ON A CONNECTION BETWEEN THE NUMERICAL RANGE AND SPECTRUM OF AN OPERATOR ON A HILBERT SPACE

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For a complex Hilbert space H we denote by $B(H)$ the algebra of continuous linear operators on H. For $T \in B(H)$, T^* denotes the adjoint operator. The *numerical range* of T, $W(T)$, is defined as

$$W(T) = \{(Tx, x) : x \in H, \|x\| = 1\},$$

and

$$v(T) = \sup \{|\lambda| : \lambda \in W(T)\}$$

is the *numerical radius* of T. $W(T)$ is a convex subset of the complex plane whose closure contains the spectrum of T, $\sigma(T)$. The set of eigenvalues of T is denoted by $\rho\sigma(T)$ and the set of approximate eigenvalues by $\pi\sigma(T)$. Co $\sigma(T)$ is the convex hull of $\sigma(T)$.

A point $\lambda \in \overline{W(T)}$ is a *bare point* of $\overline{W(T)}$ if λ lies on the perimeter of a closed circular disc containing $\overline{W(T)}$. We say $\overline{W(T)}$ has a *corner* with *vertex* λ if $\lambda \in \overline{W(T)}$ and $\overline{W(T)}$ is contained in a half-cone with vertex λ and angle less than π.

We aim to relate the vertices of corners of $\overline{W(T)}$ to points in $\sigma(T)$. The starting point is the following lemma first suggested to me by A. M. Sinclair.

LEMMA 1. *For a complex Hilbert space H and $T \in B(H)$, if $1 = v(T) \in W(T)$, then $1 \in \rho\sigma(U)$ where $U = \frac{1}{2}[T + T^*]$.*

Proof. $1 = \sup \operatorname{Re} W(T) = \sup W(U) \leqslant v(U) = \|U\| \leqslant \frac{1}{2}(v(T) + v(T^*)) = 1$; so $\|U\| = 1$. Now for some $x \in H$, $\|x\| = 1$, we have

$$1 = (Tx, x) = \operatorname{Re}(Tx, x) = (Ux, x) \leqslant \|Ux\| \, \|x\| \leqslant 1;$$

so, by the rotundity of H, $Ux = x$.

LEMMA 2. *For a complex Hilbert space H and $T \in B(H)$, if $\lambda \in \overline{W(T)}$ is a bare point of $W(T)$, then $(e^{-i\theta} T + e^{i\theta} T^*)x = (e^{-i\theta}\lambda + e^{i\theta}\bar{\lambda})x$ for some $x \in H$, $\|x\| = 1$, and θ, $0 \leqslant \theta < 2\pi$.*

Proof. Since λ is a bare point of $\overline{W(T)}$ there exists $r > 0$ and $\alpha \in C$ such that $W(T) \subseteq D = \{z \in C : |z - \alpha| \leqslant r\}$ and $\lambda \in \overline{W(T)} \cap \text{bdry } D$. Let $\lambda - \alpha = re^{i\theta}$, $0 \leqslant \theta < 2\pi$ and set $T_1 = r^{-1} e^{-i\theta}(T - \alpha I)$. Then $\overline{W(T_1)}$ is contained in the unit disc and if $x \in H$, $\|x\| = 1$, is such that $\lambda = (Tx, x)$, we have

$$1 = (T_1 x, x) = v(T_1) \in W(T_1);$$

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so, by Lemma 1, $\frac{1}{2}[T_1+T_1^*]x = x$. Therefore

$$\tfrac{1}{2}[r^{-1}e^{-i\theta}(T-\alpha I)+r^{-1}e^{i\theta}(T^*-\bar{\alpha}I)]x = x$$

or

$$\begin{aligned}\tfrac{1}{2}[e^{-i\theta}T+e^{i\theta}T^*]x &= rx+\tfrac{1}{2}(e^{i\theta}\alpha+e^{i\theta}\bar{\alpha})x\\ &= rx+\tfrac{1}{2}(e^{i\theta}\lambda-r+e^{i\theta}\lambda+r)x\\ &= \tfrac{1}{2}(e^{i\theta}\lambda+e^{i\theta}\bar{\lambda})x.\end{aligned}$$

This last lemma is similar to a result by B. A. Mirman for compact operators [**4**; sledstvie 1], and from it our first main result follows.

THEOREM 1. *For a complex Hilbert space H and $T \in B(H)$, if $\lambda \in W(T)$ is the vertex of a corner of $\overline{W(T)}$, then $\lambda \in p\sigma(T)$.*

Proof. Since λ is the vertex of a corner of $\overline{W(T)}$, λ is a bare point of $\overline{W(T)}$, and in fact we can find at least $r_1, r_2 > 0$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_1 \neq t\alpha_2$ for any $t \in R$, such that $\overline{W(T)} \subseteq D_j = \{z \in C : |z-\alpha_j| \leqslant r_j\}$ and $\lambda \in \overline{W(T)} \cap D_j$ for $j = 1, 2$. So from the proof of Lemma 2 there exist $\theta_1, \theta_2 \in (0, 2\pi)$, $0 < |\theta_1-\theta_2| < \pi$, such that

$$\tfrac{1}{2}[e^{-i\theta_j}T+e^{i\theta_j}T^*]x = \tfrac{1}{2}(e^{-i\theta_j}\lambda+e^{i\theta_j}\bar{\lambda})x$$

or

$$\tfrac{1}{2}[e^{-2i\theta_j}T+T^*]x = \tfrac{1}{2}(e^{-2i\theta_j}\lambda+\bar{\lambda})x,\ j = 1, 2.$$

Subtracting these two equations gives

$$\tfrac{1}{2}(e^{-2i\theta_1}-e^{-2i\theta_2})Tx = \tfrac{1}{2}(e^{-2i\theta_1}-e^{2i\theta_2})\lambda x$$

and so, since $\theta_1 \neq \theta_2$, $Tx = \lambda x$.

COROLLARY 1.1. *For a complex Hilbert space H and compact operator $T \in B(H)$, if $0 \neq \lambda \in W(T)$ is the vertex of a corner of $\overline{W(T)}$, then $\lambda \in p\sigma(T)$.*

Proof. Since λ is the vertex of a corner of $\overline{W(T)}$, λ is a non-zero exposed point of $\overline{W(T)}$ and so, by [**1**; Theorem 1], $\lambda \in W(T)$ and the result now follows from Theorem 1.

COROLLARY 1.2. *For a complex Hilbert space H and $T \in B(H)$, if $W(T)$ is a closed polygon then* co $\sigma(T) = W(T)$.

Proof. Let the vertices of the convex polygon $W(T)$ be $\{\lambda_i\}$. Then, by Theorem 1, $\lambda_i \in p\sigma(T)$ for all i; so

$$\text{co}\,\sigma(T) \supseteq W(T) \quad \text{but} \quad \text{co}\,\sigma(T) \subseteq \overline{W(T)} = W(T).$$

COROLLARY 1.3. *A closed bounded polygon with m vertices is the numerical range of an operator on n-dimensional Hilbert space if and only if $m \leqslant n$.*

Proof. Let the numerical range of T be the closed polygon with vertices $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then by Theorem 1 each λ_i is an eigenvalue of T and there are at most n of them.

Conversely, let $\lambda_1, \ldots, \lambda_m$ $(m \leqslant n)$ be the vertices of a closed polygon P. Then the normal operator represented by the diagonal matrix

$$\begin{aligned} a_{ij} &= \lambda_i \delta_{ij} \qquad 1 \leqslant i \leqslant m \\ &= 0 \qquad\quad m < i \leqslant n \end{aligned}$$

has $W(T) = \operatorname{co} \sigma(T) = P$.

We now consider the case when λ is the vertex of a corner of $\overline{W(T)}$ but $\lambda \in \overline{W(T)} \setminus W(T)$.

THEOREM 2. *For complex Hilbert space H and $T \in B(H)$, if $\lambda \in \overline{W(T)}$ is the vertex of a corner of $\overline{W(T)}$ then $\lambda \in \pi\sigma(T)$.*

Proof. By a construction of S. K. Berberian [**2**] and a result of Berberian and G. H. Orland [**3**] we can embed H in a larger Hilbert space K and extend T to $[T] \in B(K)$ such that $\overline{W(T)} = W([T])$ and $\pi\sigma(T) = \rho\sigma([T])$. The result now follows by applying Theorem 1 to $[T]$, since $\lambda \in W([T])$ is the vertex of a corner of $\overline{W([T])} = W([T])$.

COROLLARY 2.1. *For a complex Hilbert space H and $T \in B(H)$, if $\overline{W(T)}$ is a closed polygon, then* $\operatorname{co} \sigma(T) = \overline{W(T)}$.

Proof. Let $\{\lambda_i\}$ be the vertices of $\overline{W(T)}$. Then, by Theorem 2, $\lambda_i \in \pi\sigma(T)$ for all i; so $\operatorname{co} \sigma(T) \supseteq \overline{W(T)}$.

A result corresponding to Theorem 2 is not generally valid in a Banach algebra without further restrictions. B. Schmidt [**5, 6**] has shown that if λ is the vertex of a corner of $V(B, T)$ with angle less than $\pi/2$ then $\lambda \in \sigma(T)$ and that this is best possible.

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References

1. G. de Barra, J. R. Giles and Brailey Sims, " On the numerical range of compact operators on a Hilbert space ", *J. London Math. Soc.*, 5 (1972), 704–706.
2. S. K. Berberian, "Approximate proper vectors ", *Proc. Amer. Math. Soc.*, 13 (1962), 111–114.
3. ——— and G. H. Orland, " On the closure of the numerical range of an operator ", *Proc. Amer. Math. Soc.*, 18 (1967), 499–503.
4. B. A. Mirman, " Čislovaja oblast'vpolne neprerynovo operatora i evo spektr ", *Teor. Funcii Funkcional. Anal. i Priložen*, 8 (1969), 44–49. (M.R. 41, No. 9043).
5. B. Schmidt, " Über die Ecken des numerischen wertebereichs in einer Banachalgebra ", *Math. Z.*, 126 (1972), 47–50.
6. ———, " Über die Ecken des numerischen wertebereichs in einer Banachalgebra II ", *Math. Z.*, 127 (1972), 186–190.

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