PROPERTIES $(U\widetilde{A}_2)^*$ AND $(W\widetilde{A}_2)$ IN ORLICZ SEQUENCE SPACES AND SOME OF THEIR CONSEQUENCES

YUNAN CUI(1), HENRYK HUDZIK, AND BRAILEY SIMS(2)

Abstract. *In this paper, we introduce a new geometric property* $(U\widetilde{A}_2)^*$ *and we show that if a separable Banach space has this property, then both* X *and its dual* X^* *have the weak fixed point property. We also prove that a uniformly Gateaux differentiable Banach space has property* $(U\widetilde{A}_2)$ *and that if* X^* *has property* $(U\widetilde{A}_2)^*$, *then* X *has the* (UKK)*-property. Criteria for Orlicz spaces to have the properties* (UA_2^ε), $(UA_2^\varepsilon)^*$ *and* (NUS^*) *are given.*

Keywords and Phrases: Orlicz space, Property (A_2^ε), Fixed point property, (UKK)-property, Weak fixed point property, The weak Banach-Saks property.

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§ 1. INTRODUCTIONS

We will denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of natural and real numbers, respectively. Let X be a *Banach space* and let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively.

Given any element $x \in S(X)$ and any positive number δ, we define a w^*-slice by,

$$S^*(x,\delta) = \{x^* \in B(X^*) : x^*(x) \geq 1-\delta\}.$$

Let A be a bounded subset of X. Its Kuratowski measure of noncompactness, $\alpha(A)$, is defined as the infimum of all numbers $d > 0$ such that A may be covered by a finite family of sets with diameters smaller than d.

A Banach space X is said to be NUS^* [14] (equivalently, its dual is UKK*, [17]) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S(X)$, then $\alpha(S^*(x,\delta)) \leq \varepsilon$.

A *Banach* space X is said to have the weak Banach-Saks property whenever given any weak null sequence $\{x_n\}$ in X there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that the sequence $\{\frac{1}{k}(z_1 + z_2 + \cdots + z_k)\}$ converges strongly to zero.

A Banach space X is said to have property (A_2) if there exists a number $\Theta \in (0,2)$ such that for each weak null sequence $\{x_n\}$ in $S(X)$, there are $n_1, n_2 \in \mathcal{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$. It is well known that if X has property (A_2) then X has the weak Banach-Saks property (see [7]).

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A Banach space X is said to have property $(\widetilde{A}_2)$ if for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that for any $t \in (0, \delta)$ and each weak null sequence $\{x_n\}$ in $S(X)$, there is $k \in \mathcal{N}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ (see [14] and [15]).

Now, we introduce the notions of the $(U\widetilde{A}_2)$, $(U\widetilde{A}_2)^*$ and $(W\widetilde{A}_2)$ properties.

A Banach space X is said to have property $(U\widetilde{A}_2)$ if for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each weak null sequence $\{x_n\}$ in $S(X)$, there is $k \in \mathcal{N}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ for all $t \in (0, \delta)$.

The dual space X^* of a Banach space X is said to have property $(U\widetilde{A}_2)^*$ if for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each weak* null sequence $\{x_n^*\}$ of $S(X^*)$, there is $k \in \mathcal{N}$ satisfying $\|x_1^* + tx_k^*\| < 1 + t\varepsilon$ for all $t \in (0, \delta)$.

Notice that for reflexive Banach spaces the properties $(U\widetilde{A}_2)$ and $(U\widetilde{A}_2)^*$ coincide.

Prus (see [15]) has proved that X is NUS^* if and only if X has property $(U\widetilde{A}_2)$ and X contains no copy of l_1. He also proved that if X is NUS^*, then X has the weak Banach-Saks property (see [14] and [15]).

A natural generalization of this notion is the following property $(W\widetilde{A}_2)$ defined below.

We say a Banach space X has property $(W\widetilde{A}_2)$ whenever it satisfies the condition from the definition of property $(U\widetilde{A}_2)$ with 'for some $\varepsilon \in (0, 1)$' in place of 'for every $\varepsilon > 0$'.

Let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be nonexpansive whenever the inequality $\|Tx - Ty\| \leq \|x - y\|$ holds for every $x, y \in C$.

We will say that X has the weak fixed point property (**WFPP** for short) if every nonexpansive mapping $T : K \to K$ from a nonempty weakly compact convex subset K of X into itself has a fixed point.

R. Browder, D. Gohde, W. A. Kirk (see [9]) and other authors have established many conditions of a geometric nature on the norm of X that guarantee the **WFPP**. Uniform rotundity, uniform rotundity in every direction and normal structure are examples of such conditions.

To obtain a geometric property of a Banach space X that guarantees it has the weak fixed point property, García-Falset [7] introduced the coefficient $R(X)$ defined by the formula:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\| : \{x_n\} \subset B(X), x_n \xrightarrow{w} 0, x \in B(X)\right\}.$$

He proved in [7] that a Banach space X with $R(X) < 2$ has the weak fixed point property. This coefficient was also considered in [20].

A Banach space X with property $(W\widetilde{A}_2)$ has $R(X) < 2$ (see Note 1 below). Therefore, a Banach space X with property $(W\widetilde{A}_2)$ has the weak fixed point property.

We say that a norm $\|\cdot\|$ on X is uniformly *Frechet* differentiable (a **UF**-norm for short) if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists uniformly with respect to x and y in $S(X)$.

Let (G,Σ,μ) be a measure space with a finite and non-atomic measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G. Let l^0 stand for the space of all real sequences.

A map $\Phi:\mathcal{R}\to[0,\infty)$ is said to be an *Orlicz function* if it is even, convex, vanishes at 0, and it is not identically equal to 0.

An Orlicz function is called an *N-function* if

$$\lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty.$$

By the *Orlicz function space* L_Φ we mean the space

$$L_\Phi=\left\{x\in L^0: I_\Phi(cx)=\int_G\Phi\left(cx(t)\right)d\mu<\infty \quad for \;\; some \;\; c>0\right\}.$$

Analogously, we define the *Orlicz sequence space*

$$l_\Phi=\left\{x\in l^0: I_\Phi(cx)=\sum_{i=1}^{\infty}\Phi(cx(i))<\infty \quad for \;\; some \;\; c>0\right\}.$$

The spaces L_Φ and l_Φ are equipped with the so-called *Luxemburg norm*

$$\|x\|=\inf\{\varepsilon>0: I_\Phi(\frac{x}{\varepsilon})\le 1\}$$

or with the equivalent one

$$\|x\|_0=\inf_{k>0}\frac{1}{k}\left(1+I_\Phi(kx)\right),$$

called the *Orlicz* or the *Amemiya norm.* It is well known that if Φ is an N-function, then for any $x\neq 0$ there exists a number $k>0$ such that

$$\|x\|_0=\frac{1}{k}\left(1+I_\Phi(kx)\right).$$

(see [1]).

To simplify notations, we put $L_\Phi=\left(L_\Phi,\|\cdot\|\right)$, $l_\Phi=\left(l_\Phi,\|\cdot\|\right)$, $L_\Phi^0=\left(L_\Phi,\|\cdot\|_0\right)$ and $l_\Phi^0=(l_\Phi^0,\|\cdot\|_0)$.

For any Orlicz function Φ we define its *complementary function* $\Psi:\mathcal{R}\longrightarrow[0,\infty]$ by the formula

$$\Psi\left(v\right)=\sup_{u>0}\left\{u\left|v\right|-\Phi\left(u\right)\right\},$$

for every $v\in\mathcal{R}$. The complementary function Ψ of an Orlicz function is also a convex function vanishing at zero.

For $x \in L^0_\Phi$ (respectively l^0_Φ) we denote by $k(x)$ the set of those $k > 0$ such that $\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx))$. It is known (see [1], [2] and [19]) that $k(x) = [k*(x), k**(x)]$, whenever $k**(x) < \infty$, where,

$$k*(x) = \inf\{\lambda > 0 : I_\Psi(p(\lambda|x|)) \geq 1\}, \quad k{*}{*}(x) = \sup\{\lambda > 0 : I_\Psi(p(\lambda|x|)) \leq 1\}$$

and Psi is the function complementary to Phi. In the case when $k**(x) = \infty$ and $k*(x) < \infty$, we have $k(x) = [k*(x), k**(x))$. When $k*(x) = \infty)$,

$$\|x\|_0 \;=\; \lim_{k\to\infty} \frac{1}{k}(1 + I_\Phi(kx)) \;=\; \lim_{k\to\infty} \frac{1}{k} I_\Phi(kx).$$

We say an Orlicz function Φ *satisfies the* Δ_2*-condition* (δ_2*-condition*) if there exist constants $k \geq 2$ and $u_0 > 0$ such that $\Phi(u_0) < \infty$ (respectively, $\Phi(u_0) > 0$) and

$$\Phi(2u) \leq k\Phi(u),$$

for every $|u| \geq u_0$ (respectively, for every $|u| \leq u_0$), (see [1], [11], [12], [14] and [16]).

We say an Orlicz function Φ *satisfies the* ∇_2-condition (respectively, $\overline{\delta}_2$-condition) if its complementary function Ψ satisfies the Δ_2-condition (respectively, δ_2-condition).

An Orlicz function Φ is said to be *uniformly convex* in $[0, u_0]$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{\Phi(u) + \Phi(v)}{2}$$

for all $u, v \in [0, u_0]$ satisfying $|u - v| \geq \epsilon \max\{u, v\}$.

We say an Orlicz function Φ is *strictly convex* in $\mathbb{R}$ if for any $u, v \in \mathbb{R}$, $u \neq v$, and $\alpha \in (0,1)$ we have

$$\Phi(\alpha u + (1-\alpha)v) < \alpha\Phi(u) + (1-\alpha)\Phi(v).$$

For more details on Orlicz functions and Orlicz spaces we refer to [1], [11], [12], [14] and [18].

§2. GENERAL RESULTS

We begin with the following observation. **Note 1.** *Property* $(W\widetilde{A}_2)$ *of a Banach space* X *implies that* $R(X) < 2$.

Proof. Take any weak null sequence $\{x_n\}$ in $S(X)$ and $x \in S(X)$. Then we have that the sequence $\{x, x_1, x_2, \ldots\} \subset S(X)$ is weakly null. So, by property $(W\widetilde{A}_2)$, for some $\varepsilon > 0$ and δ which we may take to be in $(0,1)$) we can find a k_1 such that $\|x + \delta x_{k_1}\| \leq 1 + \delta\varepsilon$. Consider next the weak null sequence $\{x, x_{k_1+1}, x_{k_1+2}, \ldots\}$. There is a $k_2 > k_1$ such that $\|x + \delta x_{k_2}\| \leq 1 + \delta\varepsilon$. In this way we can inductively construct a sequence

$$k_1 < k_2 < \ldots < k_l < \ldots$$

of natural numbers such that $\|x + \delta x_{k_l}\| \le 1 + \delta\varepsilon$ for all $l \in N$. Therefore, $\|x + x_{k_l}\| = \|x + \delta x_{k_l} + (1-\delta)x_{k_l}\| \le 1 + \delta\varepsilon + (1-\delta) = \eta(\varepsilon) \in (1,2)$. Since $\eta(\varepsilon)$ is independent of $x \in S(X)$ and independent of the weakly convergent sequence $\{x_n\}$ in $S(X)$, the proof is complete.

Theorem 1. If $\|\cdot\|$ is a **UF**-norm in a Banach space X, then X has property $(U\widetilde{A}_2)$.

Proof. Since $\|\cdot\|$ is a **UF**-norm in X, it follows that X is Gateaux differentiable; that is, X is smooth. Let $f_x \in S(X^*)$ denote the unique supporting functional at $x \in S(X)$. It is known that the norm $\|\cdot\|$ is uniformly Fréchet differentiable on the space X if and only if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} = f_x(y)$$

exists uniformly with respect to $x, y \in S(X)$.

Now, for any $\varepsilon > 0$ and each weak null sequence $\{x_n\}$ in $S(X)$, there exists $n_0 \in \mathcal{N}$ such that

$$|f_x(x_n)| < \frac{\varepsilon}{2}$$

for all $n \ge n_0$. Since the norm $\|\cdot\|$ is (by assumption) **UF** on X, there exists a $\delta > 0$ such that

$$\left| \frac{\|x + tx_{n_0}\| - \|x\|}{t} - f_x(x_{n_0}) \right| < \frac{\varepsilon}{2}$$

whenever $|t| < \delta$, whence

$$\|x + tx_{n_0}\| - \|x\| < \frac{t\varepsilon}{2} + |f_x(x_{n_0})|\, t < t\varepsilon$$

uniformly with respect to $x \in S(X)$. This means that X has property $(U\widetilde{A}_2)$, as required.

Theorem 2. Suppose that a Banach space X has property $(W\widetilde{A}_2)$. Then X has the weak Banach-Saks property and the weak fixed point property.

Proof. Since X has the property $(W\widetilde{A}_2)$, there exist $\varepsilon \in (0,1)$ and $\delta > 0$ such that for any $t \in [0,\delta]$ and any weak null sequence $\{x_n\}$ in $B(X)$ there exists $k \in N, k > 1$, such that $\|x_1 + tx_k\| < 1 + \varepsilon\delta$. Hence

$$\|x_1 + x_k\| = \|x_1 + \delta x_k + (1-\delta)x_k\|$$

$$\le \|x_1 + \delta x_k\| + (1-\delta) \le 1 + \varepsilon\delta + 1 - \delta = 2 - \delta(1-\varepsilon),$$

which means that a Banach space with property $(W\widetilde{A}_2)$ has property (A_2). Consequently, a Banach space with property $(W\widetilde{A}_2)$ has the weak Banach-Saks property.

Moreover, we have by the above estimate that $R(X) \le 2 - \delta(1-\varepsilon) < 2$, so X enjoys the weak fixed point property (see [7]).

Let us recall that for a Banach space X with basis $\{x_i\}$, the basis constant of the space is the number $M = \sup_n \|P_n\|$, where P_n are the projections defined by $P_n(x) = \sum_{i=1}^{n} a_i x_i$, where $x = \sum_{i=1}^{\infty} a_i x_i$.

Theorem 3. Let X be a separable Banach space. If its dual space X^* has property $(U\widetilde{A}_2)^*$, then X has the (UKK)-property.

Proof. Let $\{x_n\}$ be a sequence in $S(X)$ with $sep(\{x_n\}) := \inf_{m \neq n} \|x_m - x_n\| > \varepsilon$ and $x_n \xrightarrow{w} x \in B(X)$. Deleting at most one element of the sequence, we can assume that $sep(\{x_n - x\}) > \varepsilon$. For any $\varepsilon_1 > 0$ let $M = 1 + \varepsilon_1$. By the Bessaga-*Pelczynski* selection principle, there exists a subsequence $\{z_n\}$ of the sequence $\{x_n - x, x\}$ with $z_1 = x$, being a basic sequence with the basis constant less than or equal to M (see [5], p. 46).

Let us consider the sequence $\{z_n^*\}$ of the *Hahn-Banach* extensions of the coefficient functionals of the basic sequence $\{\frac{z_n}{\|z_n\|}\}$ and put $X_0 = \overline{span}\{z_n : n = 1, 2, ...\}$. Then we can prove that $\langle z_n^*, z\rangle \to 0$ for any $z \in X_0$ as $n \to \infty$. Namely, for any $z \in X_0$ we have $z = \sum_{i=1}^{\infty} z_i^*(z) z_i$, whence

$$|\langle z_n^*, z\rangle| = \|z_n^*(z) z_n\| = \left\| \sum_{i=n}^{\infty} z_i^*(z) z_i - \sum_{i=n+1}^{\infty} z_i^*(z) z_i \right\|$$

$$\leq \left\| \sum_{i=n}^{\infty} z_i^*(z) z_i \right\| + \left\| \sum_{i=n+1}^{\infty} z_i^*(z) z_i \right\| \to 0.$$

Since X is separable, we can assume that $z_n^* \xrightarrow{w^*} z^*$ as $n \to \infty$.

Let us now take any $\varepsilon_2 \in (0, 1)$. Since X^* has property $(WU\widetilde{A}_2)^*$, there exists $0 < \delta_2 \leq 1$ and $k \in N$, $k > 1$, such that for any $t \in (0, \delta_2)$

$$\left\| \frac{z_1^*}{\|z_1^*\|} + t \frac{(z_k^* - z^*)}{\|z_k^* - z^*\|} \right\| < 1 + t\varepsilon_2. \tag{1}$$

It is easy to see that:

(2) For all $k \in \mathbb{N}$, $\langle z^*, z_k\rangle = 0$ and $\langle z_k^*, z_k\rangle = \|z_k\|$. In particular $\langle z^*, x\rangle = 0$,

(3) For all $k \geq 2$, $\|x + z_k\| = 1$ and $\langle z_k^*, x\rangle = 0$,

(4) For all $k \in \mathbb{N}$, $\|z_k^* - z^*\| \leq 4M$ and $\|z_1^*\| \leq M$.

Since $sep(\{x_n\}) > \varepsilon$ we can assume that $\|z_n\| \geq \frac{\varepsilon}{2}$ for $n \geq 2$. Let $k > 1$ be a natural number for which (1) holds for all $t \in (0, \delta_2)$. Then by conditions (2)-(4) and the fact that $z_1 = x$, we obtain

$$\|x\| = \langle z_1^*, x\rangle = \|z_1^*\| \langle \frac{z_1^*}{\|z_1^*\|}, x\rangle = \|z_1^*\| [\langle \frac{z_1^*}{\|z_1^*\|}, x + z_k\rangle]$$

$$= \|z_1^*\|[\langle \frac{z_1^*}{\|z_1^*\|}, x + z_k\rangle + t\langle \frac{z_k^* - z*}{\|z_k^* - z*\|}, x + z_k\rangle - t\langle \frac{z_k^* - z*}{\|z_k^* - z*\|}, x + z_k\rangle]$$

$$= \|z_1^*\|[\langle \frac{z_1^*}{\|z_1^*\|} + t\frac{z_k^* - z*}{\|z_k^* - z*\|}, x + z_k\rangle - \frac{t\|z_k\|}{\|z_k^* - z^*\|}]$$

$$\leq \|z_1^*\|[\|\frac{z_1^*}{\|z_1^*\|} + t\frac{z_k^* - z*}{\|z_k^* - z*\|}\| - \frac{t\|z_k\|}{\|z_k^* - z^*\|}]$$

$$\leq M[(1 + t\varepsilon_2) - \frac{t\varepsilon}{2\|z_k^* - z^*\|}] \leq M[(1 + t\varepsilon_2) - \frac{t\varepsilon}{8M}].$$

So, we have $\|x\| \leq M(1 + t\varepsilon_2 - \frac{t\varepsilon}{8M})$. Using $M = 1 + \varepsilon_1$, and taking the limit as $\varepsilon_1 \to 0$, we obtain

$$\|x\| \leq 1 + t(\varepsilon_2 - \frac{\varepsilon}{8}).$$

Now taking $\varepsilon_2 = \frac{\varepsilon}{16}$, and $t = \frac{\delta_2}{2}$, we get

$$\|x\| \leq 1 - \frac{\delta_2 \varepsilon}{32},$$

completing the proof.

Remark 1. It is worth noticing that separability of X in the last theorem is only necessary to ensure that $w*$- compact subsets of X are $w*$-sequentially compact. We can relax the assumption of separability of X, requiring for example that X admits an equivalent smooth norm (see [10]).

The next result follows directly from our Theorems 2 and 3.

Corollary 1. Let X be a separable Banach space. If its dual space X^* has property $(U\widetilde{A}_2)^*$, then both X and X^* have the weak fixed point property.

§ 3. THE CASE OF ORLICZ SPACES

Corollary 2. Let X be the Orlicz space L_M or L_M^0. Then the following statements are equivalent:

(1) X is uniformly smooth;

(2) X is nearly uniformly smooth;

(3) X is (NUS^*);

(4) X has property $(U\widetilde{A}_2)$;

(5) $\Psi \in \Delta_2$, Ψ is strictly convex on the whole real line and Φ is uniformly convex outside a neighborhood of zero.

Proof. This follows from our Theorem 3 and Theorem 3.15 in [1].

Lemma 1. Suppose $\Phi \in \delta_2$. Then for any $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that,

$$I_\Phi(x + ty) - I_\Phi(x) < t\varepsilon,$$

whenever $I_\Phi(x) \le L$, $I_\Phi(y) \le \delta$ and $t \in (0, 1)$.

Proof. Since $\Phi \in \delta_2$, for any $\varepsilon > 0$ and $L > 0$ there exists $\delta \in (0, 1)$ such that,

$$I_\Phi(x + y) - I_\Phi(x) < \varepsilon$$

whenever $I_\Phi(x) \le L$ and $I_\Phi(y) \le \delta$ (see [4]). So for any $t \in (0, \delta)$, we have,

$$\begin{aligned} I_\Phi(x + ty) &= I_\Phi(tx + ty + (1 - t)x) \\ &\le tI_\Phi(x + y) + (1 - t)I_\Phi(x) \\ &\le t(I_\Phi(x) + \varepsilon) + (1 - t)I_\Phi(x) = I_\Phi(x) + t\varepsilon, \end{aligned}$$

whenever $I_\Phi(x) \le L$ and $I_\Phi(y) \le \delta$.

Lemma 2. Suppose $\Phi \in \overline{\delta}_2$. Then for any $\varepsilon > 0$ and $u_0 > 0$ there exists $\delta > 0$ such that

$$\Phi(tu) \le t\varepsilon\Phi(u),$$

whenever $|u| \le u_0$ and $t \in (0, \delta)$.

Proof. Suppose that $\Phi \in \overline{\delta}_2$. Then for any $u_0 > 0$ there exists $\theta \in (0, 1)$ such that

$$\Phi\left(\frac{u}{2}\right) \le \frac{\theta}{2}\Phi(u)$$

whenever $|u| \le u_0$ (see [1] and [16]). Take $n \in \mathcal{N}$ such that $\theta^n \le \varepsilon$. Then for $\delta = \frac{1}{2^n}$, we have

$$\Phi(\delta u) = \Phi\left(\frac{u}{2^n}\right) \le \left(\frac{\theta}{2}\right)^n \Phi(u) \le \delta\varepsilon\Phi(u),$$

whenever $|u| \le u_0$.

Hence, for any $t \in (0, \delta)$, we have

$$\Phi(tu) = \Phi\left(\frac{t}{\delta}\delta u\right) \le \frac{t}{\delta}\delta\varepsilon\Phi(u) = t\varepsilon\Phi(u),$$

whenever $|u| \le u_0$, which finishes the proof.

From here on we will make use of the following parameter for an Orlicz function Φ:

$$m(\Phi) = \sup\left\{n \in \mathbb{N}\colon \sum_{i=1}^{n} \Psi(A) < 1\right\},$$

where $A := \lim\limits_{u\to\infty} (\Phi(u)/u)$ and Ψ is the function complementary to Φ in the sense of Young.

For any $x \in l_\Phi^0$, put $N(x) = \{i \in N\colon\ x(i) \ne 0\}$ and define $D(l_\Phi^0) = \{x = (x(i)) \in B(l_\Phi^0) : N(x) \text{ is finite }\}$.

Lemma 4. Let Φ be an Orlicz function with $\Phi \in \delta_2$, $m(\Phi) \leq 1$ and $\Phi \in \overline{\delta}_2$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every weak null sequence $\{x_n\}$ in $B(l_\Phi^0)$ and every $x \in D(l_\Phi^0)$ there is a natural number $k > 1$ such that

$$\|x + tx_k\|^0 \leq 1 + t\varepsilon,$$

whenever $t \in (0, \delta)$.

Proof. Case I. Assume that $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = +\infty$. Let $\varepsilon > 0$ be given. By $\Phi \in \overline{\delta}_2$, the set $Q = \{k_x : \frac{1}{2} \leq \|x\|_0 \leq 1 \text{ and } \|x\|^0 = \frac{1}{k_x}(1 + I_\Phi(k_x x))\}$ is bounded; that is, there exists $\mathbf{k} > 1$ such that $1 \leq k_x \leq \mathbf{k}$ whenever $\frac{1}{2} \leq \|x\|_0 \leq 1$ (see [1]). By Lemma 2, we know that there exists $\delta \in (0, 1)$ such that

$$\Phi(tu) \leq t\delta\Phi(u)$$

whenever $t \in (0, \delta)$ and $|u| \leq \Phi^{-1}(\mathbf{k})$. By Lemma 1, there exists $\theta > 0$ such that

$$|I_\Phi(x + ty) - I_\Phi(x)| < t\varepsilon,$$

whenever $I_\Phi(x) \leq L$, $I_\Phi(y) \leq \theta$ and $t \in (0, 1)$.

Fix $t \in (0, \frac{\delta}{\mathbf{k}})$ and let $\{x_n\}$ be an arbitrary weak null sequence in $S(l_\Phi^0)$. For any $x \in D(l_\Phi^0)$, take $i_0 \in \mathcal{N}$ such that $x(i) = 0$ for $i > i_0$. Since $x_n \xrightarrow{w} 0$, we conclude that $x_n \to 0$ coordinatewise, and so there exists $n_0 \in \mathcal{N}$ such that $\sum\limits_{i=1}^{i_0} \Phi(x_n(i)) < \theta$ for all $n \geq n_0$. Hence, we get for $l \geq 1$ satisfying $\|x\| = \frac{1}{l}(1 + I_\Phi(lx))$ that:

$$\|x + tx_n\|^0 \leq \frac{1}{l}\left[1 + I_\Phi\left(l(x + tx_n)\right)\right]$$

$$= \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(l(x(i) + tx_n(i))\right) + \sum_{i=i_0+1}^{\infty} \Phi\left(ltx_n(i)\right)\right]$$

$$\leq \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx(i)\right) + t\varepsilon + \sum_{i=i_0+1}^{\infty} \Phi\left(ltx_n(i)\right)\right]$$

$$\leq \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx(i)\right) + t\varepsilon + lt\varepsilon \sum_{i=i_0+1}^{\infty} \Phi\left(x_n(i)\right)\right]$$

$$\leq \frac{1}{l}\left[1 + \sum_{i=1}^{i_0} \Phi\left(lx(i)\right)\right] + 2t\varepsilon \leq 1 + 2t\varepsilon. \qquad \square$$

Case II.

Assume that $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = A < \infty$. Let $\{x_n\}$ be a weak null sequence in $S(\ell_\Phi^0)$ and x be in $D(\ell_\Phi^0)$. Put,

$$y_m = \left(\frac{1}{A}, \frac{1}{m}, 0, 0, ...\right),$$

where $m := m(\Phi)$. Since $x_n \xrightarrow{w} 0$, we may assume without loss of generality that $x_n(i) = 0$ for $i = 1, 2$ (because weak convergence to zero in ℓ_Φ^0 implies coordinatewise convergence to zero). By the condition $m(\Phi) \leq 1$, we know that there exists $k_m > 0$ such that

$$\|y_m\|^0 = \frac{1}{k_n}\left(1 + I_\Phi(k_m y_m)\right) \quad \forall\, m \in \mathbb{N}.$$

It is clear that the sequence $\{k_m\}$ is bounded. Hence, by virtue of Lemma 2,

$$\begin{aligned}
\|x + tx_n\|^0 &\leq \|y_m + tx_n\|^0 \\
&\leq \frac{1}{k_n}\left(1 + \sum_{i=1}^{\infty} \Phi\left(k_m\left(y_m(i) + tx_n(i)\right)\right)\right) \\
&= \frac{1}{k_n}\left(1 + \sum_{i=1}^{2} \Phi\left(k_m y_m(i)\right) + \sum_{i=3}^{\infty} \Phi\left(k_m t x_n(i)\right)\right) \\
&\leq \|y_m\|^0 + t\varepsilon \sum_{i=3}^{\infty} \Phi\left(x_n(i)\right) \\
&\leq \|y_m\|^0 + t\varepsilon
\end{aligned}$$

Passing to the limit as m tends to ∞, we obtain that

$$\|x + tx_n\|^0 \leq 1 + t\varepsilon,$$

as required.

Theorem 4. Let Φ be an N-function and $X = l_\Phi^0$ fail the Schur property. Then the following statements are equivalent:

(1) X has property $(U\widetilde{A}_2)$;

(2) X has property $(W\widetilde{A}_2)$;

(3) $R(X) < 2$;

(4) $\Phi \in \delta_2$, $m(\Phi) \leq 1$ and $\Phi \in \overline{\delta}_2$.

Proof. That (1) implies (2) is clear and by Note 1, (2) implies (3.

To see that (3) implies (4), suppose that $\Phi \notin \delta_2$, then for any $\varepsilon > 0$ there exists $x \in S(l_\Phi^0)$ such that

$$1 - \varepsilon \leq \left\|\sum_{i=n}^{\infty} x(i)e_i\right\|^0 \leq 1$$

for all $n \in \mathcal{N}$. Take a sequence $\{n_i\}$ in $\mathcal{N}$ with $n_1 < n_2 < \cdots$ such that

$$\left\|\sum_{j=n_i+1}^{n_{i+1}} x(j)e_j\right\|^0 \geq 1 - 2\varepsilon \quad \text{for all} \quad i \in \mathcal{N}.$$

Put $x_i = \sum\limits_{j=n_i+1}^{n_{i+1}} x(j)e_j$. Since Φ is an N-function,

$$\lim_{\lambda\to 0}\left(\sup_{i\in N}\frac{I_\Phi(\lambda x_i)}{\lambda}\right) \leq \lim_{\lambda\to 0}\frac{I_\Phi(\lambda x)}{\lambda} = 0,$$

so we have that $x_i \overset{l_\Psi}{\to} 0$. Notice that every singular functional vanishes on any x_i. In consequence $x_i \overset{w}{\to} 0$.

But $\liminf_{i\to\infty} \|x_i + x\|^0 \geq \liminf_{i\to\infty} 2\|x_i\|^0 \geq 2(1-2\varepsilon)$. By the arbitrariness of $\varepsilon > 0$, we get $R(l^0_\Phi) = 2$. Thus, we have proved that if $\Phi \notin \delta_2$, then (3) does not hold.

Now we need to prove the necessity of the condition $m(\Phi) \leq 1$ for $R(X) < 2$. Let us assume that $m(\Phi) \geq 2$ and for each $n \in \mathbb{N}$ define

$$x_n = \left(0, ..., 0, \frac{1}{A}, 0, ...\right),$$

where $\frac{1}{A}$ is in the n'th place and $A := \lim_{u\to\infty} \frac{\Phi(u)}{u}$. Then $\|x_n\|^0 = 1$, because $m(\Phi) \leq 2$ yields $k^*(x_n) = \infty$, and so from our earlier discussion $\|x_n\|^0 = \lim_{k\to\infty} (I_\Phi(kx_n)/k)$. Since ℓ^0_Φ fails the Schur property, we have the equality $\lim_{u\to\infty} (\Phi(u)/u) = 0$. Consequently,

$$\lim_{\lambda\to 0}\left(\sup_n \frac{I_\Phi(\lambda x_n)}{\lambda}\right) = \lim_{\lambda\to 0} \frac{\Phi\left(\frac{\lambda}{A}\right)}{\lambda} = 0.$$

Therefore, by virtue of lemma 2.3 in [3](also see, Theorem 1.69 in [1]) and $\Phi \in \delta_2$, we conclude that $\{x_n\}$ is a weak null sequence (also see the proof of Theorem 2.3 in [6]). Moreover,

$$\|x_n + x_1\|^0 = 2A \cdot \frac{1}{A} = 2,$$

so $R(\ell^0_\Phi) = 2$, which establishes the necessity of the condition $m(\Phi) \leq 1$ for $R(\ell^0_\Phi) < 2$.

Suppose that $\Phi \notin \overline{\delta}_2$. Then the Kottman constant $K(l^0_\Phi) = \sup\{d_x : x \in S(l^0_\Phi)\} = 2$ (see [1] and [18]). Hence for any $\varepsilon > 0$ there exists $x \in S(l^0_\Phi)$ such that $d_x > 2 - \varepsilon$. Furthermore, we have $d_{x,k} \geq d_x > 2 - \varepsilon$ for all $k > 1$.

Put,

$$\begin{aligned}
x_1 &= (x(1), 0, x(2), 0, x(3), 0, x(4), 0, x(5), 0, x(6), 0, ...),\\
x_2 &= (0, x(1), 0, 0, 0, x(2), 0, 0, 0, 0, 0, 0, 0, x(3), 0, 0, ...),\\
x_3 &= (0, 0, 0, x(1), 0, 0, 0, 0, 0, 0, 0, 0, x(2), 0, 0, 0, 0, ...), ...,\\
&\cdots,
\end{aligned}$$

so the supports of the x_n are pairwise disjoint and for any $n \in N$ the non-zero coordinates of x_n are precisely the coordinates of x.

Then, $\|x_n\|^0 = 1$, for any $n \in \mathcal{N}$, $x_n \overset{w}{\to} 0$ and for any $k > 1$ we have

$$\begin{aligned}
\frac{1}{k}\left(1 + I_\Phi\left(\frac{k(x_n + x_1)}{d_x}\right)\right) &\geq \frac{1}{k}\left(1 + I_\Phi\left(\frac{k(x_n + x_1)}{d_{x,k}}\right)\right)\\
&= \frac{1}{k}\left(1 + I_\Phi\left(\frac{kx}{d_{x,k}}\right) + I_\Phi\left(\frac{kx}{d_{x,k}}\right)\right) = \frac{1}{k}(1 + \frac{k-1}{2} + \frac{k-1}{2}) = 1.
\end{aligned}$$

So, we get $\left\|\frac{x_n+x_1}{d_x}\right\|^0 \geq 1$; that is, $\liminf_{n\to\infty} \|x_n + x_1\|^0 \geq d_x - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get $R(l_\Phi^0) = 2$. Therefore, we have proved that $\Phi \notin \overline{\delta}_2$ implies that (3) does not hold.

(4) $\Rightarrow$ (1). By Lemma 4, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every weak null sequence $\{x_n\}$ in $B(l_\Phi^0)$ and any $x \in D(l_\Phi^0)$, there exists a number $m > 1$ such that

$$\|x + tx_m\|^0 \leq 1 + \frac{t\varepsilon}{2},$$

whenever $t \in (0, \delta)$.

Let $t \in (0, \delta)$ be given arbitrarily. For any weak null sequence $\{x_n\}$ in $B(l_\Phi^0)$, we only need to consider the case when $N(x_1)$ is infinite. Take i_0 large enough so that $\left\|\sum_{i=i_0+1}^{\infty} x_1(i)e_i\right\|^0 \leq \frac{t\varepsilon}{2}$. Then there exists $m \in \mathcal{N}$ such that

$$\left\|\sum_{i=1}^{i_0} x_1(i)e_i + tx_m\right\|^0 \leq 1 + \frac{t\varepsilon}{2}.$$

Hence,

$$\|x_1 + tx_m\|^0 \leq \left\|\sum_{i=1}^{i_0} x_1(i)e_i + tx_m\right\|^0 + \frac{t\varepsilon}{2} \leq 1 + \frac{t\varepsilon}{2} + \frac{t\varepsilon}{2} = 1 + t\varepsilon.$$

Corollary 3. Let Φ be any Orlicz function and $X = l_\Phi^0$. Then the following statements are equivalent:

(1) X is (NUS^*);

(2) X is nearly uniformly smooth;

(3) $\Phi \in \delta_2$, $\Phi \in \overline{\delta}_2$ and $m(\Phi) \leq 1$.

Proof. (3) $\Rightarrow$ (1). If $\Phi \in \delta_2$, $\Phi \in \overline{\delta}_2$ and $m(\Phi) \leq 1$, by Theorem 4, ℓ_Φ^0 has property $(U\widetilde{A}_2)$. Moreover, ℓ_Φ^0 is then B-convex (see [1]), so ℓ_Φ^0 contains no copy of ℓ_1. Since a Banach space X has (NUS^*) if and only if has property $(U\widetilde{A}_2)$ and contains no copy of ℓ_1 (see [15]), condition (3) implies condition (1).

Again by our Theorem 4 and the result from [15] that we just mentioned, we have that (1) $\Rightarrow$ (2), because condition (1) implies reflexivity of ℓ_Φ^0 and we therefore also have (2) $\Rightarrow$ (3).

The following theorem can be proved in a similar way as for $X = \ell_\Phi^0$, so we omit its proof.

Theorem 5. For any Orlicz function Φ and $X = \ell_\Phi$ the following statements are equivalent:

(1) X has property $(U\widetilde{A}_2)$;

(2) X has property $(W\widetilde{A}_2)$;

(3) $R(X) < 2$;

(4) $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

Corollary 4. Let Φ and X be as in Theorem 5. The following statements are equivalent:

(1) X is nearly uniformly smooth;

(2) X is (NUS^*);

(3) $\Phi \in \delta_2$ and $\Phi \in \overline{\delta}_2$.

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Yunan Cui: Department of Mathematics,, Harbin University of Science and Technology, HARBIN,150080, P.R. China

E-mail address: cuiya@hrbust.edu.cn

Henryk Hudzik: Faculty of Mathematics and Computer Science, Adam Mickiewicz Uniwersity, Poznań, Poland

E-mail address: hudzik@amu.edu.pl

Brailey Sims: School of Mathematical and Physical Sciences, The University of Newcastle, NSW 2308, Australia

E-mail address: brailey.sims@newcastle.edu.au