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ON THE EXISTENCE OF SUPPORT MAPS WITH DENSE IMAGES

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Abstract

For a normed linear space X we investigate conditions for the existence of support maps under which the image of X is a dense subset of the dual. In the case of finite-dimensional spaces a complete answer is given. For more general spaces some sufficient conditions are obtained.

Throughout we will use $\|\cdot\|$ for the norm function of a normed linear space X, X' for its dual space and $S(X)$ to denote its unit sphere $\{x \in X: \|x\| = 1\}$.

We will be particularly interested in $S(X')$ regarded as a metric space under the metric $d(f,g) = \|f - g\|$ for all $f, g \in S(X')$.

Unless otherwise stated, by the interior, $\operatorname{int} A$, or the boundary, $\operatorname{bdry} A$, of a subset $A \subseteq S(X')$ we mean in the context of $(S(X'), d)$. Thus, for example, $f \in \operatorname{int} A$ if there exists $r > 0$ such that $B_r(f) = \{g \in S(X'): \|f - g\| < r\} \subseteq A$.

It is a simple consequence of the Hahn–Banach Theorem that we may define a set valued map $\mathscr{D}$ from $S(X)$ into the non-trivial subsets of $S(X')$ by

$$f \in \mathscr{D}(x) \quad \text{if} \quad f(x) = 1.$$

This map is frequently termed the duality map of X. When we want to emphasize the underlying space X we will write $\mathscr{D}_X(x)$ in place of $\mathscr{D}(x)$.

A *support map* is a selector for $\mathscr{D}$, that is a *function*

$$\phi: S(X) \to S(X'): \quad x \mapsto \phi_x \in \mathscr{D}(x).$$

The important property of subreflexivity, as established by Bishop and Phelps (1961), states that for a Banach space X, $\bigcup_{x \in S(X)} \mathscr{D}(x)$ is a dense subset of $S(X')$. We will be interested in the geometry of spaces which have a support map ϕ with $\overline{\phi(S(X))} = S(X')$. Such a support map will be referred to as having *dense image.*

Not every Banach space has a support map with dense image, a fact amply demonstrated by the space $l^2_\infty(\mathscr{R})$.

Recalling that a Banach space X is *smooth* at $x \in S(X)$ if $\mathscr{D}(x)$ is a singleton set, we see that subreflexivity establishes that for every smooth Banach space the unique support map has dense image. So a sufficient condition for a Banach space to have a support map with dense image would be the existence of a lower semi-continuous support map (norm to weak*, Cudia (1964)).

That the requirement of smoothness is over strong may be seen from the example of $\mathscr{R}^3$ equipped with norm the gauge of the "lens-shaped" set

$$\{x: \|x-(0,\tfrac{1}{2},0)\|_2 \leqq 1 \text{ and } \|x+(0,\tfrac{1}{2},0)\|_2 \leqq 1\}.$$

In this space the selection of a support map with dense image follows from the existence of a function $f: \mathscr{R} \to \mathscr{R}$ under which the image of an open neighbourhood is a dense subset of $\mathscr{R}$. Accordingly we seek weaker conditions than smoothness which will ensure the existence of support maps with dense images.

The following equivalence is an obvious consequence of subreflexivity.

Proposition 1. *A support map ϕ of the Banach space X has dense image if and only if for each $x \in S(X)$ and $f \in \mathscr{D}(x)$ there exists a sequence $\{x_n\}$ of points in $S(X)$ with $\phi_{x_n} \to f$.*

As a consequence of this proposition we have:

If for any $x \in S(X)$, $\operatorname{int} \mathscr{D}(x) \neq \varnothing$ and

$$\underline{[\operatorname{int} \mathscr{D}(x)] \cap [\textstyle\bigcup_{y \in S(X)\setminus\{x\}} \mathscr{D}(y)] = \varnothing},$$

then X does not have a support map with dense image.

The next lemma shows that the second (underlined) condition is redundant.

Lemma 2. *In the normed linear space X, if $f \in \operatorname{int} \mathscr{D}(x)$ for some $x \in S(X)$, then $f \notin \mathscr{D}(y)$ for any $y \in S(X)\setminus\{x\}$.*

Proof. Assume the contrary, that there exists $y \in S(X)\setminus\{x\}$ with $f \in \mathscr{D}_X(y)$. Let Y be the two-dimensional subspace of X spanned by x and y. Then $f|_Y \in \mathscr{D}_Y(y)$ and further in $S(Y')$, $f|_Y \in \operatorname{int} \mathscr{D}_Y(x)$ which clearly cannot be the case in a two-dimensional space unless $x = y$, a contradiction.

COROLLARY 3. *If the normed linear space X has a support map with dense image, then* $\operatorname{int} \mathscr{D}(x) = \varnothing$ *for all* $x \in S(X)$.

We now develop some partial converses to Corollary 3.

LEMMA 4. *For the Banach space X, if* $\operatorname{int} \mathscr{D}(x) = \varnothing$ *for all* $x \in E$ *a countable subset of X, then*

$$\operatorname{int}\left[\bigcup_{x \in E} \mathscr{D}(x)\right] = \varnothing.$$

PROOF. Assume the contrary, then there exists $f_0 \in S(X')$ and $r > 0$ with $B_r(f_0) = \{f \in S(X'): \|f - f_0\| < r\} \subseteq \operatorname{int}[\bigcup_{x \in E} \mathscr{D}(x)]$. Now the closed subset $B_{r/2}[f_0] = \{f \in S(X'): \|f - f_0\| \leqq \frac{1}{2} r\}$ is a complete metric space. However,

$$B_{r/2}[f_0] = \bigcup_{x \in E} (\mathscr{D}(x) \cap B_{r/2}[f_0])$$

and for each $x \in E$, $\mathscr{D}(x) \cap B_{r/2}[f_0]$ is nowhere dense, since $\mathscr{D}(x)$ is closed and in $B_{r/2}[f_0]$, $\operatorname{int}(\mathscr{D}(x) \cap B_{r/2}[f_0]) = \varnothing$, contradicting the Baire Category Theorem.

For any normed linear space X denote by $\lambda(X)$ the set of non-smooth points of the unit sphere $S(X)$ and let $\Delta = \cup\{\mathscr{D}(x): x \in S(X)\}$ and $\Lambda = \cup\{\mathscr{D}(x): x \in \lambda(X)\}$.

LEMMA 5. *Every support map of the Banach space X has dense image in* $S(X')\backslash \operatorname{int} \bar{\Lambda}$.

PROOF. For $f \in S(X')\backslash \operatorname{int} \bar{\Lambda}$, either $f \in S(X')\backslash \bar{\Lambda}$ or $f \in \operatorname{bdry} \bar{\Lambda}$. If f belongs to the open set $S(X')\backslash \bar{\Lambda}$, then by the subreflexivity of X there exists a sequence $\{f_n\}$ of functionals in $\Delta\backslash\bar{\Lambda}$ convergent to f. Now each $f_n \in \mathscr{D}(x_n)$ for some $x_n \in S(X)\backslash\lambda(X)$ in which case $\mathscr{D}(x_n)$ is the singleton set $\{\phi_{x_n}\}$ and so we have a sequence $\{x_n\}$ in $S(X)$ with $\phi_{x_n} \to f$.

On the other hand, if $f \in \operatorname{bdry} \bar{\Lambda}$, then by definition there exists a sequence $\{f_n\}$ of elements in $S(X')\backslash\bar{\Lambda}$ with $f_n \to f$. From the first half of the proof we can choose an $x_n \in S(X)$ with $\|\phi_{x_n} - f_n\| < 1/n$ in which case

$$\|f - \phi_{x_n}\| \leqq \|f - f_n\| + \|f_n - \phi_{x_n}\| \to 0$$

and again we have established the existence of a sequence $\{x_n\}$ in $S(X)$ with $\phi_{x_n} \to f$, thus establishing the result.

COROLLARY 6. *Let X be a Banach space and suppose* Λ *is nowhere dense. Then every support map on X has dense image.*

LEMMA 7. *Let X be a normed linear space. If* Λ *has empty interior in the metric subspace* Δ, *then every support map has dense image in* Δ.

PROOF. If $f \in \Delta$ then, for any $\varepsilon > 0$, $B_\varepsilon(f)$ contains a point $g \in \Delta \backslash \Lambda$. Since $g = \phi_x$ for some $x \in S(X)$, $\| f - \phi_x \| < \varepsilon$ so the image of ϕ is dense in Δ.

THEOREM 8. *Let X be a Banach space with separable dual, then X has a support map with dense image if and only if* $\operatorname{int} \mathscr{D}(x) = \emptyset$ *for each* $x \in S(X)$.

PROOF. Necessity has already been proved in Corollary 3.

To prove sufficiency, by Lemma 5, we need only ensure the image of ϕ is dense in $\operatorname{int} \bar{\Lambda}$.

Since $\operatorname{int} \bar{\Lambda}$ is an open subset of $S(X')$ we may choose $\{f_1, f_2, \cdots, f_n, \cdots\}$ to be a countable, dense subset of $\operatorname{int} \bar{\Lambda}$.

Now let $\theta: n \mapsto (\theta_1(n), \theta_2(n))$ be a 1–1 map from the set of natural numbers $\aleph$ onto $\aleph \times \aleph$, and inductively select x_n from

$$\{x \in \lambda(X) \backslash \{x_1, x_2, \cdots, x_{n-1}\}: \mathscr{D}(x) \cap B_{r_n}(f_{\theta_2(n)}) \neq \emptyset \text{ where } r_n = \theta_1(n)^{-1}\}$$

and ϕ_{x_n} from $\mathscr{D}(x_n) \cap B_{r_n}(f_{\theta_2(n)})$.

Such a selection is possible since $\operatorname{int} \bar{\Lambda}$ is an open subset of $\bar{\Lambda}$, and for any $n \in \aleph$

$$\bigcup_{x \in \lambda(X) \backslash \{x_1, \cdots, x_n\}} \mathscr{D}(x)$$

is a dense subset of $\bar{\Lambda}$ as $\operatorname{int} \mathscr{D}(x) = \emptyset$, $\mathscr{D}(x)$ is closed, and so $\bigcup_{i=1}^n \mathscr{D}(x_i)$ is nowhere dense by Lemma 4.

It is clear from the above selection procedure that $\{\phi_{x_n} : n \in \aleph\}$ is dense in $\operatorname{int} \bar{\Lambda}$. Thus assigning ϕ_x arbitrarily for $x \in \lambda(X) \backslash \{x_1, x_2, \cdots, x_n, \cdots\}$ we arrive at a support map with dense image.

We now investigate some conditions under which $\operatorname{int} \mathscr{D}(x) = \emptyset$. From the convexity of the norm in the normed linear space X it follows that for any $x, y \in S(X)$ and α real

$$g^+(x; y) = \operatorname*{Limit}_{\alpha \to 0^+} \alpha^{-1}(\| x + \alpha y \| - 1) \qquad \text{and}$$

$$g^-(x; y) = \operatorname*{Limit}_{\alpha \to 0^-} \alpha^{-1}(\| x + \alpha y \| - 1)$$

exist.

It is well known that

$$\begin{aligned} g^-(x; y) &= \inf\{\operatorname{Re} f(y): f \in \mathscr{D}(x)\} \\ &\leqq \sup\{\operatorname{Re} f(y): f \in \mathscr{D}(x)\} \\ &= g^+(x; y). \end{aligned}$$

The norm is differentiable at $x \in S(X)$ in the direction y if $g^-(x;y) = g^+(x;y)$, in which case we will denote the common value of these two limits by $g(x;y)$.

If the norm is differentiable at $x \in S(X)$ in some direction $y \in S(X)\backslash\{x, -x\}$ we say *the norm is differentiable at x in a non-radial direction, y.*

LEMMA 9. *In the normed linear space X, if the norm is differentiable at $x \in S(X)$ in a non-radial direction y, then the real linear hull of $\mathcal{D}(x)$ is a proper subset of X'.*

PROOF. It suffices to observe that $z = y - g(x;y)x$ is a non-zero element of X for which $\operatorname{Re} f(z) = 0$ for all $f \in \mathcal{D}(x)$, and so should the real linear hull of $\mathcal{D}(x)$ equal X' we would contradict the Hahn–Banach Theorem.

As a partial converse to this result we offer the following.

LEMMA 10. *If X is a finite-dimensional normed linear space and $x \in S(X)$ is such that the real linear hull of $\mathcal{D}(x)$ is a proper subset of X', then the norm is differentiable at x in a non-radial direction.*

PROOF. Let D be the real linear hull of $\mathcal{D}(x)$ then D is a proper closed subspace of $(X')_{\mathcal{R}}$ — the dual of X regarded as a linear space over $\mathcal{R}$. So by the Hahn–Banach Theorem there exists $F \in (X')'_{\mathcal{R}}$ with $\|F\| = 1$ and $F(D) = \{0\}$.

Form F' by $F'(f) = F(f) - iF(if)$ for all $f \in X'$ then $F' \in X''$ and so by the reflexivity of X, $F' = \hat{y}$ for some $y \in S(X)$, where $\hat{y}(f) = f(y)$. Clearly $y \neq x, -x$ as $f(x) = -f(-x) = 1$ for all $f \in \mathcal{D}(x)$ while $\operatorname{Re} f(y) = \operatorname{Re} \hat{y}(f) = F(f) = 0$ for all $f \in \mathcal{D}(x)$. From this it also follows that $g^-(x;y) = g^+(x;y) = 0$ and so $g(x;y)$ exists.

LEMMA 11. *If, in the normed linear space X, $x \in S(X)$ has* $\operatorname{int} \mathcal{D}(x) \neq \varnothing$, *then X' is the real linear hull of $\mathcal{D}(x)$.*

PROOF. Choose $f \in \operatorname{int} \mathcal{D}(x)$, then, for $g \in X'$ either $g = kf$ for some $k \in \mathcal{R}$ or $\{f\} \subsetneqq (\mathcal{D}(x) \cap \langle f, g \rangle_{\mathcal{R}})$ where $\langle f, g \rangle_{\mathcal{R}}$ is the real linear hull of $\{f, g\}$. So there exists $f' \in (\mathcal{D}(x)\backslash\{f\}) \cap \langle f, g \rangle_{\mathcal{R}}$ and further $f' \neq kf$ $(k \in \mathcal{R})$ since $|k| = 1$ so $k = \pm 1$ but if $f' = -f$ then $0 = \frac{1}{2}f + \frac{1}{2}f' \in \mathcal{D}(x)$ which is impossible. Thus f, f' form a basis of $\langle f, g \rangle_{\mathcal{R}}$ and so g is a real linear combination of f and f' as required.

LEMMA 12. *Let X be a normed linear space. If the norm is differentiable at $x \in S(X)$ in a non-radial direction, then* $\operatorname{int} \mathcal{D}(x) = \varnothing$.

PROOF. The result follows from Lemmas 9 and 11.

Whether this requirement of differentiability is also a necessary condition is not known. Since in general the converse of Lemma 11 may be untrue, a reversal of the above line of reasoning cannot be attempted. However in the case of finite-dimensional spaces we have the following result.

LEMMA 13. *Let X be a normed linear space of finite dimension n. If $x \in S(X)$ is such that the real linear hull of $\mathscr{D}(x)$ is X', then* int $\mathscr{D}(x) \neq \varnothing$.

PROOF. Let $f_1, f_2, \cdots, f_n \in \mathscr{D}(x)$ have X' as their real linear hull. Form $f = \Sigma_{k=1}^{n} (1/n) f_k \in \mathscr{D}(x)$ by its convexity. From the continuity of the natural projections $\pi_k : X' \to \langle f_k \rangle$ we can choose an $\varepsilon > 0$ so that, if $g = \Sigma_{k=1}^{n} \mu_k f_k$ $(\mu_k \in \mathscr{R})$ has $\| g - f \| < \varepsilon$ then $| \mu_k - 1/n | < 1/2n$ and so $\mu_k > 0$ for each k. For such an ε, take $g \in S(X)$ with $\| f - g \| < \varepsilon$, then $1 = \| g \| \leqq \Sigma_{k=1}^{n} \mu_k$ while $g' = g / \Sigma_{k=1}^{n} \mu_k$ is a convex combination of the $\{f_k\}$ and so belongs to $\mathscr{D}(x)$. Consequently g' has norm 1, whence $\Sigma_{k=1}^{n} \mu_k = 1$ and so $g = g' \in \mathscr{D}(x)$. That is $\{g \in S(X) : \| g - f \| < \varepsilon \} \subset \mathscr{D}(x)$ and so $f \in \text{int}\, \mathscr{D}(x)$.

Combining this partial converse to Lemma 11 with Lemma 10 and Theorem 8 we arrive at the following characterization in finite-dimensional spaces.

THEOREM 14. *Let X be a finite-dimensional normed linear space. Then the norm is differentiable at $x \in S(X)$ in a non-radial direction if and only if* int $D(x) = \varnothing$. *Therefore X has a support map with dense image if and only if at each point of $S(X)$ the norm is differentiable in a non-radial direction.*

PROOF. Lemmas 10, 12 and 13 establish the first equivalence, while the second equivalence follows from the first and Theorem 8.

THEOREM 15. *Let X be a Banach space with $\lambda(X)$ finite. If at each $x \in \Lambda(X)$ the norm is differentiable in some non-radial direction, then every support map has dense image.*

PROOF. Applying Lemma 12 then Lemma 4 shows that Λ is nowhere dense. Hence the conclusion follows from Corollary 6.

THEOREM 16. *Let X be a reflexive space and $\lambda(X)$ countable. If at each $x \in \Lambda(X)$ the norm is differentiable in some non-radial direction, then every support map has dense image.*

PROOF. Since X is reflexive, $S(X') = \Delta$, so the result follows from the successive application of Lemmas 12, 4 and 7.

THEOREM 17. *Let X be a Banach space with separable dual. If at each $x \in S(X)$ the norm is differentiable in some non-radial direction, then X has a support map with dense image.*

Proof. The conclusion follows from Lemma 12 and Theorem 8.

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