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THE FIXED POINT PROPERTY IN c_0

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ABSTRACT. A closed convex subset of c_0 has the fixed point property (fpp) if every nonexpansive self mapping of it has a fixed point. All nonempty weak compact convex subsets of c_0 are known to have the fpp. We show that closed convex subsets with a nonempty interior and nonempty convex subsets which are compact in a topology slightly coarser than the weak topology may fail to have the fpp.

1. **Introduction.** We say a closed convex subset of the Banach space $(X, \|\cdot\|)$ has the *fixed point property* (fpp) if every nonexpansive mapping $T: C \longrightarrow C$ has a fixed point. Here, T nonexpansive means $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We ask which nonempty closed bounded convex subsets of c_0 enjoy the fpp?

It is now well known that all nonempty weak compact convex subsets of c_0 have the fpp [Maurey, 1980]. On the other hand, closed bounded convex subsets with a nonempty interior always fail to have the fpp, Proposition 1 below. That sets without interior may also fail to have the fpp is demonstrated by $B_{c_0}^+ := \{(x_n) : 0 \leq x_n \leq 1, \text{all } n\}$ on which $T: (x_n) \longmapsto (1, x_1, x_2, \ldots)$ is a fixed point free isometry.

We refine this last example by showing that closed bounded convex subsets of c_0 which are compact in a locally convex topology only 'slightly' coarser than the weak topology may fail to have the fpp. This lends support to the following.

CONJECTURE. In c_0 the only closed bounded convex subsets with the fpp are weak compact.

PROPOSITION 1. *Let C be a closed bounded convex subset of c_0. If the set C has an interior point then C fails the* fpp.

PROOF. Without loss of generality we may suppose that $0 \in \operatorname{int}(C)$, so there exists $\varepsilon > 0$ such that $B[0, \varepsilon] \subset C$.

We define $R: C \longrightarrow B[0, \varepsilon]^+$ by

$$R\Big(\big(x(n)\big)\Big) = \Big(\big(|x(n)| \wedge \varepsilon\big)\Big)$$

where $|x(n)| \wedge \varepsilon := \min\{|x(n)|, \varepsilon\}$, and $B[0, \varepsilon]^+ = \Big\{\big(x(n)\big) \in B[0, \varepsilon] : x(n) \geq 0\Big\}$. In order to prove that R is nonexpansive, we apply the well known James-Birkhoff inequality:

$$|a \wedge \varepsilon - b \wedge \varepsilon| \leq |a - b|, \quad \text{for every } a, b, \varepsilon \in \mathbf{R}.$$

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Therefore we have:

$$\|R(x) - R(y)\| = \sup\Big\{\Big|\,|x(n)| \wedge \varepsilon - |y(n)| \wedge \varepsilon\Big| : n = 1, 2, \ldots\Big\}$$
$$\leq \sup\Big\{\Big|\,|x(n)| - |y(n)|\,\Big| : n = 1, 2, \ldots\Big\} \leq \|x - y\|.$$

Now we define the mappings $S\colon B[0, \varepsilon]^+ \longrightarrow B[0, \varepsilon]^+$ by

$$S\Big(\big(x(n)\big)\Big) = \big(\varepsilon, x(1), x(2), \ldots\big),$$

and $T\colon C \longrightarrow B[0, \varepsilon]^+$ by $T := S \circ R$.

This map T is a nonexpansive selfmapping of C. If there exists $x \in C$ with $T(x) = x$, then $x \in B[0, \varepsilon]^+$, $R(x) = x$, and $T(x) = S(x) = x$, a contradiction. ∎

2. **The $\mathcal{E}$-topology on $\mathbf{c}_0$.** Let $d := (1, 1, 1, \ldots, 1, \ldots) \in \ell_\infty = c_0^{**}$, and let E be the closed subspace of ℓ_1 given by $E := \ker(d)$. That is, $E = \Big\{\big(y(n)\big) \in \ell_1 : \sum y(n) = 0\Big\}$. By [Guerre-Delabrière, 1992, Lemma 1.1.11] E is a norming subspace for c_0. Alternatively it is easily verified by direct calculation (see, for example, Lemma 2.8 below) that in this case

$$\frac{1}{2}\|x\|_\infty \leq \sup\{\langle x, y\rangle : y \in E, \|y\|_1 \leq 1\} \leq \|x\|_\infty,$$

where $\langle x, y\rangle = \sum x(k)y(k)$, as usual. Consequently E separates points of c_0 and so, by [Jameson, 1974, 27.3], the set E is dense in $c_0^* = \ell_1$ with respect to the weak* topology. We consider c_0 equipped with the topology $\mathcal{E} := \sigma(c_0, E)$. That is, $\mathcal{E}$ is the smallest locally convex linear topology on c_0 making continuous all the elements of E (as linear functionals on c_0).

The topology $\mathcal{E}$ may be seen as 'slightly' coarser than the weak topology on c_0, being induced by a norming codimension one subspace of c_0^*. It displays some unusual, though not too pathological, properties. For example, the following five propositions can be proved by more or less standard methods of locally convex space theory.

PROPOSITION 2.1. *The topology $\mathcal{E}$ consist of $\emptyset$, c_0, all finite intersections of the sets*

$$\Big\{\big(x(n)\big) \in c_0 : a < \sum x(n)y(n) < b, \sum y(n) = 0\Big\}$$

and all arbitrary unions of these finite intersections.

PROPOSITION 2.2. *$\mathcal{E}$ is Hausdorff.*

PROPOSITION 2.3. *A sequence (x_n) in c_0 is $\mathcal{E}$ convergent to $x \in c_0$ if and only if for every $y \in E$,*

$$\langle x_n, y\rangle \longrightarrow \langle x, y\rangle.$$

PROPOSITION 2.4. *Every $\mathcal{E}$-convergent sequence is bounded.*

PROPOSITION 2.5. *Let M be a bounded subset of c_0 and let $x \in \mathcal{E}$-*cl *M. Then there exists a sequence (x_n) in M such that $x_n \overset{\mathcal{E}}{\longrightarrow} x$.*

On the other hand, we have some results which are specific for the topology $\mathcal{E}$.

REMARK 2.6. The sequence (d_n) in c_o given by

$$d_n := (\underbrace{1,\dots,1}_{n},0,0,\dots)$$

$\mathcal{E}$-converges to 0, but (d_n) does not have weakly null subsequences. Indeed, for $y = \big(y(n)\big) \in E$,

$$\langle d_n, y\rangle = \sum_{j=1}^{n} y(j) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Note that (d_n) is the standard *summing basis* for c_0.

REMARK 2.7. Let (x_n) be a sequence in c_0 which is $\mathcal{E}$-convergent to $x \in c_0$. Since, the vector $y := (\underbrace{1,\dots,1}_{k},-k,0,\dots)$ belongs to E, we have

$$x_n(1) + \cdots + x_n(k) - kx_n(k+1) \longrightarrow x(1) + \cdots + x(k) - kx(k+1)$$

and so

$$\frac{x_n(1) + \cdots + x_n(k)}{k} - x_n(k+1) \longrightarrow \frac{x(1) + \cdots + x(k)}{k} - x(k+1).$$

Necessary conditions such as this help provide a better understanding of $\mathcal{E}$-convergence.

LEMMA 2.8. *For every element $x = \big(x(n)\big) \in c_0$ there exists a sequence (y_n) in E such that $\|y_n\|_1 = 2$ and*

$$|\langle x, y_n\rangle| \longrightarrow \|x\|.$$

PROOF. Take $x(l) \in \{x(n) : n \in \mathbf{N}\}$ such that $|x(l)| = \|x\|$ and define

$$y_n := (\underbrace{0,\dots,0}_{l},1,\underbrace{0,\dots,0}_{n},-1,0,\dots)$$

Clearly $\|y_n\|_1 = 2$ and

$$|\langle x, y_n\rangle| = |x(l) - x(n+l)| \longrightarrow |x(l)| = \|x\|, \quad \text{as } n \longrightarrow \infty.$$

■

PROPOSITION 2.9. *If a sequence (x_n) in c_o is $\mathcal{E}$-convergent to $x \in c_0$ then $\|x\| \leq 2\liminf_n \|x_n\|$.*

PROOF. Take $y \in E$. We have

$$|\langle x, y\rangle| = \lim |\langle x_n, y\rangle| \leq \|y\|_1 \liminf \|x_n\|$$

We now apply the above lemma, to obtain a sequence (y_n) in E with $\|y_n\|_1 = 2$ such that

$$|\langle x, y_n\rangle| \longrightarrow \|x\|, \quad \text{as } n \longrightarrow \infty$$

and therefore the last inequality gives, for $n = 1, 2, \dots$

$$|\langle x, y_n \rangle| \leq \|y_n\|_1 \liminf_m \|x_m\|.$$

Taking limits we obtain the conclusion:

$$\|x\| = \lim_{n \to \infty} |\langle x, y_n \rangle| \leq 2 \liminf_m \|x_m\| \qquad \blacksquare$$

REMARK 2.10. The bound 2 in the last inequality cannot be improved. For example, if we consider the sequence $(d_n) \subset c_0$ defined above in Remark 2.6, then for $e_1 := (1, 0, \dots .)$ we have $d_n - 2e_1 \xrightarrow{\mathcal{E}} -2e_1$, but

$$\| - 2e_1\| = 2 = 2 \liminf \|d_n - 2e_1\|.$$

REMARK 2.11. There exist bounded, convex, norm-closed sets which are not $\mathcal{E}$-closed (That is, we do not have a Mazur's theorem for the $\mathcal{E}$-topology). To see this, let K be the norm closed convex hull of the set $D = \{d_n : n = 1, \dots\}$. Obviously every convex combination y of vectors d_n must verify $y(1) = 1$, and so $\|y\| = 1$. Therefore

$$0 = \mathcal{E} - \lim d_n \notin K,$$

and K is not $\mathcal{E}$-closed.

REMARK 2.12. The right shift $S: c_0 \longrightarrow c_0$ is not $\mathcal{E}$-continuous. Indeed, the sequence (d_n) is $\mathcal{E}$-convergent to 0 but for $y \in E$ with $y(1) \neq 0$ we have

$$\langle S(d_n), y \rangle = \sum_{j=2}^{n} y(j) = \left(\sum_{j=1}^{n} y(j) \right) - y(1) \longrightarrow -y(1), \quad \text{as } n \longrightarrow \infty$$

and so $\big(S(d_n)\big)$ does not converges to $S(0)$.

PROPOSITION 2.13. *A sequence (x_n) in c_0 is weakly convergent to $x \in c_0$ if and only if $\big(S(x_n)\big)$ is $\mathcal{E}$-convergent to $S(x)$.*

PROOF. Since the right shift S is weak continuous we have that if $x_n \xrightarrow{w} x$ then $S(x_n) \xrightarrow{w} S(x)$, and so $S(x_n) \xrightarrow{\mathcal{E}} S(x)$. Conversely, for every $y = (y(1), y(2), \dots) \in \ell_1$ we have that

$$\tilde{y} := \left(- \sum y(j), y(1), y(2), \dots \right) \in E$$

If $S(x_n) \xrightarrow{\mathcal{E}} S(x)$ then $\langle S(x_n), \tilde{y} \rangle \longrightarrow \langle S(x), \tilde{y} \rangle$. But it is easy to see that $\langle S(x_n), \tilde{y} \rangle = \langle x_n, y \rangle$ and $\langle S(x), \tilde{y} \rangle = \langle x, y \rangle$, which yields the conclusion. $\blacksquare$

$\mathcal{E}$-convergence can also be related to weak* convergence in $c_0^{**} = \ell_\infty$.

PROPOSITION 2.14. *For a bounded sequence (x_n) in c_0 we have for the following conditions that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) For some λ_1 we have $\hat{x}_n \xrightarrow{w^*} \lambda_1 d$.

(ii) $x_n \xrightarrow{\mathcal{E}} 0$.

(iii) There exists a subsequence (x_{n_k}) with $\hat{x}_{n_k} \xrightarrow{w^*} \lambda_2 d$, for some $\lambda_2 \in \mathbf{R}$.

PROOF. If $\hat{x}_n \xrightarrow{w^*} \lambda_1 d$, then for $f \in \ker d$ we have $f(x_n) = \hat{x}_n(f) \longrightarrow \lambda_1 d(f) = 0$, so (i) $\Rightarrow$ (ii).

Suppose $x_n \xrightarrow{\mathcal{E}} 0$ and let $f_0 := (1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots) \in \ell_1$, so $d(f_0) = 1$. Choose a subsequence x_{n_k} such that $\lim_k f_0(x_{n_k})$ exists, and equals λ_2 say. Then for $f \in c_0^* = \ell_1$ we have $f = d(f)f_0 + g$, where $g = f - d(f)f_0 \in E = \ker(d)$, and so $\hat{x}_{n_k}(f) = f(x_{n_k}) \longrightarrow d(f)\lambda_2 = \lambda_2 d(f)$. Thus (ii) $\Rightarrow$ (iii). ∎

3. **c_0 fails the $\mathcal{E}$-fpp.** Let $d_0 := 0$ and for $n = 1, 2, 3, \ldots$ define d_n as above;

$$d_n := (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots)$$

To demonstrate the failure of the $\mathcal{E}$-fpp in c_0, we show that

$$K := \overline{\text{co}}\{d_n\}_{n=0}^{\infty}$$

consisting of vectors of the form

$$\sum_{n=0}^{\infty} \lambda_n d_n = \big(1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), 1 - (\lambda_0 + \lambda_1 + \lambda_2), \ldots\big),$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$, is a $\mathcal{E}$-compact convex set which admits a fixed point free affine isometry. Indeed T defined by

$$T\big(1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \ldots\big) := \big(1, 1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \ldots\big)$$

is such a map. The proof of these claims occupies the remainder of this section and is contained in the following lemmas.

LEMMA 3.1. *For the mapping T defined above we have*

(i) T maps K into K,

(ii) T is an isometry,

(iii) T is fixed point free in K.

PROOF. To establish (i) it suffices to note that for $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$, we have

$$T\Big(\sum_{n=0}^{\infty} \lambda_n d_n\Big) = \big(1, 1 - \lambda_0, 1 - (\lambda_0 + \lambda_1), \ldots\big)$$
$$= \sum_{n=1}^{\infty} \lambda_{n-1} d_n \in K.$$

(ii) follows, since for $x = \big(1-\lambda_0, 1-(\lambda_0+\lambda_1), \ldots\big)$ and $y = \big(1-\mu_0, 1-(\mu_0+\mu_1), \ldots\big)$ we have that

$$\begin{aligned}\|Tx - Ty\| &= \|(0, \mu_0 - \lambda_0, \mu_0 + \mu_1 - \lambda_0 - \lambda_1, \ldots)\| \\ &= \|(\mu_0 - \lambda_0, \mu_0 + \mu_1 - \lambda_0 - \lambda_1, \ldots)\| \\ &= \|x - y\|.\end{aligned}$$

Finally, if $x = \big(1-\lambda_0, 1-(\lambda_0+\lambda_1), \ldots\big)$ were such that $x = Tx = \big(1, 1-\lambda_0, 1-(\lambda_0+\lambda_1), \ldots\big)$ then we would have $\lambda_0 = 0$, $\lambda_1 = 0, \ldots$ contradicting the requirement that $\sum_{n=0}^{\infty} \lambda_n = 1$. Indeed, $T(0) = (1, 0, 0, \ldots) \neq 0$, and so we have (iii). ■

LEMMA 3.2. *K is $\mathcal{E}$-closed.*

PROOF. For $n = 1, 2, \ldots$ let

$$x_n = \sum_{k=0}^{\infty} \lambda_k^{(n)} d_k = (1 - \lambda_0^{(n)}, 1 - \lambda_0^{(n)} - \lambda_1^{(n)}, \ldots),$$

where $\lambda_k^{(n)} \geq 0$ and $\sum_{k=0}^{\infty} \lambda_k^{(n)} = 1$, be such that $x_n \xrightarrow{\mathcal{E}} x = (\mu_1, \mu_2, \ldots)$.

Choosing $f := (1, -1, 0, 0, \ldots) \in E$ we have

$$f(x_n - x) = (1 - \lambda_0^{(n)} - \mu_1) - (1 - \lambda_0^{(n)} - \lambda_1^{(n)} - \mu_2) \longrightarrow 0.$$

That is,

$$\lambda_1^{(n)} \longrightarrow \mu_1 - \mu_2.$$

Similarly, choosing $f := (0, 1, -1, 0, 0, \ldots)$ we obtain

$$\lambda_2^{(n)} \longrightarrow \mu_2 - \mu_3,$$

and in general

$$\lambda_k^{(n)} \longrightarrow \mu_k - \mu_{k+1}.$$

Thus, for $k = 1, 2, \ldots$

$$\lambda_k := \mu_k - \mu_{k+1} = \lim_n \lambda_k^{(n)} \geq 0$$

and

$$x = (\mu_1, \mu_1 - \lambda_1, \mu_1 - \lambda_1 - \lambda_2, \ldots) \in c_0.$$

So we must have

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k \geq 0,$$

and then, provided $\mu_1 \leq 1$,

$$x = \sum_{k=1}^{\infty} \lambda_k d_k \in K$$

But, given $\epsilon > 0$ there exists N so that

$$\mu_1 = \sum_{k=1}^{\infty} \lambda_k < \sum_{k=1}^{N} \lambda_k + \epsilon/2,$$

and there exists n for which

$$|\lambda_k - \lambda_k^{(n)}| \leq \epsilon/2N, \quad \text{for } k = 1, 2, \ldots, N.$$

Thus,

$$\mu_1 \leq \sum_{k=1}^{N} \lambda_k^{(n)} + \epsilon \leq 1 + \epsilon, \quad \text{as } \sum_{k=0}^{\infty} \lambda_k^{(n)} = 1,$$

and so $\mu_1 \leq 1$, as required. ∎

Since $d_n \xrightarrow{\mathcal{E}} d_0$, we have that $\{d_n\}_{n=0}^{\infty}$ is $\mathcal{E}$-compact. The $\mathcal{E}$-compactness of K then follows from Lemma 3.2, the definition of $\mathcal{E}$, and the following general result.

LEMMA 3.3. *Let X be a separable Banach space and let M be a closed norming subspace of X^*. If $D \subset X$ is $\sigma(X,M)$-compact then* $\mathrm{co}(D)$ *is $\sigma(X,M)$-precompact.*

PROOF. Since M is closed and norming, D is bounded and, equipped with the relative $\sigma(X,M)$ topology, is a compact Hausdorff space. Let $C := C\big(D, \sigma(X,M)\big)$, the space of continuous real valued functions on D with this topology. Then V defined by

$$V(f)(m) := f(m|_D), \quad \text{for } f \in C^* \text{ and } m \in M$$

is a weak* to weak*; that is, $\sigma(C^*, C)$ to $\sigma(M^*, M)$, continuous linear operator from C^* to M^*. Since M is norming, X may be identified with a closed subspace of M^* (the space $(X, \|\cdot\|')$ is complete, where $\|x\|' := \sup\{m(x) : m \in M, \|m\| \leq 1\}$). It suffices to show that $V(C^*) \subseteq X$, as then $V(B_{C^*})$ is a $\sigma(X,M)$-compact convex subset of X containing D (for $d \in D$ consider the action of V on d regarded as a point measure in B_{C^*}).

To establish that $V(C^*) \subseteq X$ we first note that if $f \in C^*$ then $V(f)$ is $\sigma(M,X)$ boundedly continuous. Indeed, since X is separable, bounded subsets of M are $\sigma(M,X)$ metrizable. So, if (m_n) is a bounded sequence in M with $m_n \longrightarrow m$ in the $\sigma(M,X)$ topology then the Lebesgue dominated convergence theorem gives that $f(m_n|_D) \longrightarrow f(m|_D)$, as required.

Now, suppose there is an $f \in C^*$ with $g := V(f) \notin X$. Then there exists $F \in M^{**}$ with $\|F\| = 1$, $F(g) \neq 0$, and $F|_X = 0$. B_M is $\sigma(M^{**}, M^*)$ dense in $B_{M^{**}}$, so there is a net $(m_i) \subset B_M$ with $\hat{m}_i(m^*) \longrightarrow F(m^*)$, for all $m^* \in M^*$. In particular $\hat{m}_i(x) \longrightarrow F(x) = 0$, for all $x \in X \leq M^{**}$. That is, $m_i \longrightarrow 0$ in the $\sigma(M,X)$ topology, and so since (m_i) is bounded $g(m_i) \longrightarrow g(0) = 0$. But, $g \in M^*$ so $g(m_i) = \hat{m}_i(g) \longrightarrow F(g) \neq 0$, a contradiction establishing the result. ∎

4. **Further results.** In this section we note that the construction of the $\mathcal{E}$-topology can be generalized to obtain a family of similar topologies for some of which compact convex sets C may fail to have the fpp even for *contractive* mappings; that is, mappings $T: C \longrightarrow C$ satisfying $\|Tx - Ty\| < \|x - y\|$, whenever $x \neq y$. Most of the proofs require only minor modifications to those given in sections 2 and 3 for the $\mathcal{E}$-topology, and so will be omitted.

To effect the generalization let $a = \big(a(n)\big) \in \ell_\infty$ be any sequence of 'weights' satisfying $\alpha \leq a(n) \leq \beta$, for some $0 < \alpha \leq \beta < \infty$, and take

$$\mathcal{E}_a := \sigma\big(c_0, \ker(a)\big),$$

the coarsest (locally convex linear topology) on c_0 making each functional in E_a continuous, where $E_a := \{y(n) \in \ell_\infty : \sum a(n)y(n) = 0\}$.

Proposition 2.1 remains true with the obvious modifications, namely:

PROPOSITION 4.1. *The topology $\mathcal{E}_a$ consists of $\emptyset$, c_0, all finite intersections of the sets*

$$\left\{\big(x(n)\big) \in c_0 : a < \sum x(n)y(n) < b, \sum a(n)y(n) = 0\right\}$$

and all arbitrary unions of these finite intersections.

Again E_a is a norming subspace for c_0, indeed

$$\frac{\beta}{\alpha+\beta}\|x\|_\infty \le \sup\{\langle x,y\rangle : y \in E_a, \|y\|_1 \le 1\} \le \|x\|_\infty,$$

so $\mathcal{E}_a$ is Hausdorff.

Similarly one can verify Propositions 2.3, 2.4 and 2.5 with E replaced by E_a and $\mathcal{E}$ replaced by $\mathcal{E}_a$.

The sequence (d_n) need not convege to 0 with respect to the $\mathcal{E}_a$ topology. Indeed, for $y = \big(y(n)\big) \in E_a$

$$\langle d_n, y\rangle = \sum_{j=1}^{n} y(j)$$

and it is generally untrue that the above sum converges to 0 as $n \longrightarrow \infty$. On the other hand, if we replace (d_n) by the sequence (a_n) given by

$$a_n := \big(a(1), \dots, a(n), 0, 0, \dots\big)$$

we have

PROPOSITION 4.2. *The sequence a_n is $\mathcal{E}_a$-convergent to 0 and does not have weakly null subsequences. Indeed, for $y = \big(y(n)\big) \in E_a$,*

$$\langle a_n, y\rangle = \sum_{j=1}^{n} a(j)y(j) \longrightarrow 0, \quad \textit{as } n \longrightarrow \infty.$$

Variants of Lemma 2.8 and Proposition 2.9 also hold for $\mathcal{E}_a$ as do analogues of Remarks 2.10, 2.11 and 2.12.

LEMMA 4.3. *For every element $x = \big(x(n)\big) \in c_0$ there exists (y_n) in E_a such that*

$$1 + \frac{\alpha}{\beta} \le \|y_n\|_1 \le 1 + \frac{\beta}{\alpha}$$

and

$$|\langle x, y_n\rangle| \longrightarrow \|x\|.$$

The proof is essentially the same as that for Lemma 2.8 if the -1 in the definition of y_n is replaced by $-a(l)/a(n+l)$.

Using this lemma we can prove the following in a way similar to that for Proposition 2.9.

PROPOSITION 4.4. *If a sequence (x_n) in c_0 is $\mathcal{E}_a$-convergent to $x \in c_0$ then*

$$\|x\| \leq \left(1 + \frac{\beta}{\alpha}\right) \liminf_n \|x_n\|.$$

To obtain instances where the $\mathcal{E}_a$-fpp fails we put $a_0 := 0$ and take

$$K_a := \overline{\mathrm{co}}\{a_n\}_{n=0}^{\infty}.$$

Then K_a consists of vectors of the form

$$\sum_{n=0}^{\infty} \lambda_n a_n = \Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), a(3)\big(1-(\lambda_0+\lambda_1+\lambda_2)\big), \dots\Big),$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.

That K_a is $\mathcal{E}_a$-closed follows by effectively the same argument as that used for Lemma 3.2 with the functional f employed at the n-th step of the induction replaced by $f := \big(0, \dots, 1, -a(n)/a(n+1), 0, \dots\big)$, where the 1 occurs in the n-th position. This, in combination with Proposition 4.2 and Lemma 3.3, establishes the following.

PROPOSITION 4.5. *K_a is an $\mathcal{E}_a$-compact convex set.*

Now define T_a to be the affine map given by

$$T_a\Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), \dots\Big) := \Big(a(1), a(2)(1-\lambda_0), a(3)\big(1-(\lambda_0+\lambda_1)\big), \dots\Big).$$

In other words,

$$T_a\Big(\sum_{n=0}^{\infty} \lambda_n a_n\Big) := \sum_{n=1}^{\infty} \lambda_{n-1} a_n.$$

It is clear that T_a maps K_a into K_a. Moreover, if

$$x = \Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), \dots\Big)$$

were such that

$$x = T_a(x) = \Big(a(1), a(2)(1-\lambda_0), a(3)\big(1-(\lambda_0+\lambda_1)\big), \dots\Big)$$

then we would have $\lambda_0 = 0, \lambda_1 = \lambda_0, \dots$ contradicting the requirement that $\sum_{n=0}^{\infty} \lambda_n = 1$, so T_a is fixed point free in K_a.

Further, if

$$x = \Big(a(1)(1-\lambda_0), a(2)\big(1-(\lambda_0+\lambda_1)\big), \dots\Big)$$

and

$$y = \Big(a(1)(1-\mu_0), a(2)\big(1-(\mu_0+\mu_1)\big), \dots\Big)$$

are elements of K_a then

$$\|x - y\| = \max\{a(1)|\mu_0 - \lambda_0|, a(2)|\mu_0 - \lambda_0 + \mu_1 - \lambda_1|, \dots\}.$$

On the other hand

$$Tx = \Big(a(1), a(2)(1-\lambda_0), a(3)\big(1-(\lambda_0+\lambda_1)\big), \ldots\Big),$$
$$Ty = \Big(a(1), a(2)(1-\mu_0), a(3)\big(1-(\mu_0+\mu_1)\big), \ldots\Big)$$

and so,

$$\|Tx - Ty\| = \max\{a(2)|\mu_0 - \lambda_0|, a(3)|\mu_0 - \lambda_0 + \mu_1 - \lambda_1|, \ldots\}.$$

We therefore arrive at the following conclusion.

PROPOSITION 4.6. *$T_a\colon K_a \longrightarrow K_a$ is a fixed point free (contractive) nonexpansive mapping of the nonempty $\mathcal{E}_a$-compact convex set K_a whenever the sequence of weights $a = (a_n)$ is (strictly) decreasing.*

REMARK 4.7. Similar constructions and conclusions can be achieved in the James space J and in various equivalent renormings of c_0. This leads us to ask the following.

QUESTION. To what extent can the above construction and results be extended
(a) in c_0, and
(b) into other Banach spaces?
We also reiterate our earlier conjecture.

QUESTION. Does the nonexpansive-fpp for a closed bounded convex set in c_0 characterize the set being weak compact?

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