ON THE NUMERICAL RANGE OF COMPACT OPERATORS ON HILBERT SPACES

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In the study of the numerical range of operators on Hilbert spaces, P. R. Halmos has raised the problem of determining those continuous linear operators T on infinite-dimensional Hilbert spaces whose numerical range $W(T)$ is closed, [**1**, problem 168, p. 111]. He has given the example of the operator T on l^2 defined by

$$T(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\alpha_1, \tfrac{1}{2}\alpha_2, \ldots, \frac{1}{n}\alpha_n, \ldots\right),$$

with $W(T) = (0, 1]$ to show that " the numerical range may fail to be closed even for compact operators ", [**1**, solution 168, p. 320]. It is the aim of this note to contribute further to our knowledge of the closure properties of the numerical range of compact operators.

THEOREM 1. *For a compact operator T on an infinite-dimensional Hilbert space,*

(i) *if $0 \in W(T)$ then $W(T)$ is closed,*

(ii) *if $0 \notin W(T)$ then 0 is an extreme point of $\overline{W(T)}$, and $\overline{W(T)}\setminus W(T)$ consists at most of line segments in $\partial W(T)$ which contain 0 but no other extreme point of $\overline{W(T)}$.*

Proof. If λ is a cluster point of $W(T)$, then there exists a sequence $\{(Tx_n, x_n)\}$, where $\|x_n\| = 1$ for all n, converging to λ. Since the unit ball in a Hilbert space is weakly sequentially compact, there exists a subsequence $\{x_{n_k}\}$ which is weakly convergent to an x where $\|x\| \leqslant 1$. Since T is a compact operator, $\{Tx_{n_k}\}$ is strongly convergent to Tx.
However,

$$\begin{aligned}|(Tx_{n_k}, x_{n_k}) - (Tx, x)| &\leqslant |(Tx_{n_k}, x_{n_k}) - (Tx, x_{n_k})| + |(Tx, x_{n_k}) - (Tx, x)| \\ &\leqslant \|x_{n_k}\| \, \|Tx_{n_k} - Tx\| + |(x_{n_k}, Tx) - (x, Tx)|.\end{aligned}$$

Therefore $\{(Tx_{n_k}, x_{n_k})\}$ converges to (Tx, x) and so $(Tx, x) = \lambda$. If $\lambda \neq 0$, clearly $x \neq 0$, so

$$\frac{\lambda}{\|x\|^2} = \left(T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right) \in W(T).$$

Since $\|x\| \leqslant 1$ it follows that λ belongs to the interval $(0, \lambda/\|x\|^2]$, using an obvious notation for line segments in the complex plane.

(i) If $0 \in W(T)$ we have from the convexity of $W(T)$ that $\lambda \in W(T)$ and so $W(T)$ is closed.

(ii) If λ, $(\lambda \neq 0)$, is an extreme point of the intersection of a ray from 0 with $\overline{W(T)}$, then, since $0 \in \overline{W(T)}$, we have that $\lambda = \lambda/\|x\|^2$ and so $\|x\| = 1$ and $\lambda \in W(T)$. If 0 is in the interior of a line segment in $\partial W(T)$ then the intersection of the line with

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$\overline{W(T)}$ has two extreme points and these are in $W(T)$; but these are on either side of 0, so from the convexity of $W(T)$ we have that $0 \in W(T)$; we conclude that if $0 \notin W(T)$ then 0 is an extreme point of $\overline{W(T)}$. If $0 \notin W(T)$ and 0 does not belong to a line segment in $\partial W(T)$ then every point in $\partial W(T) \setminus \{0\}$ is an extreme point of the intersection of a ray from 0 with $\overline{W(T)}$, and so $\overline{W(T)} \setminus W(T) = \{0\}$.

COROLLARY. *For a compact operator T on a non-separable Hilbert space, $W(T)$ is closed.*

Proof. Since the range of a compact operator is separable, 0 is an eigenvalue of T, so $0 \in W(T)$.

It remains to examine the case of a compact operator T on an infinite-dimensional Hilbert space where $0 \notin W(T)$ and 0 is contained in a line segment of $\partial W(T)$. The authors wish to thank Dr. A. M. Sinclair for drawing their attention to examples of compact normal operators which exhibit the exceptional behaviour of the numerical range in such a case. Such examples can be constructed from the following characterisation of the numerical range of a compact normal operator.

THEOREM 2. *For a compact normal operator T on a Hilbert space, $W(T) = \mathrm{co}(\mathrm{p}\sigma(T))$, the convex hull of the point spectrum of T.*

Proof. Clearly $W(T) \supset \mathrm{co}(\mathrm{p}\sigma(T))$ so it is sufficient to show that $W(T) \subset \mathrm{co}(\mathrm{p}\sigma(T))$. Suppose that there exists a $\lambda \in W(T) \setminus \mathrm{co}(\mathrm{p}\sigma(T))$; then $0 \notin \mathrm{co}(\mathrm{p}\sigma(T)) - \lambda \equiv A$, say. Then for any $z \in A$, $\theta \leqslant \arg z \leqslant \theta + \pi$ for some θ, so for any $z \in \exp(-i\theta) A$, $\operatorname{Im} z \geqslant 0$. Now there exists an x, $\|x\| = 1$ such that $\lambda = (Tx, x)$. We may choose an orthonormal basis for the space such that if $x = \sum_1^\infty \alpha_n e_n$, $Tx = \sum_1^\infty \mu_n \alpha_n e_n$ where $\{\mu_n\} \subset \mathrm{p}\sigma(T)$. So $\lambda = \sum_1^\infty \mu_n |\alpha_n|^2$ where $\sum_1^\infty |\alpha_n|^2 = 1$, and $\sum_1^\infty \exp(-i\theta)(\mu_n - \lambda)|\alpha_n|^2 = 0$. If we write $\exp(-i\theta)(\mu_n - \lambda) = \gamma_n + i\delta_n$ then $\sum_1^\infty \gamma_n |\alpha_n|^2 + i \sum_1^\infty \delta_n |\alpha_n|^2 = 0$, where $\delta_n \geqslant 0$ for all n. We may assume that $\alpha_n \neq 0$ for each n, so $\delta_n = 0$ for all n. We may choose n_1 and n_2 where γ_{n_1} and γ_{n_2} have opposite signs to get $\lambda \in [\mu_{n_1}, \mu_{n_2}]$. But this contradicts the convexity of $\mathrm{co}(\mathrm{p}\sigma(T))$. Therefore, $W(T) \subset \mathrm{co}(\mathrm{p}\sigma(T))$.

Examples

1. The following compact normal operators illustrate the different sorts of exceptional behaviour of $W(T)$ for a compact operator T when $0 \notin W(T)$ and 0 is contained in a line segment of $\partial W(T)$. The operators on l^2 defined by

$$T_1(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\alpha_1, i\alpha_2, \ldots, \frac{i}{n-1}\alpha_n, \ldots\right),$$

$$T_2(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\tfrac{1}{2}\alpha_1, \alpha_2, i\alpha_3, \ldots, \frac{i}{n-2}\alpha_n, \ldots\right),$$

$$T_3(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\alpha_1, i\alpha_2, \frac{i+1}{2}\alpha_3, \ldots, \frac{i+1}{n-1}\alpha_n, \ldots\right),$$

$$T_4(\alpha_1, \alpha_2, \ldots, \alpha_n. \ldots) = \left(\alpha_1, i\alpha_2, \ldots, \frac{\alpha_{2n-1}}{2n-2}, \frac{i\alpha_{2n}}{2n-1}, \ldots\right),$$

have numerical ranges with the same closure, the triangle in the complex plane with vertices 0, 1 and i, but $\overline{W(T_1)}\setminus W(T_1) = [0, 1)$, $\overline{W(T_2)}\setminus W(T_2) = [0, \frac{1}{2})$, $\overline{W(T_3)}\setminus W(T_3) = [0, i) \cup [0, 1)$ and $W(T_4)\setminus \overline{W(T_4)} = \{0\}$.

2. The following compact operator shows that the line segments in $\overline{W(T)}\setminus W(T)$ which contain 0 need not have eigenvalues as end points. The operator T defined on l^2 by

$$T(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\alpha_1, \alpha_1+\alpha_2, \alpha_3, \tfrac{1}{2}\alpha_4, \ldots, \frac{1}{n-2}\alpha_n, \ldots\right)$$

can be regarded as the direct sum of operators T_1 on $l_2{}^2$ defined by

$$T_1(\alpha_1, \alpha_2) = (\alpha_1, \alpha_1+\alpha_2)$$

with $\sigma(T_1) = \{1\}$ and $W(T_1)$ the closed disc with centre 1 and radius $\frac{1}{2}$, and T_2 on l^2 defined by

$$T_2(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \left(\alpha_1, \tfrac{1}{2}\alpha_2, \ldots, \frac{1}{n}\alpha_n, \ldots\right)$$

with $\sigma(T_1) = \{0, 1, \frac{1}{2}, \ldots, 1/n, \ldots\}$ and $W(T_2) = (0, 1]$. As such we can compute $W(T) = \text{co}\,\{W(T_1) \cup W(T_2)\}$, [**1**, p. 113]. We have then that $\overline{W(T)}\setminus W(T)$ consists of two half-open line segments containing 0, and tangent to the disc $W(T_1)$, but $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ is contained in the real line.

Reference

1. P. R. Halmos, *Hilbert space problem book* (Van Nostrand, 1967).

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