SOME REMARKS CONCERNING D – METRIC SPACES

Zead Mustafa and Brailey Sims

November 2003

Abstract

In this note we make some remarks concerning D–metric spaces, and present some examples which show that many of the basic claims concerning the topological structure of such spaces are incorrect, thus nullifying many of the results claimed for these spaces.

1 Introduction

In 1992 B. C. Dhage [1] proposed the notion of a D–metric space in an attempt to obtain analogous results to those for metric spaces, but in a more general setting. In a subsequent series of papers (including: [2], [3], [4], and [5]) Dhage presented topological structures in such spaces together with several fixed point results. These works have been the basis for a substantial number of results by other authors. Unfortunately, as we will show, most of the claims concerning the fundamental topological properties of D–metric spaces are incorrect, nullifying the validity of many results obtained in these spaces.

We begin by recalling the axioms of a D–metric space.

Definition: let X be a nonempty set, and let $\mathbf{R}$ denote the real numbers. A function $D : X \times X \times X \to R$ satisfying the following axioms:

(D1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$,

(D2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(D3) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry in all three variables),

(D4) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$. (rectangle inequality),

is called a *generalized metric*, or a *D–metric* on X. The set X together with such a generalized metric, D, is called a *generalized metric space*, or *D–metric space*, and denoted by (X, D).

An additional property sometimes imposed on a D–metric (see [3]) is,

(D5) $D(x, y, y) \leq D(x, z, z) + D(z, y, y)$ for all $x, y, z \in X$

If $D(x, x, y) = D(x, y, y)$ for all $x, y \in X$ then D is referred to as a *symmetric D–metric*.

The following constructions show that intuitively $D(x, y, z)$ may be thought of as providing some measure of the perimeter of the triangle with vertices at x, y and z.

If $\rho : X \times X \to R$ is any semi-metric on X (that is, ρ is a positive, symmetric function with $\rho(x, y) = 0$ if and only if $x = y$) then it is easily verified that D defined by either,

(M1) $D(x, y, z) := \rho(x, y) + \rho(x, z) + \rho(y, z)$, or

(M2) $D(x, y, z) := \max\{\rho(x, y),\ \rho(x, z),\ \rho(y, z)\}$,

is a symmetric D–metric on X, but need not satisfy (D5). In particular every metric space can be equipped with a D–metric in either of these two ways and in this case (D5) is also satisfied. It should, however, be noted that the rectangle inequality (D4) satisfied by D–metrics arising in this way does not depend on the presence of a triangle inequality for the underlying metric, and is rarely sharp.

In a D–metric space (X, D), three possible notions for the convergence of a sequence (x_n) to a point x suggest themselves:

(C1) $x_n \to x$ if $D(x_n, x, x) \to 0$ as $n \to \infty$,

(C2) $x_n \to x$ if $D(x_n, x_n, x) \to 0$ as $n \to \infty$,

(C3) $x_n \to x$ if $D(x_m, x_n, x) \to 0$ as $m, n \to \infty$,

Clearly, $(C3) \Rightarrow (C2)$ and if D is symmetric then $(C1) \Leftrightarrow (C2)$. Further, if D arrises from a metric according to either (M1) or (M2) then it is easily checked that all three types of convergence are equivalent and correspond to convergence with respect to the underlying metric.

No other implications are true in general. The following example shows that (C1) and (C2) need not imply (C3), even for D–metrics arising from semi-metrics.

Example 1: Let $A = \{1/n : n = 1, 2, 3, \cdots\}$ and $X = A \cup \{0\}$ and let ρ be the semi-metric on X defined by:

$$\begin{array}{ll} \rho(x,x) := 0, & \text{for all } x \in X, \\ \rho(0,1/n) := \rho(1/n,0) = 1/n, & \text{for } n = 2,3,\cdots, \\ \rho(x,y) := 1, & \text{for } x,y \in A \text{ with } x \neq y. \end{array}$$

Then, for $D(x, y, z) := \max\{\rho(x, y),\ \rho(x, z),\ \rho(y, z)\}$ we have

$$D(1/n, 1/n, 0) = 1/n \to 0, \text{ so } 1/n \overset{(C1),(C2)}{\longrightarrow} 0,$$

but the sequence $(1/n)$ does not converge in the sense of (C3), as $D(x_n, x_m, 0) = 1$ for all n, m with $n \neq m$.

The next example demonstrates that (C2) convergence of a sequence need not imply its convergence in the sense of (C1) or (C3).

Example 2: For X as above, define D by,

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ 1/n, & \text{if one of } x,y,z \\ & \text{is equal to 0 and the other two are equal to 1/n,} \\ 1, & \text{otherwise,} \end{cases}$$

then it is readily seen that D is a generalized metric which also satisfies (D5). Further, $1/n \stackrel{(C2)}{\longrightarrow} 0$, but the sequence $(1/n)$ does not converge in the sense of (C1) or (C3), as $D(1/n,0,0) = 1 = D(1/n,1/m,0)$ for all n,m, $(m \neq n)$. This also shows that a generalized metric need not be a continuous function of its variables with respect to convergence in the sense of (C2), as $D(1/n,0,0) \not\to D(0,0,0) = 0$.

The next example demonstrates that (C1) convergence of a sequence need not imply its convergence in the sense of (C2) or (C3).

Example 3: For X as above, define D by,

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ 1/n, & \text{if two of } x,y,z \\ & \text{are equal to 0 and the other} \\ & \text{is equal to } 1/n, \\ 1, & \text{otherwise.} \end{cases}$$

Then, it is readily seen that D is a generalized metric satisfying (D5). Further, $1/n \stackrel{(C1)}{\longrightarrow} 0$, but the sequence $(1/n)$ does not converge in the sense of (C2) or (C3) , as $D(1/n,1/m,0) = 1$ for all n,m. This also shows that a generalized metric need not be a continuous function of its variables with respect to convergence in the sense of (C1), as $D(1/n,1/n,0) \not\to D(0,0,0) = 0$.

Our next example shows that even a symmetric D–metric arising from a semi-metric, need not be a continuous function of its variables with respect to convergence in the sense of (C3), contrary to the claim in [1] lemma 2.1.

Example 4: For X again as in example 1, but with semi-metric ρ defined by:

$$\begin{aligned} &\rho(0,1) := \rho(1,0) = 1, && \\ &\rho(1,1/n) := \rho(1/n,1) = 1/2, && \text{for } n = 2,3,\cdots, \\ &\rho(1,1) := 0, && \\ &\rho(x,y) := |x-y|, && \text{for } x,y \in X\backslash\{1\}. \end{aligned}$$

and $D(x,y,z) := \rho(x,y) + \rho(x,z) + \rho(y,z)$ we have that the sequence $(1/n)$ converges to 0 in each of the senses (C1), (C2) and (C3), but $D(1/n,1,1) = 1 \not\to D(0,1,1) = 2$, so D is not continuous with respect to convergence in any of these ways.

In [1], Dhage defined Cauchy sequences in a D–metric space as follows.

Definition: A sequence (x_n) of points in a D–metric space is said to be a *D–Cauchy sequence* if for all $\epsilon > 0$, there exists an $n_0 \in N$ such that for all $m, n, p \geq n_0$, $D(x_n, x_m, x_p) < \epsilon$.

In [1], Dhage mentioned the possibility of defining two topologies, denoted by τ^* and τ, in any D–metric space, with convergence in the sense of (C3) corresponding to convergence in the τ topology. Details were presented in two subsequent papers; [2] and [3].

The τ^* topology is generated by the family of 'open balls' of the form

$$\text{(B1)} \qquad B^*(x, r) := \{y \in X \,:\, D(x, y, y) < r\},$$

where $x \in X$ and $r > 0$. To ensure that these form a base for the topology it was necessary to assume that the D–metric satisfied the additional condition (D5), see [3]. However, with the exception of [4], [5] and [2], no subsequent paper dealing with D-metric spaces acknowledges the need for this extra axiom when considering (C2) convergence, or its associated topology.

Clearly, convergence of a sequence in the τ^* topology is equivalent to its (C2) convergence. But, in [3], where the τ^* topology was discussed, D-metric convergence of a sequence is taken to mean that it converges in both the sense of (C2) and (C1), and it is claimed that "the D–metric topology (here the τ^*–topology) is the same as the topology of D–metric convergence of sequences in X" in this sense of (C1) and (C2) convergence. But, this claim is not true, as we have already seen in example 2 that (C2) convergence of a sequence need not imply its (C1) convergence, even in the presence of (D5). Thus, this notion of D–metric convergence is stronger than convergence in the τ^* topology. If, however, we try to correct this by taking convergence to mean only in the sense of (C2), then we encounter a new problem; namely, that the sequence (x_n) of example 2 is now convergent, but is not D-Cauchy, since $D(x_m, x_n, x_p) = 1$ $(= D(x_m, x_n, x_n)$ whenever m, n and p are distinct.

In an attempt to specify a base for the τ topology Dhage introduced a new class of 'open balls', denoted by $B(x, r)$. In [1] we find the definition,

$$B(x, r) = \bigcap_{z \in X} \{y, z \in X \,:\, D(x, y, z) < r\},$$

while in [2] we find this modified to become,

$$\begin{aligned} B(x,r) = \ & \{y \in B^*(x,r) : \ \text{if } y,z \in B^*(x,r) \text{ are any} \\ & \qquad\qquad\qquad\quad \text{two points then } D(x,y,z) < r\} \\ = \ & \{y,z \in X : D(x,y,z) < r\} \,. \end{aligned}$$

We take these ill-formed statements to mean,

(B2) $B(x,r) := \{y \in B^*(x,r) \ : \ \text{for all } z \in B^*(x,r) \text{ we have } D(x,y,z) < r\}$,

an interpretation consistent with Dhage's calculations concerning these balls found in [2].

Example 5: Let $X = \mathbf{R}$ and define $D(x,y,z) := \rho(x,y) + \rho(y,z) + \rho(x,z)$, where $\rho(x,y) := |x-y|$ is the usual metric on $\mathbf{R}$. In this case all three types of convergence (C1) – (C3) reduce to the usual convergence in $\mathbf{R}$, so the balls $B(x,r)$ should be a base for the usual topology. Indeed, Dhage ([2], theorem 3.2) claimed that $B(x,r)$ equaled the interval $(x - r/4, x + r/4)$. What his proof actually establishes is that $B(x,r) \subseteq (x - r/4, x + r/4)$, which is certainly true, as a quick calculation reveals that in fact $B(x,r) = \{x\}$, for all $x \in \mathbf{R}$ and $r > 0$, and so τ is the discrete topology. Similar problems arise when D is defined via (M2) showing that Theorem (3.1) in [2] is false.

If in a generalized metric space (X, D) convergence in the sense of (C3) corresponds to convergence with respect to a topology τ, then we would have (at least in the first countable case) that the sequential closure of a set $A \subseteq X$; that is,

$$\overline{A} := \{x \in X \ : \ \text{ there exists } (a_n) \subseteq A \text{ with } a_n \xrightarrow{(C3)} x\},$$

coincides with its τ-closure, $\mathrm{cl}(A)$, in particular then we must have $\overline{\overline{A}} = \overline{A}$, see for example [6]. The following example shows that this need not be the case in a general D–metric space and so the sought after topology τ may fail to exist.

Example 6: let $p_0, p_1, p_2, p_3, \cdots$ be the prime numbers enumerated in ascending order, and define,

$A_0 := \{\frac{1}{p_0}, \frac{1}{p_0^2}, \frac{1}{p_0^3}, \cdots, \frac{1}{p_0^k}, \cdots\}$,
$A_1 := \{\frac{1}{p_0} + \frac{1}{p_1}, \frac{1}{p_0} + \frac{1}{p_1^2}, \frac{1}{p_0} + \frac{1}{p_1^3}, \cdots, \frac{1}{p_0} + \frac{1}{p_1^k}, \cdots, \}$,
$A_2 := \{\frac{1}{p_0^2} + \frac{1}{p_2}, \frac{1}{p_0^2} + \frac{1}{p_2^2}, \frac{1}{p_0^2} + \frac{1}{p_2^3}, \cdots, \frac{1}{p_0^2} + \frac{1}{p_2^k}, \cdots\}$,
and in general,
$A_n := \{\frac{1}{p_0^n} + \frac{1}{p_n}, \frac{1}{p_0^n} + \frac{1}{p_n^2}, \frac{1}{p_0^n} + \frac{1}{p_n^3}, \cdots, \frac{1}{p_0^n} + \frac{1}{p_n^k}, \cdots\}$.

Let $X = \{0\} \cup \bigcup_{n=0}^{\infty} A_n$, define a semi-metric on X by,

$$\rho(x,y) := \begin{cases} 0, & \text{whenever } x = y, \\ |x-y|, & \text{if } x, y \in A_0 \cup \{0\}, \text{ or} \\ & \text{for some } n = 1,2,3,\cdots, \ x, y \in A_n \cup \{\frac{1}{p_0^n}\}, \\ 1, & \text{otherwise,} \end{cases}$$

and let $D(x,y,z) := \rho(x,y) + \rho(y,z) + \rho(x,z)$, then D is a symmetric generalized metric, not satisfying (D5).

For $A := \cup_{n=1}^{\infty} A_n$ we have $A_0 \subset \overline{A}$, since $\frac{1}{p_0^n} + \frac{1}{p_n^k} \xrightarrow{(C3)} \frac{1}{p_0^n}$, as $k \to \infty$. Thus, $0 \in \overline{A_0} \subset \overline{\overline{A}}$, but it is readily seen that $0 \notin \overline{A}$, so $\overline{A} \neq \overline{\overline{A}}$.

This same example also shows that in the absence of the (D5) condition convergence in the sense of (C1), or (C2) may fail to correspond to convergence in any topology.

For X and A as above, but with,

$$D(x,y,z) := \begin{cases} |x-y| + |y-z| + |x-z|, & \text{if } x, y, z \in A_0 \cup \{0\}, \text{ or} \\ & \text{if } x, y, z \in A_n \cup \{\frac{1}{p_0^n}\}, \text{ for} \\ & \text{some } n = 1,2,3,\cdots, \text{ or} \\ & \text{if two or more of } x, y \text{ and } z \\ & \text{are equal,} \\ 1, & \text{otherwise,} \end{cases}$$

we see that (X, D) is a generalized metric space with D both symmetric and satisfying (D5), so (C1) = (C2) convergence corresponds to convergence in the τ^*-topology (indeed, the (C2)-sequential closure of A contains 0 and is itself (C2)-sequentially closed), but for which convergence in the sense of (C3) still fails, as above, to correspond to convergence with respect to any topology.

The following example confirms just how aberrant the 'balls' defined by (B2) can be.

Example 7: let X = **R** and define,

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ |x| + |y| + |z|, & \text{otherwise.} \end{cases}$$

then it is readily checked that (X, D) is a D–metric space satisfying (D5) in which $B^*(3,7) = \{3\} \cup (-2,2)$, while $B(3,7) = (-1,1) \not\ni 3$. So, balls defined by (B2) need not contain their centres.

Further, if $0 < r_1 < r_2$ it need not follow that $B(x, r_1) \subseteq B(x, r_2)$; for instance, $B(3,3) = \{3\} \not\subseteq B(3,7)$. By taking $a = 0$ this provides a counter-example to lemma (3.1) in [2], which implies that *if* $a \in B(x,r)$ *then* $B(x,r_1) \subseteq B(x,r)$, *where* $r_1 := D(x,a,a) < r$. Similarly, it also invalidates arguments on which the proof of theorem (3.5) in [2] is based, since the proof relies on the fact that if $a \in B(x,r)$, and $r_1 := D(x,a,a)$ then we can find $\epsilon > 0$ such that $B^*(x, r_1 + \epsilon) \subseteq B(x,r)$, but we have $D(3,0,0) = 3$, while $\{3\} = B^*(3,3) \subseteq B^*(3, 3+\epsilon)$, for all $\epsilon > 0$, so $B^*(3, 3+\epsilon) \not\subseteq B(3,7)$, for any $\epsilon \geq 0$.

Example 8 : let $A := \{\frac{1}{2^n}, n = 1,2,3,\ldots\}$ and let $B := \{\frac{1}{3^n}, n = 1,2,3,\ldots\}$, let $X := A \cup B \cup \{0\}$, and define $D : X \times X \times X \to \mathbf{R}^+$ by

$$D(x,y,z) := \begin{cases} 0, & \text{if } x = y = z \\ \frac{1}{2}, & \text{if one of } x, y \text{ and } z \text{ is zero and} \\ & \text{one of the other two points belongs to } A \\ & \text{ and the other to } B \\ x + y + z, & \text{otherwise.} \end{cases}$$

Then (X, D) is a D–metric space for which (D5) holds.

We note that D is not jointly continuous with respect to convergence in the sense of (C1), (C2) or (C3), since if we take $x_n := \frac{1}{2^n}$ and $y_n := \frac{1}{3^n}$ then both sequences (x_n) and (y_n) converge to 0, in all three senses, but $\lim_{n\to\infty} D(0, x_n, y_n) = \frac{1}{2} \neq D(0,0,0) = 0$

Dhage claimed (Theorem 6.1.2 in [2]) that "if a D–Cauchy sequence of points in a D–metric space contains a convergent (in the sense of (C3)) subsequence, then the sequence is itself convergent", however, the previous example shows that this is not generally valid. Define the sequence (z_n) by

$$z_n := \begin{cases} \frac{1}{2^n}, & \text{if n even,} \\ \frac{1}{3^n}, & \text{if n odd,} \end{cases}$$

then (z_n) is a D–Cauchy sequence, with convergent subsequences, but z_n is not itself convergent, since $D(0, z_n, z_m) = \frac{1}{2}$, whenever $n \neq m$.

In [2] Dhage took the distance between a point x and a subset A of (X, D) to be

$$d(x, x, A) := \inf\{D(x, x, a) : a \in A\}.$$

and claimed that the function $f(x) := d(x, x, A)$ is continuous in both the τ–topology [2], and the τ^*– topology [3]. However, the proofs of lemma (5.1) in [2] and lemma (1.2) in [3] rely on the continuity of D in the respective topologies and also contain logical errors. These results, and hence theorem (5.4) in [2] and (2.2) in [3] are therefore not necessarily true in general. Indeed, for (X, D) as in example (2), let $A := \{\frac{1}{n} : n \in N\}$ then

$$f(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x = 0, \end{cases}$$

and so $f(0) = d(0, 0, A) = \inf\{D(0, 0, a), a \in A\} = \inf\{1\} = 1$, on the other hand the function f is not continuous at $x = 0$ with respect to the topology τ^*. For example, $E := B^*(1, \frac{1}{2}) = (\frac{3}{4}, \frac{5}{4})$ is open in the τ^*– topology, but $f^{-1}(E) = \{0\}$ is not open in (X, D) with respect the τ^*–topology, since for any $\delta > 0$ the ball $B^*(0, \delta)$ contains some element of A.

In [5] Dhage worked with D–metric spaces satisfying the $(D5)$ condition, he denote by 2^X the class of nonempty τ-closed and bounded subsets of X and proceeded to try to develop a notion of Hausdorff D–metric on 2^X. Here we find yet another attempt to define τ ϵ-neighbourhoods of a point; namely,

$$N(a, \epsilon) = \left\{ \begin{array}{ll} x \in X : & D(a, x, x) < \epsilon \text{ and if } y, z \in N(a, \epsilon) \text{ are any two} \\ & \text{points then } D(a, y, z) < \epsilon \end{array} \right\}.$$

This makes no sense, and together with the many problems associated with the τ topology already pointed out, renders most of the results in [5] invalid.

In generalized metric spaces, the proofs of most fixed point results claimed by Dhage and others relied, either directly or indirectly, on the continuity of D with respect to the τ-topology (or at least, convergence in the sense of (C3)), or the τ^*-topology (convergence in the sense of (C2)). However, as we have seen (example 4, and more generally example 8) this need not be the case, even in the presence of (D5), nullifying the validity of these arguments.

References

[1] Dhage, B.C., Generalized Metric Space and Mapping With Fixed Point, *Bull. Cal. Math. Soc.* **84**, (1992), 329–336.

[2] Dhage, B.C., Generalized Metric Space and Topological Structure I, *An. ştiinţ. Univ. Al.I. Cuza Iaşi. Mat(N.S)* **46**, (2000), 3–24.

[3] Dhage,B.C., On Generalized Metric Spaces and Topological Structure II, *Pure. Appl. Math. Sci.* **40**,1994, 37–41.

[4] Dhage,B.C., On Continuity of Mappings in D–metric Spaces, *Bull. Cal. Math. Soc.* **86**, (1994), 503–508.

[5] Dhage,B.C., Generalized D–Metric Spaces and Multi–valued Contraction Mappings, *An. ştiinţ. Univ. Al.I. Cuza Iaşi. Mat(N.S)*, **44**, (1998), 179–200.

[6] Uryshon, P., Sur Les Classes ($\mathcal{L}$) de M. Fréchet, *L'Ens. Math.*, (1926), 77–83.

School of Mathematical and Physical Sciences,
The University of Newcastle,
NSW 2308, Australia.
Brailey.Sims@newcastle.edu.au