepsilon\delta + 1 - \delta = 2 - \delta(1-\varepsilon),$$

That is, a Banach space with property (WA_2^ε) has property (A_2). Consequently, a Banach space with property (WA_2^ε) has the weak Banach-Saks property.

Moreover, we have $R(X) \leq 2 - \delta(1-\varepsilon) < 2$, so X enjoys the weak fixed point property.

Theorem 3. Let X be a separable Banach space. If X^*has property $(A_2^\varepsilon)^*$, then X has the (UKK)-property.

Proof: Let $\{x_n\}$ be a sequence in $S(X)$ with $sep(\{x_n\}) > \varepsilon$ and $x_n \stackrel{w}{\to} x \in B(X)$, deleting at most one element of the sequence, we can assume that $sep(\{x_n - x\}) > \varepsilon$. For any $\varepsilon_1 > 0$ let $M = 1 + \varepsilon_1$. By the Bessaga-*Pelczynski* selection principle, there exists a subsequence $\{z_n\}$ of $\{x_n - x, x\}$ with $z_1 = x$ that is a basic sequence with basic constant less than or equal to M. (See [[2]] p 46)

Let us consider the sequence $\{z_n^*\}$ of the *Hahn-Banach* extensions of the coefficient functionals of the basic sequence $\{\frac{z_n}{\|z_n\|}\}$. Put $X_0 = \overline{span}\{z_n : n = 1, 2, ...\}$. Then $\langle z_n^*, z\rangle \to 0$ for any $z \in X_0$ as $n \to \infty$. In fact, for any $z \in X_0$ we have $z = \sum_{i=1}^{\infty} z_i^*(z) z_i.$, hence

$$|\langle z_n^*, z\rangle| = \|z_n^*(z) z_n\| = \left\| \sum_{i=n}^{\infty} z_i^*(z) z_i - \sum_{i=n+1}^{\infty} z_i^*(z) z_i \right\|$$

$$\leq \left\| \sum_{i=n}^{\infty} z_i^*(z) z_i \right\| + \left\| \sum_{i=n+1}^{\infty} z_i^*(z) z_i \right\| \to 0.$$

Since X is separable, we can assume that $z_n^* \stackrel{w^*}{\to} z^*$ as $n \to \infty$.

Now, for any $\varepsilon_2 > 0$. Since X^* has property $(WA_2^\varepsilon)^*$, there exists $0 < \delta_2 \leq 1$ such that for any $t \in (0, \delta_2)$ there exists $k > 1$ such that

(1) $$\left\| \frac{z_1^*}{\|z_1^*\|} + t \frac{(z_k^* - z^*)}{\|z_k^* - z^*\|} \right\| < 1 + t\varepsilon_2,$$

It is easy to see that

(2) For all $k \in \mathbb{N}$, $\langle z^*, z_k\rangle = 0$ and $\langle z_k^*, z_k\rangle = \|z_k\|$. In particular $\langle z^*, x\rangle = 0$

(3) For all $k \geq 2$, $\|x + z_k\| = 1$ and $\langle z_k^*, x\rangle = 0$

(4) For all $k \in \mathbb{N}$, $\|z_k^* - z^*\| \leq 4M$, and $\|z_1^*\| \leq M$.

We can assume that $\|z_n\| \geq \frac{\varepsilon}{2}$ for $n \geq 2$, because $sep(\{x_n\}) > \varepsilon$

Let $t \in (0, \delta_2)$ and let $k > 1$ be such that (1) holds , by (2)- (4) we obtain

$$\|x\| = \langle z_1^*, x\rangle = \|z_1^*\| \langle \frac{z_1^*}{\|z_1^*\|}, x\rangle = \|z_1^*\| [\langle \frac{z_1^*}{\|z_1^*\|}, x + z_k\rangle]$$

$$= \|z_1^*\|[\langle\frac{z_1^*}{\|z_1^*\|}, x+z_k\rangle + t\langle\frac{z_k^*-z*}{\|z_k^*-z*\|}, x+z_k\rangle - t\langle\frac{z_k^*-z*}{\|z_k^*-z*\|}, x+z_k\rangle]$$

$$= \|z_1^*\|[\langle\frac{z_1^*}{\|z_1^*\|} + t\frac{z_k^*-z*}{\|z_k^*-z*\|}, x+z_k\rangle - \frac{t\|z_k\|}{\|z_k^*-z^*\|}]$$

$$\leq \|z_1^*\|[\|\frac{z_1^*}{\|z_1^*\|} + t\frac{z_k^*-z*}{\|z_k^*-z*\|}\| - \frac{t\|z_k\|}{\|z_k^*-z^*\|}]$$

$$\leq M[(1+t\varepsilon_2) - \frac{t\varepsilon}{2\|z_k^*-z^*\|}] \leq M[(1+t\varepsilon_2) - \frac{t\varepsilon}{8M}]$$

So far we have $\|x\| \leq M(1+t\varepsilon_2 - \frac{t\varepsilon}{8M})$.Using $M = 1+\varepsilon_1$, and taking the limit as $\varepsilon_1 \to 0$ and obtain

$$\|x\| \leq 1 + t(\varepsilon_2 - \frac{\varepsilon}{8})$$

Now take $\varepsilon_2 = \frac{\varepsilon}{16}$, and $t = \frac{\delta_2}{2}$, and get

$$\|x\| \leq 1 - \frac{\delta_2\varepsilon}{32}$$

Completing the proof of the theorem.

Remark 1. It worth noting that separability of X in the last theorem is only necessary to ensure that $w*$- compact subsets are $w*$-sequentially compact. We can relax the assumption of separability of X to, for example, requiring X admit an equivalent smooth norm [13].

Corollary 1. Let X be a separable Banach space. If X^*has property $(A_2^\varepsilon)^*$, then both X and X^* have the weak fixed point property.

Proof: The result follows from theorem 2, Theorem 3 and Theorem 1 in [ ].

Corollary 2. Let X be the Orlicz space L_M or L_M^0. The following statements are equivalent:

(1) X is uniformly smooth;

(2) X is nearly uniformly smooth;

(3) X is (**NUS***):

(4) X has property (A_2^ε) ;

(5)$\Psi \in \Delta_2$, Ψ is strictly convex on the whole real line and Φ is uniformly convex outside a neighborhood of zero.

Proof: It follows from Theorem 3 and Theorem 3.15 in [1].

Lemma 1. Suppose $\Phi \in \delta_2$. Then for any $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that

$$I_\Phi(x+ty) - I_\Phi(x) < t\varepsilon$$

whenever $I_\Phi(x) \le L$, $I_\Phi(y) \le \delta$ and $t \in (0,1)$.

Proof: Since $\Phi \in \delta_2$, for any $\varepsilon > 0$ and $L > 0$ there exists $\delta \in (0,1)$ such that

$$I_\Phi(x+y) - I_\Phi(x) < \varepsilon$$

whenever $I_\Phi(x) \le L$ and $I_\Phi(y) \le \delta$ (see [ ]).

So for any $t \in (0,\delta)$

$$\begin{aligned} &I_\Phi(x+ty) = I_\Phi(tx+ty+(1-t)x) \\ &\le tI_\Phi(x+y) + (1-t)I_\Phi(x) \\ &\le t(I_\Phi(x)+\varepsilon) + (1-t)I_\Phi(x) = I_\Phi(x) + t\varepsilon \end{aligned}$$

whenever $I_\Phi(x) \le L$ and $I_\Phi(y) \le \delta$.

Lemma 2. Suppose $\Phi \in \overline{\delta}_2$. Then for any $\varepsilon > 0$ and $u_0 > 0$ there exists $\delta > 0$ such that

$$\Phi\left(tu\right) \le t\varepsilon\Phi\left(u\right)$$

whenever $|u| \le u_0$ and $t \in (0,\delta)$.

Proof: Suppose that $\Phi \in \overline{\delta}_2$. Then for any $u_0 > 0$ there exists $\theta \in (0,1)$ such that

$$\Phi\left(\frac{u}{2}\right) \le \frac{\theta}{2}\Phi\left(u\right)$$

whenever $|u| \le u_0$ (see [ ]). Take $n \in \mathcal{N}$ such that $\theta^n \le \varepsilon$. Then for $\delta = \frac{1}{2^n}$, we have

$$\Phi(\delta u) = \Phi\left(\frac{u}{2^n}\right) \le \left(\frac{\theta}{2}\right)^n \Phi\left(u\right) \le \delta\varepsilon\Phi\left(u\right)$$

whenever $|u| \le u_0$.

Hence for any $t \in (0,\delta)$, we have

$$\Phi(tu) = \Phi\left(\frac{t}{\delta}\delta u\right) \le \frac{t}{\delta}\delta\varepsilon\Phi\left(u\right) = t\varepsilon\Phi\left(u\right)$$

whenever $|u| \le u_0$.

For any $x \in l_\Phi^0$, put $N(x) = \{i \in N : x(i) \ne 0\}$. Define $D(l_\Phi^0) = \{x = (x(i)) \in B(l_\Phi^0) : N(x) \text{ is finite }\}$.

Lemma 4. Let Φ be an N-function such $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every weakly null sequence $\{x_n\}$ in $B(l_\Phi^0)$ and $x \in D(l_\Phi^0)$ there exists $k > 1$ such that

$$\|x + tx_k\|^0 \le 1 + t\varepsilon$$

whenever $t \in (0,\delta)$.

Proof: Let $\varepsilon > 0$ be given. By $\Phi \in \overline{\delta}_2$, the set $Q = \{k_x : \frac{1}{2} \le \|x\| \le 1\ \}$ is bounded, i.e., there exists $\mathbf{k} > 1$ such that $1 \le k_x \le \mathbf{k}$. By Lemma 2, we know that there exists $\delta \in (0,1)$ such that

$$\Phi\left(tu\right) \le t\delta\Phi\left(u\right)$$

whenever $t \in (0,\delta)$ and $|u| \le \Phi^{-1}(\mathbf{k})$.

By the Lemma 1, there exists $\theta > 0$ such that

$$I_\Phi(x+ty) - I_\Phi(x) < t\varepsilon$$

whenever $I_\Phi(x) \le L$, $I_\Phi(y) \le \theta$ and $t \in (0,1)$.

Let $t \in (0, \frac{\delta}{\mathbf{k}})$ be fixed and $\{x_n\}$ be arbitrary weakly null sequence in $S(l_\Phi^0)$. For any $x \in D(l_\Phi^0)$, take $i_0 \in \mathcal{N}$ such that $x(i) = 0$ when $i > i_0$. Since $x_n \xrightarrow{w} 0$, there exists $n_0 \in \mathcal{N}$ such that $\sum_{i=1}^{i_0} \Phi(x_n(i)) < \theta$ for all $n \ge n_0$. Hence, we get for $l \ge 1$ satisfying $\|x_1\| = \frac{1}{l}(1 + I_\Phi(lx_1))$:

$$\|x_1 + tx_n\|^0 \le \frac{1}{l}\left[1 + I_\Phi\left(l(x_1 + tx_n)\right)\right]$$

$$= \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(l(x_1(i) + tx_n(i))\right) + \sum_{i=i_0+1}^{\infty} \Phi\left(ltx_n(i)\right)\right]$$

$$\le \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx_1(i)\right) + t\varepsilon + \sum_{i=i_0+1}^{\infty} \Phi\left(ltx_n(i)\right)\right]$$

$$\le \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx_1(i)\right) + t\varepsilon + tl\varepsilon \sum_{i=i_0+1}^{\infty} \Phi\left(x_n(i)\right)\right]$$

$$\le \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx_1(i)\right)\right] + 2t\varepsilon \le 1 + 2t\varepsilon. \qquad \square$$

Assume that $\Phi \in \delta_2$. Then for any $x \in S(l_\Phi^0)$ and $k > 1$, there exists a unique $d_{x,k} > 0$ such that $I_\Phi\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$. Define $d_x = \inf\{d_{x,k} : k > 1\}$.

Theorem 4. Let Φ be an Orlicz function satisfying $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $X = l_\Phi^0$. The following statements are equivalent:

(1) X has property (A_2^ε);

(2) X has property (WA_2^ε);

(3) $R(X) < 2$;

(4) $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

Proof: (3) $\Rightarrow$ (4). Suppose that $\Phi \notin \delta_2$. Then for any $\varepsilon > 0$ there exists $x \in S(l_\Phi^0)$ such that

$$1 - \varepsilon \le \left\|\sum_{i=n}^{\infty} x(i)e_i\right\|^0 \le 1$$

for all $n \in \mathcal{N}$. Take $n_1 < n_2 < \cdots$ of $\mathcal{N}$ such that

$$\left\|\sum_{j=n_i+1}^{n_{i+1}} x(j)e_j\right\|^0 \ge 1 - 2\varepsilon \quad \text{for all} \quad i \in \mathcal{N}.$$

Put $x_i = \sum\limits_{j=n_i+1}^{n_{i+1}} x(j)e_j$. Since

$$\limsup_{\lambda\to 0} \frac{I_\Phi(\lambda x_n)}{\lambda} \le \lim_{\lambda\to 0} \frac{I_\Phi(\lambda x)}{\lambda} = 0,$$

we have $x_i \overset{l_\Psi}{\to} 0$. Notice that every singular functional vanishes on any x_i. So, we have $x_i \overset{w}{\to} 0$.

But $\liminf\limits_{i\to\infty} \|x_i + x\|^0 \ge \liminf\limits_{i\to\infty} 2\|x_i\|^0 \ge 2(1-2\varepsilon)$. By the arbitrariness of ε, we get $R(l_\Phi^0) = 2$. In such a way we proved that if $\Phi \notin \delta_2$ then (3) does not hold.

Suppose that $\Phi \notin \overline{\delta}_2$.Then the Kottman constant $K(l_\Phi^0) = \sup\{d_x : x \in S(l_\Phi^0)\} = 2$.(see [1] and [11]). Hence for any $\varepsilon > 0$ there exists $x \in S(l_\Phi^0)$ such that $d_x > 2-\varepsilon$. Furthermore, we have $d_{x,k} \ge d_x > 2-\varepsilon$ for all $k > 1$. Put

$x_1 = (x(1), 0, x(2), 0, x(3), 0, x(4), 0, x(5), 0, x(6), 0, ...)$,

$x_2 = (0, x(1), 0, 0, 0, x(2), 0, 0, 0, 0, 0, 0, 0, x(3), 0, 0, ...)$,

$x_3 = (0, 0, 0, x(1), 0, 0, 0, 0, 0, 0, 0, 0, x(2), 0, 0, 0, 0, ...), ...$.

Then $\|x_n\|^0 = 1$, $x_n \overset{w}{\to} 0$ and for any $k > 1$ we have

$$\frac{1}{k}\left(1 + I_\Phi\left(\frac{k(x_n + x_1)}{d_x}\right)\right) \ge \frac{1}{k}\left(1 + I_\Phi\left(\frac{k(x_n + x_1)}{d_{x,k}}\right)\right)$$
$$= \frac{1}{k}\left(1 + I_\Phi\left(\frac{kx}{d_{x,k}}\right) + I_\Phi\left(\frac{kx}{d_{x,k}}\right)\right) = \frac{1}{k}(1 + \frac{k-1}{2} + \frac{k-1}{2}) = 1.$$

So, we get $\left\|\frac{x_n+x_1}{d_x}\right\|^0 \ge 1$, i.e., $\liminf\limits_{n\to\infty} \|x_n + x_1\|^0 \ge d_x - \varepsilon$. By the arbitrariness of ε, we get $R(l_\Phi^0) = 2$. Therefore, we proved that $\Phi \notin \overline{\delta}_2$ implies that (3) does not hold.

(4) $\Rightarrow$ (1). By Lemma 4, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every weak null sequence $\{x_n\}$ in $B(l_\Phi^0)$ and any $x \in D(l_\Phi^0)$, there exists a number $m > 1$ such that

$$\|x + tx_m\|^0 \le 1 + \frac{t\varepsilon}{2}$$

whenever $t \in (0, \delta)$.

Let $t \in (0, \delta)$ be given arbitrary. For any weakly null sequence $\{x_n\}$ in $B(l_\Phi^0)$, we only need to consider the case when $N(x_1)$ is infinite. Take i_0 large enough such that $\left\|\sum\limits_{i=i_0+1}^{\infty} x_1(i)e_i\right\|^0 \le \frac{t\varepsilon}{2}$. Then there exists $m \in \mathcal{N}$ such that

$$\left\|\sum_{i=1}^{i_0} x_1(i)e_i + tx_m\right\|^0 \le 1 + \frac{t\varepsilon}{2}.$$

Hence

$$\|x_1 + tx_m\|^0 \le \left\|\sum_{i=1}^{i_0} x_1(i)e_i + tx_m\right\|^0 + \frac{t\varepsilon}{2} \le 1 + \frac{t\varepsilon}{2} + \frac{t\varepsilon}{2} = 1 + t\varepsilon. \qquad \square$$

Corollary 3. Let Φ be an Orlicz function with $\lim\limits_{n\to 0} \frac{\Phi(u)}{u} = 0$ and $X = l^0_\Phi$. The following statements are equivalent:

(1) X is nearly uniformly smooth;

(2) X is **(NUS*)**;

(3) $M \le 1$, $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

In same way, we can get the following result.

Theorem 5. For any Orlicz function Φ and X the following statements are equivalent:

(1) X has property (A_2^ε);

(2) X has property (WA_2^ε);

(3) $R(X) < 2$;

(4) $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

Corollary 4. Let Φ and X be as in Theorem 5. The following statements are equivalent:

(1) X is nearly uniformly smooth;

(2) X is **(NUS*)**;

(3) $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

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