The Douglas-Rachford algorithm in the absence of convexity

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Abstract

We provide convergence results for a proto-typical non-convex iteration of Douglas-Rachford type.

1 Introduction

In recent times variations of alternating projection algorithms have been applied in Hilbert space to various important applied problems—from optical abberation correction to three satisfiability, protein folding and construction of giant *Sodoku* puzzles [8]. While the theory of such methods is well understood in the convex case [3] and [11, 4, 5, 6], there is little corresponding theory when some of the sets involved are non-convex—and that is the case of the examples mentioned above [8, 9].

Our intention is to analyze the simplest non-convex prototype in Euclidean space: that of finding a point on the intersection of a sphere and a line or hyperplane. The sphere provides an accessible model of many reconstruction problems in which the magnitude of a signal is measured.

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1.1 Preliminaries

For any closed subset A of a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ we say that a mapping $P_A : D_A \subseteq X \longrightarrow A$ is a *closest point projection* of D_A onto A if $A \subseteq D_A$, $P_A^2 = P_A$ and

$$\|x - P_A(x)\| = \ \mathrm{dist}(x, A) \ := \inf\{\|x - a\| : a \in A\},$$

for all $x \in D_A$.

For a given closest point projection, P_A, onto A we take the *reflection* of x in A (relative to P_A) to be,

$$R_A := 2P_A - I.$$

In this note we will focus on the cases when the subset A is a sphere, which without loss of generality we take to be the unit sphere of the Hilbert space; $S := \{x : \|x\| = 1\}$, or a line $L := \{x : x = \lambda a + hb\}$ where, without loss of generality, we take $\|a\| = \|b\| = 1$, $a \perp b$ and $h > 0$.

The closest point projection of $x \neq 0$ onto the unit sphere S is,

$$P_S(x) := x/\|x\|$$

and so,

$$R_S(x) = 2x/\|x\| \ - \ x,$$

while the closest point projection of $x \in X$ onto L is the orthogonal projection,

$$P_L(x) := \langle x,\, a\rangle\, a \ + \ hb$$

and so,

$$R_L(x) = 2\langle x,\, a\rangle\, a \ + \ 2hb \ - \ x.$$

Given two closed sets A and B together with closest point projections P_A and P_B, starting from an arbitrary initial point $x_0 \in D_A$ the *Douglas-Rachford iteration scheme* (reflect-reflect-average), introduced in [7]) for numerical solution of partial differential equations, is a method for finding a point in the intersection of the two sets. That is, it aims to find a feasible point for the possibly non-convex constraint $x \in A \cap B$). Explicitly it is,

$$x_{n+1} := T_{A,B}(x_n),$$

where $T_{A,B}$ is the operator $T_{A,B} := \frac{1}{2}(R_B R_A + I)$. This method has many other names, see [5].

With our particular S and L we have for $x \neq 0$ that,

$$T_{S,L}(x) = \left(1 \; - \; \frac{1}{\|x\|}\right) x \; + \; \left(\frac{2}{\|x\|} - 1\right) \langle x,\, a\rangle \; a \; + \; hb.$$

Thus, if X is N-dimensional and $(x(1),\, x(2),\, x(3),\, \cdots, x(N))$ denotes the coordinates of x relative to an orthonormal basis B whose first two elements are respectively a and b we have,

$$T_{S,L}(x) \;=\; \left(\frac{x(1)}{\rho},\, \left(1 - \frac{1}{\rho}\right) x(2) + h,\, \left(1 - \frac{1}{\rho}\right) x(3),\, \cdots,\, \left(1 - \frac{1}{\rho}\right) x(N)\right),$$

where $\rho = \|x\| = \sqrt{x(1)^2 + \cdots + x(N)^2}$.

In this case the *Douglas-Rachford* scheme becomes,

$$x_{n+1}(1) = x_n(1)/\rho_n, \tag{1.1}$$

$$x_{n+1}(2) = h + [1 - 1/\rho_n\,]\, x_n(2), \quad \text{and} \tag{1.2}$$

$$x_{n+1}(k) = (1 - 1/\rho_n\,)\, x_n(k), \quad \text{for } k = 3,\ \cdots,\ N, \tag{1.3}$$

where $\rho_n := \|x_n\| = \sqrt{x_n(1)^2 + \cdots + x_n(N)^2}$.

From this it is clear that if the initial point x_0 lies in the hyperplane $\langle x,\, a\rangle = 0$; that is $x_0(1) = 0$, then all of the iterates remain in the hyperplane, which we will refer to as a *singular manifold* for the problem. We will analyze this case in greater detail in a subsequent section. Similarly, if the initial point lies in either of the two open half-spaces $\langle x,\, a\rangle > 0$ or $\langle x,\, a\rangle < 0$; that is, $x_0(1) > 0$ or $x_0(1) < 0$ respectively, then all subsequent iterates will remain in the same open half space. Further, by symmetry, it suffices to only consider initial points lying in the positive open half-space $x_0(1) > 0$.

Figure 1 shows two steps of the underlying geometric construction: the smaller (green) points are the intermediate reflections in the sphere. Most figures were constructed in *Cinderella* (`www.cinderella.de`). A web applet version of the underlying Cinderella construction is available at
`http://www.carma.newcastle.edu.au/~jb616/reflection.html`. Indeed, many of the insights for the proofs below came from examining the constructions. The number of iterations N, the height of the line (α in the interface), and the the initial point are all dynamic—changing one changes the entire visible trajectory.

Success of the Douglas-Rachford scheme relies on convergence of the (*Picard*) iterates, $x_n = T^n_{A,B}(x_0)$, to a fixed point of the generally nonlinear operator $T_{A,B}$ in $A \cap B$, as

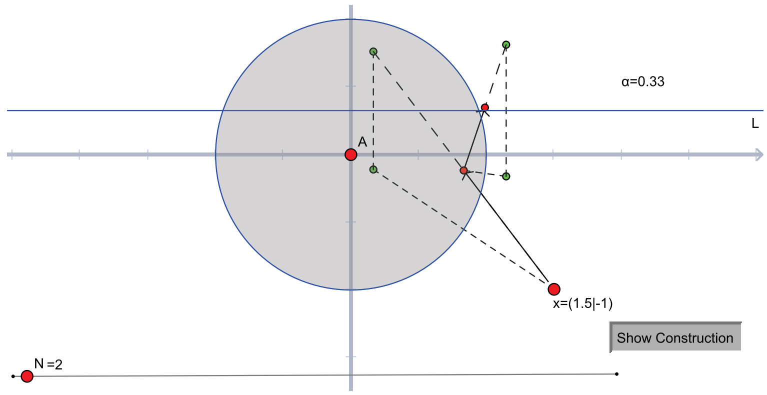

Figure 1: **Two steps showing the construction.**

$n \to \infty$. When both A and B are closed convex sets convergence of the scheme (in the weak topology) from any initial point in X to some point in $A \cap B$ was established by Lions and Mercier [11].

However,as noted, many practical situations yield feasibility problems in which one or more of the constraint sets is non-convex. That the Douglas-Rachford scheme works well in many of these situations has been observed and exploited for some years, despite the absence of any really satisfactory theoretical underpinning.

Remark 1 (divide-and-concur)**.** If one wishes to find a point in the intersection of M sets A_k in X we can instead consider

$$A := \prod_{k=1}^{N} A_k$$

and the linear set

$$B := \{x = (x_1, x_2, \ldots, x_M) \colon x_1 = x_2 = \cdots = x_M\}.$$

Then we observe that

$$R_A(x) = \prod R_{A_k}(x_k)$$

so that the reflections may be 'divided' up and

$$P_B(x) = \left(\frac{x_1 + x_2 + \cdots + x_M}{M}, \ldots, \frac{x_1 + x_2 + \cdots + x_M}{M} \right)$$

so that the projection and reflection on B are averaging ('concurrences'). Hence the name [9]. $\Diamond$

Example 1 (linear equations)**.** For the hyperplane $H_a := \{x \colon \langle a, x\rangle = b\}$ the projection is

$$x \mapsto x + \{\langle a, x\rangle - b\} \frac{a}{\|a\|^2}.$$

The consequent averaged-reflection version of the Douglas-Rachford recursion for a point in the intersection of N distinct hyperplanes is:

$$x \mapsto x + \frac{2}{N} \sum_{k=1}^{N} \{\langle a_k, x\rangle - b_k\} \frac{a_k}{\|a_k\|^2}. \tag{1.4}$$

The corresponding-averaged projection algorithm is:

$$x \mapsto x + \frac{1}{N} \sum_{k=1}^{N} \{\langle a_k, x\rangle - b_k\} \frac{a_k}{\|a_k\|^2} \tag{1.5}$$

In more generality, projection and reflection lead to greater differences. $\Diamond$

For any two closed sets A and B and feasible point $p \in A \cap B$ we say that the Douglas-Rachford scheme is *locally convergent* at p if there is a neighbourhood, N_p of p such that starting from any point x_0 in N_p the iterates $T^n_{A,B}(x_0)$ converge to p. The set comprising all initial points x_0 for which the iterates converge to p is the *basin of attraction* for p.

Remark 2 (The case of a half-line or segment)**.** Note, even in two dimensions, alternating projections, alternating reflections, project-project and average, and reflect-reflect and average will all often converge to (locally nearest) infeasible points even when A is simply the ray $R := \{[x, 0] \colon x \geqslant -1/2\}$ and B is the circle as before. They can also behave quite 'chaotically'. (See Figure 2 for a periodic illustration in *Maple* and Figure 3) for more complex behaviour. So the affine nature of the convex set seems quite important. $\Diamond$

2 Local convergence when $0 \leq h < 1$

In this note, as a first step toward an understanding of the Douglas-Rachford scheme in the absence of convexity, we analyze its behaviour in the indicative situation when

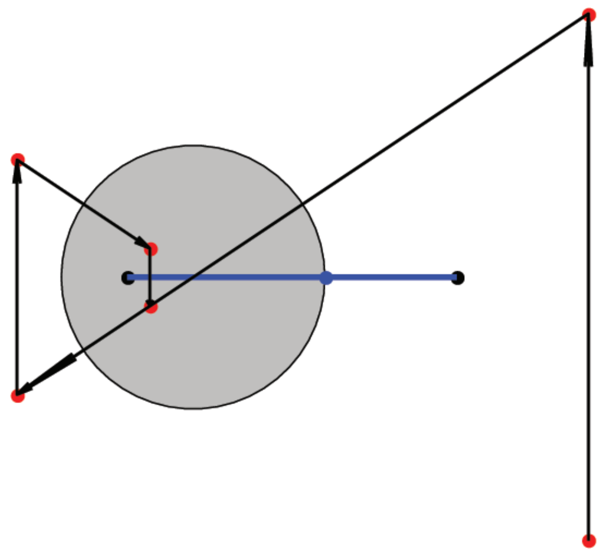

Figure 2: **Iterated reflection with a line segment.**

one of the sets is the non-convex sphere S and the other is the affine line L. We begin by establishing local convergence of the scheme when $0 \leq h < 1$.

In this section we show, at least when X is finite dimensional, that for $0 \leq h < 1$ local convergence at each of the feasible points is a consequence of the following theorem from the stability theory of difference equations. We appeal to:

Theorem 1 (Perron, [10])**.** *If $f : \boldsymbol{N} \times \boldsymbol{R}^m \longrightarrow \boldsymbol{R}^m$ satisfies,*

$$\lim_{x \to 0} \frac{\|f(n,x)\|}{\|x\|} \;=\; 0,$$

uniformly in n and M is a constant $n \times n$ matrix all of whose eigenvalues lie inside the unit disk, then the zero solution [provided it is an isolated solution] of the difference equation,

$$x_{n+1} = M x_n \;+\; f(n, x_n),$$

is exponentially asymptotically stable*; that is, there exists $\delta > 0$, $K > 0$ and $\zeta \in (0,1)$ such that if $\|x_0\| < \delta$ then $\|x_n\| \leq K\|x_0\|\zeta^n$.*

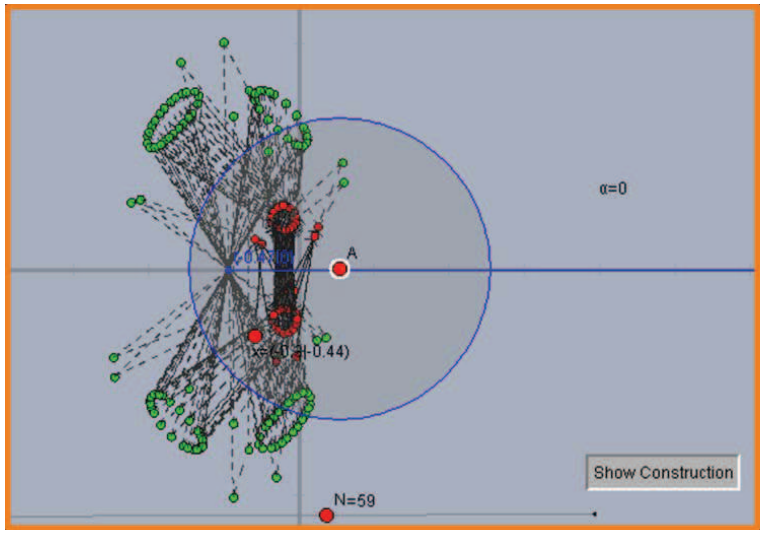

Figure 3: **More complex behaviour for a ray and circle.**

Proof. We begin by noting that for our particular S and L the operator $T_{S,L}$ is differentiable at any non-zero point y with derivative the linear operator,

$$T'_y(x) = \left\langle \left(\frac{2}{\|y\|} - 1\right) x - 2\frac{\langle x, y\rangle}{\|y\|^3} y,\ a \right\rangle a + \left(1 - \frac{1}{\|y\|}\right) x + \frac{\langle x, y\rangle}{\|y\|^3} y.$$

By symmetry it suffices to consider local convergence at the unique point p of $S \cap L$ lying in the positive open half-space $\langle x,\, a\rangle > 0$; that is $p = \sqrt{1-h^2}a + hb$, Then, p is an isolated fixed point of $T_{S,L}$ and, using $\|p\| = 1$ and $\langle p, a\rangle = \sqrt{1-h^2}$, we obtain,

$$T'_p(x) = \left\langle x,\ h^2 a - h\sqrt{1-h^2} b \right\rangle a \ + \ \left\langle x, h\sqrt{1-h^2} a + h^2 b \right\rangle b.$$

Which, relative to the basis B corresponds to the $n \times n$ Hessian matrix,

$$\begin{pmatrix} h^2 & -h\sqrt{1-h^2} & 0 & \cdots & 0 \\ h\sqrt{1-h^2} & h^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

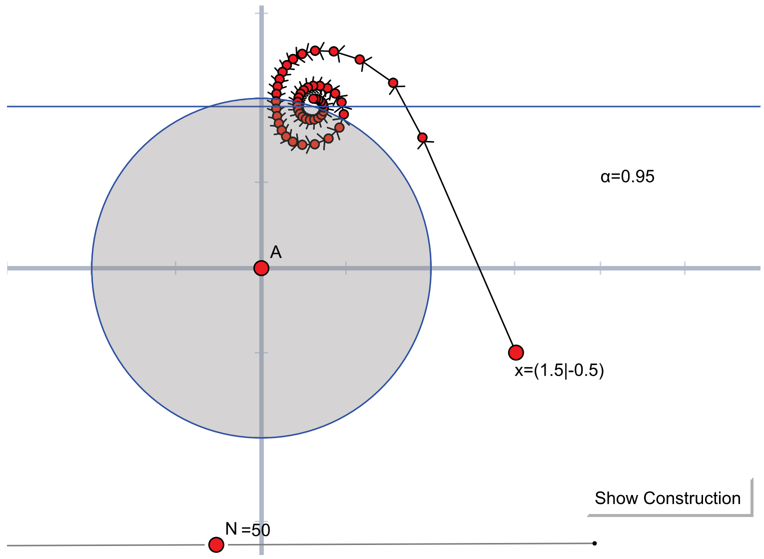

Figure 4: **Case with** $\alpha = 0.95$**.**

From this we immediately deduce that the only points in the spectrum of T'_p are the eigenvalues 0, and $h^2 \pm ih\sqrt{1-h^2}$.

Introducing the change of variable $\xi := x - p$ and defining f by,

$$f(\xi) := T_{S,L}(p+\xi) \; - \; T_{S,L}(p) \; - \; T'_p(\xi),$$

we see that the Douglas-Rachford scheme becomes,

$$\xi_{n+1} = T_{S,L}(p+\xi_n) - p = T_{S,L}(p+\xi_n) - T_{S,L}(p) = T'_p(\xi) + f(\xi).$$

Further, by the very definition of the derivative we have,

$$\lim_{\xi \to 0} \frac{\|f(\xi)\|}{\|\xi\|} = \lim_{\xi \to 0} \frac{\left\|T_{S,L}(p+\xi) - T_{S,L}(p) \; - \; T'_p(\xi)\right\|}{\|\xi\|} \; = \; 0.$$

Thus, all the conditions of Perron's theorem are satisfied, provided T'_p has its spectrum contained in the open unit disk. But, this follows immediately since the both non-zero eigenvalues have modulus equal to $h \; < \; 1$, establishing that locally the Douglas-Rachford scheme converges exponentially to $\xi = 0$; that is, to $x = p$. $\square$

Remark 3 (Explaining the spiral)**.** It is also worthy of note that the non-zero eigenvalues both have arguments whose cosines have absolute value h, so 'spiraling' illustrated in Figure 4 should be less rapid the larger the value of h, an observation born

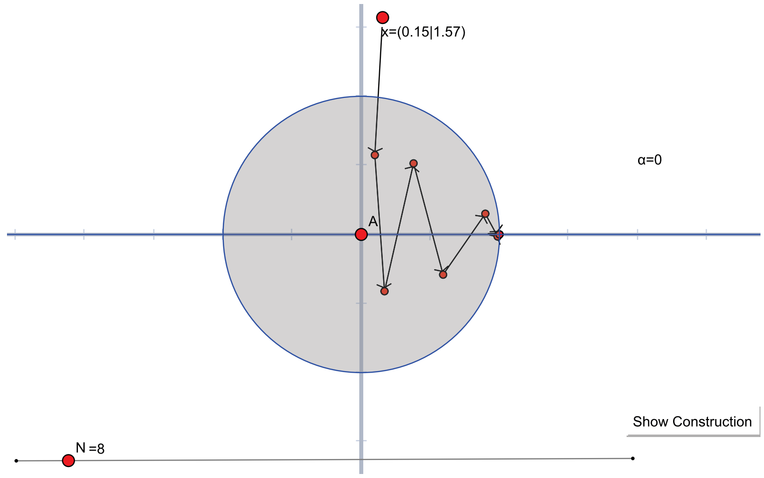

Figure 5: **Case with $h = 0$.**

out by experiment. It should also be noted that when $h = 1$; that is, the line L is tangential to the sphere S, Perron's theorem fails to apply. The spiral never begins, and indeed the conclusion is false as we will shortly show. $\Diamond$.

3 Convergence when $h = 0$

We show that starting from any initial point with $x_0(1) > 0$ the Douglas-Rachford scheme converges to the feasible point $a = (1, 0, 0, \cdots, 0)$, as illustrated in Figure 5. In this case the scheme (1.1), (1.2), (1.3) reduces to,

$$\begin{aligned} x_{n+1}(1) &= x_n(1)/\rho_n, \quad \text{and} \\ x_{n+1}(k) &= (1 - 1/\rho_n\,)\, x_n(k), \quad \text{for } k = 2, \ \cdots, \ N, \end{aligned}$$

with $\rho_n = \|x_n\| = \sqrt{x_n(1)^2 + \cdots + x_n(N)^2} \geq x_n(1) > 0$.

Proposition 1. *If $\rho_n > 1$ then $\rho_{n+1}^2 < \rho_n^2$.*

Proof. We may estimate as follows.

$$\begin{aligned}
\rho_{n+1}^2 &= \frac{x_n(1)^2}{\rho_n^2} + \left(1 - \frac{1}{\rho_n}\right)^2 \sum_{k=2}^{N} x_n(k)^2 \\
&= \frac{x_n(1)^2 + x_n(2)^2 + \cdots + x_n(N)^2}{\rho_n^2} + \left(1 - \frac{2}{\rho_n}\right) \sum_{k=2}^{N} x_n(k)^2 \\
&= 1 + \left(1 - \frac{2}{\rho_n}\right) \sum_{k=2}^{N} x_n(k)^2 \\
&\leq 1 + \left(1 - \frac{2}{\rho_n} + \frac{1}{\rho_n^2}\right) \sum_{k=2}^{N} x_n(k)^2 \\
&= 1 + \left(1 - \frac{1}{\rho_n}\right)^2 \sum_{k=2}^{N} x_n(k)^2 \\
&\leq 1 + \left(1 - \frac{1}{\rho_n}\right)^2 \rho_n^2 \\
&= 1 + (\rho_n - 1)^2 \\
&= \rho_n^2 + 2(1 - \rho_n) \\
&< \rho_n^2, \quad \text{as } \rho_n > 1.
\end{aligned}$$

□

Corollary 1. *If $\rho_n > 1$ for all n then $\rho_n \longrightarrow 1$.*

Proof. By the above proposition, the ρ_n are decreasing and so converge to some limit $\rho \geq 1$. But then, taking limits in $\rho_{n+1}^2 \leq \rho_n^2 + 2(1 - \rho_n)$ leads to $\rho \leq 1$, so $\rho = 1$. □

Proposition 2. *If $\rho_n \leq 1$ then so too is $\rho_{n+1} \leq 1$.*

Proof. From the first three lines in the proof of the above proposition we have

$$\begin{aligned}
\rho_{n+1}^2 &= 1 + \left(1 - \frac{2}{\rho_n}\right) \sum_{k=2}^{N} x_n(k)^2 \\
&\leq 1 - \sum_{k=2}^{N} x_n(k)^2, \quad \text{provided } \rho_n \leq 1 \\
&\leq 1.
\end{aligned}$$

$\square$

Theorem 2. *If $h = 0$ and the initial point has $x_0(1) > 0$ then the Douglas-Rachford scheme converges to the feasible point* $(1, 0, 0, \cdots, 0)$.

Proof. In case $\rho_n > 1$ for all n then, by the above corollary, $\rho_n \to 1$, so by the recurrence $x_n(k) \to 0$ for $k = 2, \cdots, N$ and $x_n \to (1, 0, 0, \cdots, 0)$.

On-the-other-hand, if this is not the case then there is a smallest n_0 with $\rho_{n_0} \leq 1$ and then either $\rho_{n'} = 1$ for some $n' \geq n_0$, in which case we have $x_{n'+1}(k) = 0$ for $k = 2, \cdots, N$, so $x_{n'+1} = (1, 0, \cdots, 0)$ and we have arrived at the feasible point after a finite number of steps, or alternatively from the last proposition $\rho_n < 1$ for all $n \geq n_0$. Consequently, the sequence $(x_n(1))_{n=n_0}^{\infty}$ is strictly increasing (hence convergent to some $x(1) \leq 1$) and so for $n \geq n_0$ we have $\rho_n \geq x_n(1) \geq x_{n_0} > 0$. But then, for each integer $k \geq 2$ and $n \geq n_0$ we see from the recurrence that,

$$\begin{aligned}\left|\frac{x_{n+1}(k)}{x_{n+1}(1)}\right| &= (1 - \rho_n)\left|\frac{x_n(k)}{x_n(1)}\right| \\ &\leq (1 - x_{n_0}(1))\left|\frac{x_n(k)}{x_n(1)}\right|.\end{aligned}$$

Hence, $\dfrac{x_n(k)}{x_n(1)}$ converges to 0 and we conclude that $x_n \longrightarrow (1, 0, \cdots, 0)$. $\square$

Remark 4 (Hilbert space analogues)**.** It is not essential that X be finite dimensional for any of the arguments in this section, though the convergence established in the last theorem will only be weak convergence of the iterates to the feasible point. $\Diamond$

4 The tangential case when $h = 1$

When $h = 1$ the only feasible point is $b = (0, 1, 0, \cdots, 0)$, however we show that starting from an initial point with $x_0(1) > 0$ the Douglas-Rachford scheme converges to a point $yb = (0, y, 0, \cdots, 0)$ with $y > 1$, whose projection onto either S or L is the feasible point. The following result will be needed.

Proposition 3. *If $\rho_n > 2$ then $\rho_{n+1} \leq \rho_n$.*

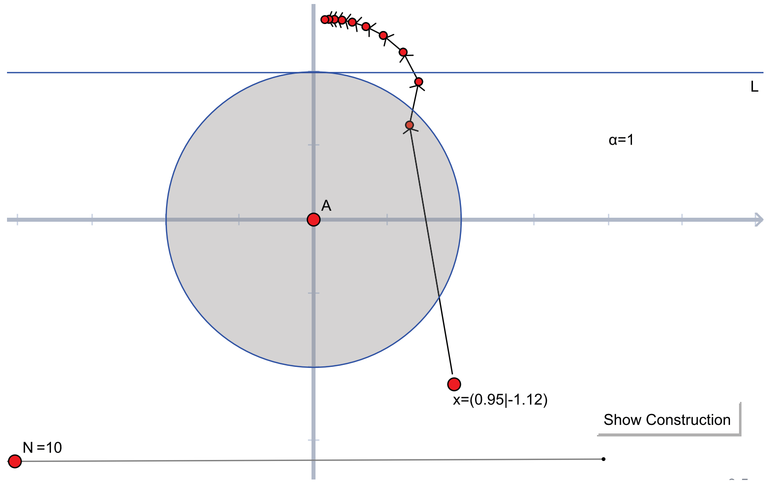

Figure 6: **Case with $h = 1$.**

Proof. The proof is similar to that of Proposition 1. We may estimate as follows.

$$
\begin{aligned}
\rho_{n+1}^2 &= \frac{x_n(1)^2}{\rho_n^2} + \left(\left(1 - \frac{1}{\rho_n}\right) x_n(2) + 1\right)^2 + \left(1 - \frac{1}{\rho_n}\right)^2 \sum_{k=3}^{N} x_n(k)^2 \\
&= \frac{x_n(1)^2 + x_n(2)^2 + \cdots + x_n(N)^2}{\rho_n^2} + \left(1 - \frac{2}{\rho_n}\right) \sum_{k=2}^{N} x_n(k)^2 + 2\left(1 - \frac{1}{\rho_n}\right) x_n(2) + 1 \\
&= 2 + \left(1 - \frac{2}{\rho_n}\right) \sum_{k=2}^{N} x_n(k)^2 + 2\left(1 - \frac{1}{\rho_n}\right) x_n(2) \\
&\le 2 + \left(1 - \frac{2}{\rho_n}\right) \rho_n^2 + 2\left(1 - \frac{1}{\rho_n}\right) \rho_n, \quad \text{as } \rho_n > 2 \\
&= \rho_n^2.
\end{aligned}
$$

□

To show the asserted behaviour, we begin by noting that from the recurrence,

$$x_{n+1}(2) = x_n(2) \; + \; 1 - \frac{x_n(2)}{\rho_n} \ge x_n(2), \tag{4.1}$$

since $\dfrac{x_n(2)}{\rho_n} \le 1$. Thus, the $x_n(2)$ are increasing and so either they converge to a finite limit, y say, or they diverge to $+\infty$.

In the first case, taking limits in the above equation (4.1) yields $y = \lim_n x_n(2) = \lim_n \rho_n$ and so $x_n \longrightarrow (0,\, y,\, 0,\, \cdots,\, 0)$. Too see that $y > 1$ we argue as follows. We have $x_n(1) \to 0$. But (1.1) shows $x_{n+1}(1) = x_n(1)/\rho_n$ so there must be some $\rho_n > 1$. On using (4.1) again, his implies that $x_{n+1}(2) - 1 = x_n(2)(1 - 1/\rho_n) > 1$ which in turn implies that $y \geqslant x_{n+1}(2) > 1$.

To show that the second divergent case is impossible we appeal to Proposition 3. to deduce that if the $x_n(2)$ diverge to $+\infty$, we must have for all sufficiently large n that $2 < x_n(2) \leq \rho_n$ and so eventually the ρ_n are decreasing and hence convergent to a finite limit which is necessarily greater than or equal to $\lim_n x_n(2)$ which cannot therefore be infinite; a contradiction.

Consequently, we have proved,

Theorem 3. *When L is tangential to S at b (that is, when $h = 1$), starting from any initial point with $x_0(1) \neq 0$, the Douglas-Rachford scheme converges to a point yb with $y > 1$.*

This is consistent with the behaviour in the convex case [11, 5].

5 Behaviour in the infeasible case when $h > 1$

Satisfyingly, when there are no feasible solutions the Douglas-Rachford scheme diverges. More precisely,

Theorem 4. *If there are no feasible solutions (that is, when $h > 1$) then for any non-zero initial point $x_n(2)$ and hence ρ_n diverge at at least linear rate to $+\infty$.*

Proof. From the recursion we have,

$$\begin{aligned} x_{n+1}(2) - x_n(2) &= h - \frac{x_n(2)}{\rho_n} \\ &> h - 1, \quad \text{as } x_n(2) < \rho_n \\ &> 0, \end{aligned}$$

from which the result follows.

It is also worth noting that, as a consequence of the above theorem and the recurrence, $x_n(1) \to 0$ and so asymptotically the iterates approach the hyperplane $\langle x,\, a\rangle = 0$. $\square$

6 Behaviour on the singular manifold, $\langle x,\, a\rangle = 0$

Here we consider the iterates of a non-zero initial point with $x_0(1) = 0$ and so $x_n(1) = 0$ for all n.

We again distinguish the cases; $h = 0$, $0 < h < 1$, $h = 1$. The case $h > 1$ having already been dealt with in the previous section.

When $h = 0$ it is readily seen that any non-zero in the singular manifold $T_{S,L}(x) = \left(1 - \frac{1}{\|x\|}\right) x$. If $\|x\| = 1$ then the scheme breaks down after the first iterate. At points with $\|x\| < 1$ we see that $T_{S,L}$ has period two (that is, $T^2_{S,L}(x) = x$), while for $\|x\| > 1$ we have $T^2_{S,L}(x) = \left(1 - \frac{2}{\|x\|}\right) x$, so the scheme breaks down after two iterations if $\|x\| = 2$.

We observe that the iterates of any non-zero point on the line $\{x : x = \lambda b,\ \lambda \in \mathbf{R}\}$ remain on this line and that when $h = 1$ (that is, L is tangential to S at b) non-zero points on this line remain fixed under $T_{S,L}$.

In the other cases the scheme can exhibit periodic behaviour when rational commensurability is present while in other situations the behaviour may be quite chaotic. To make this precise we need to consider interval-valued mappings to deal with the jump at the origin. Luckily, the work in [2, 1] shows that various interval mapping analogues of Sharkovskii's theorem are operative.

7 Some final remarks

A wealth of experimental evidence, using both *Maple* and the dynamic geometry package *Cinderella*, leads to the conclusion that the basin of attraction for $p = \sqrt{1-h^2}a + hb$ is the open half space $\{x : \langle x,\, a\rangle\ >\ 0\}$ – the largest region possible. See also `http://www.carma.newcastle.edu.au/∼jb616/expansion.html`.

Moreover, we found that for stable computation in Cinderella it was necessary to have access to precision beyond *Cinderella's* built-in double precision. This was achieved by taking input directly from *Maple*. We illustrate in Figure 7 which show various spurious red points on the left and accurate data on the right. The figures show the effect of roughly ten steps of the Douglas-Ratchford iteration for 400 different starting points—where the points are coloured by their original distance from the

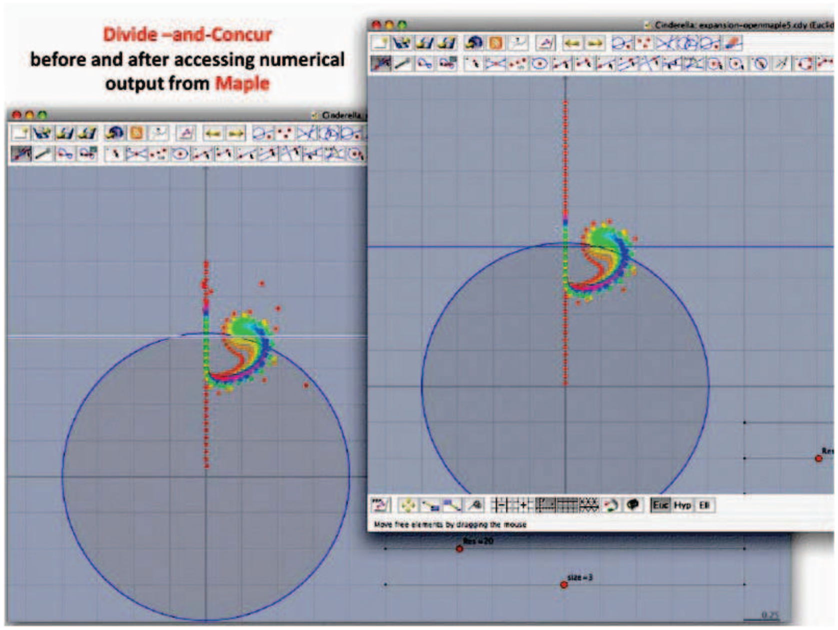

Figure 7: **Multiple iterations in Cinderella.**

vertical axis with red closest.

However, we are as yet unable to furnish a proof of this, leaving open the following conjecture:

Conjecture 1. *In the simple example of a sphere and a line with two intersection points, the basin of attraction is the two open half-spaces forming the complement of the singular manifold.*

Remark 5 (The case of a sphere and a hyperplane)**.** If we replace the line L by a hyperplane, say $H := \{x : \langle x, a\rangle = h\}$, where $\|a\| = 1$ and $0 \leq h < 1$, then, except in the 2-dimensional case where the situation is identical to the one analyzed above, the feasible points are no longer isolated, so local convergence in the sense described above is impossible.

However, a similar analysis shows firstly that for any non-feasible initial point $x_0 \neq 0$ the sequence of iterates, $x_n = T^n_{S,H}(x_0)$ is confined to the 2-dimensional subspace $M(x_0, a)$ spanned by x_0 and a. So, if the Douglas-Rachford scheme converges it will converge to a point in $S \cap H \cap M(x_0, a)$. And then secondly, we have 'local convergence'in the following sense. For any feasible point $p \in S \cap H$ there is a neighbourhood, N_p of p in $M(p, a)$ such that starting from any point x_0 in N_p the iterates converge to p.

Additionally, we may derive similar conclusions to those obtained above in the cases when $h = 0$, $h = 1$ and $h > 1$. Further, in this case the singular manifold is the line $\{x : x = \lambda a,\ \lambda \in \mathbf{R}\}$. $\Diamond$

In conclusion, our analysis sheds some light on the behaviour of non-convex Douglas-Ratchford schemes but much remains to be studied.

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