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The Knaster–Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications[1]

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1. Introduction

Sine [30] and Soardi [33] proved independently that the fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then hyperconvex metric spaces have been widely studied and many interesting results for nonexpansive mappings have been established within this framework, e.g., see Baillon [2], Goebel and Kirk [13], Khamsi et al. [19,20], Kirk [22,23], Lin and Sine [27], Sine [31,32] and others. More recently, Khasmi [18] established a hyperconvex version of the famous KKM-Fan principle due to Fan [8]. Kirk [22] has obtained a constructive fixed point theorem which arises from interval analysis in compact hyperconvex metric spaces; and also Kirk and Shin [24] have established a number of fixed point theorems for both condensing and nonexpansive mappings in hyperconvex spaces. In this paper, we first establish a characterization of the Knaster–Kuratowski and Mazurkiewicz principle in hyperconvex metric spaces which in turn leads to a characterization theorem for a family of subsets with the finite intersection property in such a setting. As applications

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we give hyperconvex versions of Fan's celebrated minimax principle and Fan's best approximation theorem for set-valued mappings. These in turn are applied to obtain formulations of the Browder–Fan fixed point theorem and the Schauder–Tychonoff fixed point theorem in hyperconvex metric spaces for set-valued mappings. Finally, existence theorems for saddle points, intersection theorems and Nash equilibria are also obtained. Our results unify and extend several of the results cited above.

The basic definition is due to Aronszajn and Panitchpakdi [1]. ($B(x,r)$ denotes the closed ball centered at $x\in X$ with radius $r\geq 0$.)

Definition 1.1. A metric space (X,d) is said to be a *hyperconvex* space if for any collection of points $\{x_\alpha\}$ of X and any collection $\{r_\alpha\}$ of nonnegative real numbers for which $d(x_\alpha,x_\beta)\leq r_\alpha + r_\beta$, it is the case that $\bigcap_\alpha B(x_\alpha,r_\alpha)\neq\emptyset$.

This definition can be seen as equivalently a binary intersection property plus metric convexity in the sense that for each given $x,y\in X$ and $\alpha\in[0,1]$, there exists $z\in X$ such that $d(x,z)=\alpha d(x,y)$ and $d(y,z)=(1-\alpha)d(x,y)$. The corresponding linear theory for hyperconvex spaces can be found in Lacey [25]. As Sine [32] points out, the nonlinear theory is still developing. Hyperconvex spaces can have quite strange aspects. For example, a hyperconvex subset need not be convex in even $\mathbb{R}^2$ (with the l_∞ norm). Also convex sets in linear spaces may fail to be hyperconvex (but for this one must go to at least $\mathbb{R}^3$, e.g., see Sine [32]). Some nice spaces such as Hilbert space fail to be hyperconvex. However hyperconvexity enjoys some properties similar to convexity and others much like compactness. More convincing analogies hold for hyperconvex sets which are ball intersections. The Nachbin–Kelley–Goodner and Hasumi Theorem in Lacey [25, p. 92] (see also Isbell [16], Nachbin [28] or Kelley [17]) says that a Banach space is hyperconvex if and only if it is linearly isometric to $C(K)$, where $C(K)$ is the space of all continuous real functions defined on some stonian space K (i.e., K is Hausdorff, compact and extremally disconnected). Thus, the space $l_\infty(I)$ for any set I and the space $L_\infty(\mu)$ for a finite measure μ are examples of hyperconvex spaces. Secondly, order intervals in L_∞ are hyperconvex, but weak compact convex sets of L_∞ need not be hyperconvex. Moreover, hyperconvex sets of L_∞ may not be convex (e.g., see Lin and Sine [27, p. 943]). On the other hand any hyperconvex space (indeed any metric space) embeds isometrically in some $l_\infty(I)$, e.g., see Lacey [25] or Khamsi [18, pp. 300–301].

The relationship between hyperconvex metric spaces and nonexpansive mappings is an important one as shown independently by the work of Sine [30] and Soardi [33]. On the other hand, we also know how important the famous Fan-KKM principle is in the study of nonlinear analysis, in particular for the study of topological fixed point theory, e.g., see [3,5–12,14,26,29,34–37] and references therein. In [18] Khamsi introduced a version of the KKM principle in hyperconvex spaces and as an application, he gave hyperconvex versions of Fan's best approximation theorem for single-valued mappings and the Schauder–Tychonoff fixed point theorem. The main thrust of this paper is in the same direction, i.e., our aim is to give a comprehensive study of KKM theory in hyperconvex spaces and its related applications to fixed point theorems, to characterize the KKM principle in hyperconvex spaces, to obtain Fan's minimax principle in hyperconvex spaces, the existence of saddle points, the intersection of sets and the existence of Nash equilibria in game theory. In order to study the KKM theory in hyperconvex spaces, we first recall some notation and basic facts about hyperconvex spaces which will be used later in the paper.

Definition 1.2. Let A be a bounded subset of a metric space (M,d). Then:
(1) $co(A)=\cap\{B\subset M: B$ is a closed ball in M such that $A\subset B\}$;
(2) $\mathscr{A}(M)=\{A\subset M: A=co(A)\}$, i.e., $A\in\mathscr{A}(M)$ if and only if A is an intersection of closed balls. In this case, we shall say that A is an *admissible* subset of M. We also note that if M is a hyperconvex space, then each admissible set in M is also hyperconvex.

For convenience we summarize the following facts.

Proposition 1.1. *Let M be a metric space. Then*:
(1) *There exists an index set I and a natural isometric embedding from M to $l_\infty(I)$.*
(2) *If M is hyperconvex, then it is complete.*
(3) *M is hyperconvex if and only if for each metric space N which contains M isometrically, there exists a nonexpansive retraction $r: N\to M$; i.e., r is nonexpansive and $r(x)=x$ for each $x\in M$. In particular, if N is a normed space, for any nonempty finite set of points $\{y_1,y_2,\ldots,y_n\subset M\}$, $r(conv\{y_1,\ldots,y_n\})\subset co\{y_1, y_2,\ldots,y_n\}$, where $co\{y_1,y_2,\ldots,y_n\}$ is given by Definition 1.2 above.*
(4) *M is hyperconvex if and only if for each metric space N which is contained metrically in any space D, and any nonexpansive mapping $T: N\to M$, there exist an extension $T^*: D\to M$ which is nonexpansive, i.e., $T(x)=T^*(x)$ for each $x\in N$.*

Proof. For example, see Proposition 1 of Khamsi [18, p. 300]. □

Let M be a metric space and consider the natural embedding into $l_\infty(I)$ given by statement (1) in Proposition 1.1. If $M_\infty := co(M)\in\mathscr{A}(l_\infty(I))$ clearly M_∞ then is a hyperconvex subset of $l_\infty(I)$, and M_∞ is also a convex subset of the linear space $l_\infty(I)$.

2. The KKM theory in hyperconvex metric spaces

Let X be a nonempty set. We denote by $\mathscr{F}(X)$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively. If A is a subset of a linear space E, the notation '$conv(A)$' always means the *convex hull* of A.

Let (M,d) be a metric space. Following Khamsi [20], a subset $S\subset M$ is said to be *finitely metrically closed* if for each $F\in\mathscr{F}(M)$, the set $co(F)\cap S$ is closed. Note that $co(F)$ is always defined and belongs to $\mathscr{A}(M)$. Thus if S is closed in M it is obviously finitely metrically closed. We also recall that a family $\{A_\alpha\}_{\alpha\in D}$ of M has the *finite intersection property* if the intersection of each of its nonempty finite subfamilies is not empty.

Definition 2.1. Let X be any nonempty set and let M be a metric space. A set-valued mapping $G:X\to 2^M\setminus\{\emptyset\}$ is said to be a *generalized metric KKM* mapping (GMKKM) if for each nonempty finite set $\{x_1,\dots,x_n\}\subset X$, there exists a set $\{y_1,\dots,y_n\}$ of points of M, not necessarily all different, such that for each subset $\{y_{i_1},\dots,y_{i_k}\}$ of $\{y_1,\dots,y_n\}$ we have

$$co\{y_{i_j}\colon j=1,\dots,k\}\subset\bigcup_{j=1}^{k}G(x_{i_j}).$$

As a special case of a generalized metric KKM mapping, we have the following definition of KKM mappings given essentially by Khamsi in [18].

Definition 2.2. Let X be a nonempty subset of a metric space M. Suppose $G:X\to 2^M$ is a set-valued mapping with nonempty values. Then G is said to be a *metric KKM* (MKKM) mapping if for each finite subset $F\in\mathscr{F}(X)$, $co(F)\subset\bigcup_{x\in F}G(x)$.

Remark 2.1. *It is clear that each metric KKM mapping is a generalized metric KKM mapping but in general the converse is not true. When X is a subset of a linear space M, if 'co' is replaced with 'conv', the usual 'convex hull', in a linear space M, then our definition of GMKKM becomes that of Chang and Zhang* [5] *and Yuan* [36] (*see also Bardaro and Ceppitelli* [3], *Fan* [12], *Granas* [14], *Lassonde* [26], *Park* [29], *Tan and Yuan* [34], *Tarafdar* [35] *and Yuan* [36] *for more recent developments in the study of KKM theory in topological vector spaces*).

Now, we give a characterization of the generalized metric KKM mapping principle in hyperconvex metric spaces.

Theorem 2.1. *Let X be a nonempty set and let M be a hyperconvex metric space. Suppose $G:X\to 2^M\setminus\{\emptyset\}$ has finitely metrically closed values. Then the family $\{G(x)\colon x\in X\}$ has the finite intersection property if and only if the mapping G is a generalized metric KKM mapping.*

Proof. *Necessity*: (Hyperconvexity is not needed for this implication.) If the family $\{G(x)\colon x\in X\}$ has the finite intersection property then for each finite subset $\{x_1,\dots,x_n\}\subset X$, $\bigcap_{i=1}^n G(x_i)\neq\emptyset$. Take any point $x^*\in\bigcap_{i=1}^n G(x_i)$ and set $y_i\equiv x^*$ for $i=1,\dots,n$. Then for any $1\le k\le n$ and any subsequence $y_{i_1},\dots,y_{i_k}$, it follows that $co(\{y_{i_j}:j=1,\dots,k\})=co(\{x^*\})=\{x^*\}\subset\cup_{j=1}^k G(x_{i_j})$.This proves that G is a GMKKM mapping.

Sufficiency: Suppose that $G:X\to 2^M\setminus\{\emptyset\}$ is a GMKKM mapping and suppose the family $\{G(x)\colon x\in X\}$ does not have the finite intersection property. Then there exists a nonempty finite set $\{x_1,\dots,x_n\}$ for which $\bigcap_{i=1}^n G(x_i)=\emptyset$. Since G is a GMKKM mapping there exist corresponding points $y_1,\dots,y_n$ of M such that for each subsequence $y_{i_1},\dots,y_{i_k}$, we have $co(\{y_{i_1},\dots,y_{i_k}\})\subset\bigcup_{j=1}^k G(x_{i_j})$. Since M is hyperconvex there exists a nonexpansive retraction $r:M_\infty\to M$. In particular, if we identify M with its isometric copy in the Banach space M_∞ then r maps the linear span L of points $\{y_1,\dots,y_n\}$ into M. Let $Y:=co(\{y_1,\dots,y_n\})$ and $S:=conv(\{y_1,\dots,y_n\})$.

Then $r(S) \subset M$ (indeed, $r(S) \subset Y$) and $r(x) = x$ for each $x \in M$. The assumption that $G(x)$ is finitely metrically closed for each $x \in X$ implies that $Y \cap G(x_i)$ is closed for $i = 1, \dots, n$. Note also that $Y \cap G(x_i) \neq \emptyset$ since, in particular, $y_i \in Y \cap G(x_i)$. However, $\bigcap_{i=1}^n G(x_i) = \emptyset$, so for each $s \in S$ there exists $i_s \in \{1, \dots, n\}$ such that $r(s) \notin Y \cap G(x_{i_s})$. Hence $dist(r(s),\ Y \cap G(x_{i_s})) > 0$. Therefore if the mapping $f : S \to [0, \infty)$ is defined by setting

$$f(s) := \sum_{i=1}^{n} dist(r(s),\ Y \cap G(x_i))$$

for each $s \in S$, it must be the case that $f(s) > 0$ for each $s \in S$ and also f is obviously continuous.

Now define a (single-valued) mapping $F : S \to S$ by setting

$$F(s) := \frac{1}{f(s)} \sum_{i=1}^{n} dist(r(s),\ Y \cap G(x_i)) y_i$$

for each $s \in S$. Then F is also continuous. Since S is a bounded closed and convex subset of the finite-dimensional space L, by Brouwer's fixed point theorem there exists $s_0 \in S$ such that $F(s_0) = s_0$; i.e.,

$$s_0 = F(s_0) = \frac{1}{f(s_0)} \sum_{i=1}^{n} dist(r(s_0),\ Y \cap G(x_i)) y_i. \tag{2.1}$$

If

$$I := \{i_1, \dots, i_k\} = \{i \in \{1, \dots, n\} : dist(r(s_0),\ Y \cap G(x_i)) > 0\}, \tag{2.2}$$

then $I \neq \emptyset$, and for each $i \in I$, $r(s_0) \notin Y \cap G(x_i)$. Note that since $r(S) \subseteq Y$ by Proposition 1.1 (3), $r(s_0) \in Y$. (Indeed, $s_0 \in conv\{y_1, \dots, y_n\}$. To see this, note that B is a closed ball centered at a point of M which contains the set $\{y_{i_j} : j = 1, \dots, n\}$. Since r is nonexpansive and leaves points of M fixed, it follows that $r(y) \in B$. This in turn implies $r(y) \in co(\{y_{i_1}, \dots, y_{i_k}\})$). Thus, it must be the case that $r(s_0) \notin G(x_i)$ for each $i \in I$, i.e.,

$$r(s_0) \notin \bigcup_{i \in I} G(x_i). \tag{2.3}$$

Then by definition of F, we have

$$s_0 = F(s_0) = \frac{1}{f(s_0)} \sum_{j=1}^{k} dist(r(s_0),\ Y \cap G(x_{i_j})) y_{i_j} \in conv(\{y_{i_1}, \dots, y_{i_k}\}).$$

This in turn implies $r(s_0) \in co(\{y_{i_1}, \dots, y_{i_k}\})$. Thus we are able to conclude that $r(s_0) \in co(\{y_{i_1}, \dots, y_{i_k}\}) \subseteq \bigcup_{j=1}^{k} G(x_{i_j})$ and this contradicts Eq. (2.3), completing the proof. □

As an application of Theorem 2.1 we have the following characterization giving the existence of a nonempty intersection for the values of a set-valued mapping in a hyperconvex metric space.

Theorem 2.2. *Let X be a non-empty set and M be a hyperconvex metric space. Suppose $G: X \to 2^M \backslash \{\emptyset\}$ is a set-valued mapping with nonempty closed values and suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized metric KKM mapping.*

Proof. *Necessity*: Since $\bigcap_{x \in X} G(x) \neq \emptyset$, it follows that the family $\{G(x): x \in X\}$ has the finite intersection property. Since $G(x)$ is closed for each $x \in X$ it is finitely metrically closed. Thus by Theorem 2.1, G is a generalized metric KKM mapping.

Sufficiency: Since G is a generalized metric KKM mapping, it follows by Theorem 2.1 that the family $\{G(x): x \in X\}$ has the finite intersection property. Rewriting this as $\{G(x) \cap (x_0): x \in X\}$ and noting $G(x_0)$ is compact, we have

$$\emptyset \neq \bigcap_{x \in X} G(x) \cap G(x_0) = G(x_0) \bigcap_{x \in X} G(x) = \bigcap_{x \in X} G(x).$$

This completes the proof. □

As a special cases of Theorem 2.1 we also have the following result which extends Theorem 3 of Khamsi [18, p. 303].

Corollary 2.3. *Let X be a nonempty subset of a hyperconvex metric space M. Suppose $G: X \to 2^M \backslash \{\emptyset\}$ is a metric KKM mapping with finitely metrically closed values. Then the family $\{G(x): x \in X\}$ has the finite intersection property.*

Definition 2.3. Let X be a nonempty set and let Y be a topological space. A mapping $G: X \to 2^Y$ is said to be *transfer closed valued* if for each $(x, y) \in X \times Y$ with $y \notin G(x)$, there exist $x' \in X$ and a nonempty open neighborhood $N(y)$ of y such that $y' \notin G(x')$ for all $y' \in N(y)$.

Definition 2.4. Let Y be a nonempty set and X a topological space. A mapping $F: X \to 2^Y$ is said to be *transfer open inversed valued* if for each $(x, y) \in X \times Y$ with $y \in F(x)$, there exists $y' \in Y$ and a nonempty open neighborhood $N(x)$ of x such that $y' \in F(z)$ for all $z \in N(x)$.

Remark 2.2. *Let X be a nonempty set and Y a topological space. Suppose $G: X \to 2^Y$. Then it is clear to see that G is transfer closed valued if and only if the mapping $F: Y \to 2^X$ defined by $F(y) = X \backslash G^{-1}(y)$ for each $y \in Y$ is transfer open inversed valued.*

The following simple example shows that set-valued mappings with transfer open inversed values may not be open inversed valued.

Let $X := [0, 1]$ and the set-valued mapping $F: X \to 2^X$ be defined by

$$F(x) = \begin{cases} [x, 1] & \text{if } x \text{ is rational}, \\ [0, 1] & \text{if } x \text{ is irrational}. \end{cases}$$

Then it is clear that F is transfer open inversed valued yet not open inversed valued.

Lemma 2.4. *Let X be a nonempty set and Y be a topological space. Suppose $F:X\to 2^Y\backslash\{\emptyset\}$. Then $\bigcap_{x\in X}F(x)=\bigcap_{x\in X}clF(x)$ if and only if F is transfer closed valued.*

Proof. This is Lemma 2.4 of Yuan [38, p. 137]. □

By Theorem 2.1 and Lemma 2.4, we have the following characterization of the finite intersection property for set-valued mappings.

Theorem 2.5. *Let X be a nonempty set and let M be a hyperconvex metric space. Suppose $G:X\to 2^M\backslash\{\emptyset\}$ is transfer closed valued and suppose there exists a finite subset X_0 of X such that $\bigcap_{x\in X_0}clG(x)$ is nonempty and compact. Then $\bigcap_{x\in X}G(x)$ is nonempty if and only if the mapping clG is a generalized metric KKM mapping.*

Proof. By Theorem 2.1 the family $\{clG(x):\ x\in X\}$ has the finite intersection property if and only if clG is a GMKKM mapping. Since there exists a nonempty finite subset X_0 of X such that $\bigcap_{x\in X_0}clG(x)$ is nonempty compact, it follows that $\bigcap_{x\in X}clG(x)$ is nonempty if and only if clG is a GMKKM mapping. As an application of Lemma 2.4 (by noting that G is transfer closed valued), it follows that $\bigcap_{x\in X}G(x)$ is nonempty if and only if clG is a generalized metric KKM mapping. □

Corollary 2.6. *Let X be a nonempty subset of a hyperconvex metric space M and let $G:X\to 2^M\backslash\{\emptyset\}$ be a transfer closed valued and metric KKM mapping. Suppose there exists a nonempty finite subset X_0 of X such that $\bigcap_{x\in X_0}clG(x)$ is compact. Then $\bigcap_{x\in X}G(x)=\bigcap_{x\in X}clG(x)\neq\emptyset$.*

Proof. Each metric KKM mapping is a generalized metric KKM mapping; thus, the conclusion follows from Theorem 2.5. □

Theorems 2.1, 2.2 and 2.5 tell us that in hyperconvex metric spaces the finite intersection property for a family of subsets is equivalent to a nonempty intersection property for a set-valued mapping which is a generalized metric KKM mapping. In order to study in which situations such generalized metric KKM mappings can be derived, we now introduce the following definitions.

Definition 2.5. Let $X\in\mathscr{F}(M)$ be a nonempty subset of a hyperconvex space M. Then:

(1) the function $f:X\to[-\infty,+\infty]$ is said to be *hyper quasi-convex* (resp., *concave*) if the set $\{x\in X:\ f(x)\le\lambda\}$ (resp., the set $\{x\in X:\ f(x)\ge\lambda\}$) for each $\lambda\in\mathbb{R}$ is an admissible set in M;

(2) the function $\psi(x,y):X\times X\to[-\infty,+\infty]$ is said to be *hyper diagonal quasi-convex* (resp., *concave*) in y if for each $A\in\mathscr{F}(X)$ and any $y_0\in co(A)$, $\psi(y_0,y_0)\le\max_{y_i\in A}\psi(y_0,y_i)$ (resp., $\psi(y_0,y_0)\ge\min_{y_i\in A}\psi(y_0,y_i)$);

(3) $\psi(x,y)$ is said to be *hyper γ-diagonal quasi-convex* (resp., *concave*) in y for some $\gamma\in[-\infty,+\infty]$ if for any $A\in\mathscr{F}(X)$ and $y_0\in co(A)$, $\gamma\le\max_{y_i\in A}\psi(y_0,y_i)$ (resp., $\gamma\ge\min_{y_i\in A}\psi(y_0,y_i)$).

Remark 2.3. *Note that the inequality '$\leq$' (resp., '$\geq$') could be replaced equivalently by the strict inequality '$<$' (resp., '$>$') in Definition* 2.5(1). *It is clear that if $\psi(x,y)$ is hyperdiagonal quasi-convex (resp., concave) in y, then $\psi(x,y)$ must be hyper γ-diagonally quasi-convex (resp., concave) in y, where $\gamma=\inf_{x\in X}\psi(x,x)$ (resp., $\gamma=\sup_{x\in X}\psi(x,x)$).*

Definition 2.6. Let $X\in\mathscr{F}(M)$ be a nonempty subset of a hyperconvex space metric space M and let $\gamma\in(-\infty,+\infty]$. Suppose $\psi:M\times X\to(-\infty,+\infty]$. Then ψ is said to be *hyper γ-generalized quasi-convex* (resp., *concave*) in M if for each nonempty finite subset $\{x_1,\dots,x_n\}\subset X$, there exists a set $\{y_1,\dots,y_n\}$ in M such that for each subset $\{y_{i_1},\dots,y_{i_k}\}$ of $\{y_1,\dots,y_n\}$ and any $x_0\in co\{x_{i_1},\dots,x_{i_k}\}$, $\gamma\leq\max_{1\leq j\leq k}\psi(x_0,y_{i_j})$ (resp., $\gamma\geq\min_{1\leq j\leq k}\psi(x_0,y_{i_j})$).

Remark 2.4. *When M is a linear metric space and X is a nonempty convex set in M, Definition* 2.6 *reduces to the corresponding definition given by Chang and Zhang* [5], *Zhou and Chen* [40] *and others. We now have the following result.*

Lemma 2.7. *Let X be a nonempty set, M a hyperconvex metric space and let $\gamma\in\mathbb{R}$ be a given real number. Suppose $\psi:M\times X\to(-\infty,+\infty]$. Then the following are equivalent*:

(1) *The mapping $G:X\to 2^M\backslash\{\emptyset\}$ defined by $G(x):=\{y\in M:\ \psi(y,x)\leq\gamma\}$ (resp., $G(x):=\{y\in M:\ \psi(y,x)\geq\gamma\}$) for each $x\in X$ is a generalized metric KKM mapping.*

(2) *The function ψ is hyper γ-generalized quasi-concave (resp., convex) in M.*

Proof. We only prove the conclusion for the case ψ is hyper γ-generalized quasi-concave. The convex case is proved similarly.

(1) $\Rightarrow$ (2): As $G:X\to 2^M\backslash\{\emptyset\}$ is a GMKKM mapping, for each finite set $\{x_1,\dots,x_n\}\subset X$, there exist $y_1,\dots,y_n$ in M such that for any subset $\{y_{i_1},\dots,y_{i_k}\}$ of $\{y_1,\dots,y_n\}$ and any $y_0\in co\{y_{i_1},\dots,y_{i_k}\}$, we have $y_0\in\bigcup_{j=1}^k G(x_{i_j})$. Hence there exists some $m\in\{1,\dots,k\}$ such that $y_0\in G(x_{i_m})$. It follows that $\psi(y_0,x_{i_m})\leq\gamma$. Therefore, $\min_{1\leq j\leq k}\psi(y_0,x_{i_j})\leq\gamma$, which implies that ψ is hyper γ-generalized quasi-concave in M.

(2) $\Rightarrow$ (1): As ψ is hyper γ-generalized quasi-concave in M, for any nonempty finite set $\{x_1,\dots,x_n\}\subset X$, there exist $y_1,\dots,y_n$ in M such that for any subset $\{y_{i_1},\dots,y_{i_k}\}$ of $\{y_1,\dots,y_n\}$ and $y_0\in co\{y_{i_1},\dots,y_{i_k}\}$ we have $\min_{1\leq j\leq k}\psi(y_0,x_{i_j})\leq\gamma$. Hence there exists some $m\in\{1,\dots,k\}$ such that $\psi(y_0,x_{i_m})\leq\gamma$. This implies that $y_0\in G(x_{i_m})$. By the arbitrariness of $y_0\in co\{y_{i_1},\dots,y_{i_k}\}$, it follows that $co\{y_{i_1},\dots,y_{i_k}\}\subset\bigcup_{j=1}^k G(x_{i_j})$ which implies that $G:X\to 2^M\backslash\{\emptyset\}$ is a generalized metric KKM mapping. $\square$

The following minimax inequality is a hyperconvex version of Fan's celebrated minimax inequality principle in [10].

Theorem 2.8. *Let $X\in\mathscr{A}(M)$ be a compact subset of a hyperconvex space M. Suppose $\psi:X\times X\to(-\infty,+\infty]$ satisfies*

(1) *for each fixed* $x \in X$, $y \mapsto \psi(x,y)$ *is hyper 0-generalized quasi-concave*; *and*
(2) *for each fixed* $y \in X$, $x \mapsto \psi(x,y)$ *is lower semicontinous.*
Then there exists $x_0 \in X$ *such that* $\sup_{y\in X} \psi(x_0,y) \leq 0$.

Proof. We define a mapping $G: X \to 2^X$ by

$$G(x) := \{y \in X:\ \psi(y,x) \leq 0\}$$

for each $x \in X$. Then it is clear that G is a GMKKM mapping by Lemma 2.7. Now by Theorem 2.2, it follows that $\bigcap_{x\in X} G(x) \neq \emptyset$. Take any point $x_0 \in \bigcap_{x\in X} G(x)$. Then $\sup_{y\in X} \psi(x_0,y) \leq 0$ completing the proof. □

As an application of the generalized metric MKKM principle established above we have the following version of Fan's best approximation in hyperconvex spaces for set-valued mappings. Note that here and henceforth we are adopting the notation $d(x,A) = dist(x,A)$ for $x \in X$ and $A \subset X$.

Theorem 2.9. *Let* M *be a hyperconvex metric space and* $X \in \mathscr{A}(M)$ *be a nonempty compact subset of* M. *Suppose* $F: X \to \mathscr{A}(M)$ *is a set-valued continuous mapping. Then there exists* $x_0 \in X$ *such that*

$$d(x_0, F(x_0)) = \inf_{x\in X} d(x, F(x_0)).$$

Proof. Define a mapping $G: X \to 2^M \setminus \{\emptyset\}$ by

$$G(x) := \{y \in X:\ d\ (y, F(y)) \leq d(x, F(y))\}$$

for each $x \in X$. As F is continuous, $G(x)$ is closed and nonempty for each $x \in X$. Now we claim that G is a metric KKM mapping. Suppose it were not. Then there exists a nonempty and finite subset $\{x_1,\ldots,x_n\}$ and $y \in co(\{x_i:\ i=1,\ldots,n\})$ such that

$$d(x_i, F(y)) < d(y, F(y))$$

for $i=1,\ldots,n$. Let $\varepsilon > 0$ be such that $d(x_i,F(y)) \leq d(y,F(y)) - \varepsilon$ for $i=1,\ldots,n$. Let $r = (y,F(y)) - \varepsilon$. Then for $i=1,\ldots,n$, $x_i \in F(y) + r$, where $F(y) + r := \bigcup\{B(a;r):\ a \in F(y)\}$. Note that since $F(y) \in \mathscr{A}(M)$, it follows that $F(y) + r \in \mathscr{A}(M)$ (e.g., see [32, p. 864]). Thus $co\{x_1,x_2,\ldots,x_n\} \subset F(y) + r$. This in turn implies $y \in F(y) + r$, that is, $(y,F(y)) \leq r = (y,F(y)) - \varepsilon$, which is impossible. Therefore, G must be a metric KKM-mapping.

Note that since X is compact, $\bigcap_{x\in X} G(x) \neq \emptyset$. Take any point $x_0 \in \bigcap_{x\in X} G(x)$. Then it is clear that $d(x_0,F(x_0)) \leq d(x,F(x_0))$ for all $x \in X$, which implies that $d(x_0,F(x_0)) = \inf_{x\in X} d(x,F(x_0))$. □

As a special case of our best approximation theorem we have the following best approximation result for single-valued mappings, obtained by Khamsi [18].

Corollary 2.10. *Let M be a hyperconvex metric space and let $X \in \mathscr{A}(m)$ be a nonempty compact subset of M. Suppose $F: X \to M$ is continuous. Then there exists $x_0 \in X$ such that*

$$d(x_0.F(x_0)) = \inf_{x \in X} d(x, F(x_0)).$$

3. Fixed point theorems in hyperconvex spaces

In this section, we shall use the generalized metric KKM principle and Fan's best approximation theorem in hyperconvex metric space established in Section 2 to derive hyperconvex versions of the Browder–Fan fixed point theorem and the Schauder–Tychonoff fixed point theorem for both set-valued and single-valued mappings.

Theorem 3.1. *Let M be a hyperconvex metric space and let $X \in \mathscr{A}(M)$ be compact. Suppose $F: X \to 2^X \setminus \{\phi\}$ satisfies*
(1) F is transfer open inversed valued;
(2) for each $x \in X$, $F(x)$ is admissible.
Then there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

Proof. Define $G: X \to 2^X$ by

$$G(x) = X \setminus F^{-1}(x)$$

for each $x \in X$. Since $F(x)$ is nonempty and transfer closed valued, $X = \bigcup_{y \in X} F^{-1}(y)$. Therefore $\bigcap_{x \in X} G(x) = \emptyset$. By Corollary 2.6, G must not be a metric KKM mapping; thus, there exists a subset $\{x_1, \ldots, x_n\} \subset X$ and a point $x_0 \in co\{x_1, \ldots, x_n\}$ such that $x_0 \notin G(x_i)$ for $i = 1, 2, \ldots, n$. It follows that $x_i \in F(x_0)$ for all $i = 1, \ldots, n$. Note that $F(x_0)$ is admissible so it follows that $x_0 \in co\{x_i,\ i = 1, \ldots, n\} \subset F(x_0)$. Therefore x_0 is a fixed point of F. □

As an application of Theorem 3.1 we have the following result which is Fan's geometric lemma 8 in hyperconvex metric space:

Theorem 3.2. *Let M be a hyperconvex metric space and let $X \in \mathscr{A}(M)$ be compact. Suppose C is a nonempty subset of $X \times X$ such that*
(1) for each fixed $x \in X$, the set $\{y \in X: (x, y) \notin C\}$ is either empty or admissible;
(2) for each fixed $y \in X$, the set $\{x \in X: (x, y) \in C\}$ is closed;
(3) for each $x \in X$, $(x,x) \in C$.
Then there exists $x_0 \in X$ such that $\{x_0\} \times X \subset C$.

Proof. In order to apply Theorem 3.1, we define $F: X \to 2^X$ by

$$F(x) := \{y \in X: (x, y) \notin C\}$$

for each $x \in X$. Then F satisfies all the hypotheses of Theorem 3.1. Since F has no fixed point, by condition (3) it follows that there must exist $x_0 \in X$ such that $F(x_0) = \emptyset$. Thus we must have $\{x_0\} \times X \subset C$ by the definition of F. □

As another application of Theorem 3.2 we have the following:

Theorem 3.3. *Let $X \in \mathscr{A}(M)$ be a nonempty compact subset of a hyperconvex metric space M. Suppose $F: X \to 2^X$ satisfies*
(1) for each $y \in X$, $X \backslash F^{-1}(y)$ is admissible;
(2) for each $x \in X$, the set $F(x)$ is closed;
(3) for each $x \in X$, $x \in F(x)$.
Then there exists $x_0 \in X$ such that $x_0 \in \bigcap_{x \in X} F(x)$.

Proof. In order to apply Theorem 3.2, let $C := \{(x, y) \in X \times X\colon x \in F(y)\}$. Then:
(1) for each $x \in X$, the set $\{y \in X\colon (x, y) \notin C\} = \{y \in X\colon x \notin F(y)\} = X \backslash F^{-1}(x)$ which is admissible by the condition (1);
(2) for each fixed $y \in X$, the set $\{x \in X\colon (x, y) \in C\} = F(y)$ is closed in X by a hypothesis (2);
(3) for each $x \in X$, it is clear that $(x,x) \in X$.

Thus all hypotheses of Theorem 3.2 are satisfied and it follows that there exists $x_0 \in X$ with $\{x_0\} \times X \subset C$. This implies that $x_0 \in \bigcap_{x \in X} F(x)$. □

Remark 3.4. *Theorems 3.2 and 3.3 are equivalent. To see this it suffices to prove Theorem 3.2 from Theorem 3.3. Define $F: X \to 2^X$ by $F(y) = \{x \in X : (x, y) \in C\}$ for each $y \in X$. Then the hypotheses of Theorem 3.2 ensure that F satisfies all the hypotheses of Theorem 3.3. By Theorem 3.3, $\bigcap_{x \in X} F(x) \neq \emptyset$. Take any $x_0 \in \bigcap_{x \in X} F(x)$ and $\{x_0\} \times X \subset C$ as required.*

The following existence theorem for maximal elements in hyperconvex spaces is an equivalent statement of Theorem 3.1.

Theorem 3.4. *Let $X \in \mathscr{A}(M)$ be a compact hyperconvex subset of a hyperconvex metric space M. Assume $F: X \to 2^X$ satisfies:*
(1) F is transfer open inversed valued;
(2) for each $x \in X$, $x \notin F(x)$ and $F(x)$ is admissible.
Then there exists $x_0 \in X$ such that $F(x_0) = \emptyset$.

As an application of Theorem 2.9 we have the following Schauder–Tychonoff fixed point theorem for set-valued mappings which includes a result of Khamsi [18].

Theorem 3.5. *Let $X \in \mathscr{F}(M)$ be a compact subset of a hyperconvex metric space M. Suppose $F: X \to \mathscr{A}(M)$ is a set-valued continuous mapping with nonempty closed values such that for each $x \in X$ with $x \notin F(x)$, there exists $z \in X$ such that $d(z, F(x)) < d(x, F(x))$. Then F has a fixed point in X.*

Proof. By Theorem 2.9 there exists $x_0 \in X$ such that

$$d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0)).$$

We claim that x_0 is a fixed point of F. Indeed, assume this were not true, i.e., $x_0 \notin F(x_0)$. Then it follows that $d(x_0, F(x_0)) > 0$. Then by assumption there exists $z_0 \in X$ such that $d(z_0, F(x_0)) < d(x_0, F(x_0))$. On the other hand, note that $d(z_0, F(x_0)) \geq d(x_0, F(x_0)) > 0$. This implies that $0 < d(z_0, F(x_0)) < d(z_0, F(x_0))$, which is impossible and thus x_0 must be a fixed point of F. □

As an immediate consequence of Theorem 3.5 we have the following fixed point theorem which is Corollary 3.5 of Kirk and Shin [24, p. 180].

Corollary 3.6. *Let $X \in \mathscr{F}(M)$ be a compact subset of a hyperconvex metric space M. Suppose $F : X \to \mathscr{A}(M) \setminus \{\phi\}$ has closed values and satisfies $F(x) \cap X \neq \emptyset$ for all $x \in X$. Then F has a fixed point in X.*

Proof. Note that $F(x) \cap X \neq \emptyset$ for each $x \in X$. It follows that for any $x \in X$ with $x \notin F(x)$, if we take $z \in F(x) \cap X$, then $d(z, F(x)) = 0 < d(x, F(x))$. Thus all hypotheses of Theorem 3.5 are satisfied and the conclusion follows. □

Remark 3.2. *Theorem* 3.5 *is a natural extension of Theorem* 6 *of Khamsi* [18].

4. Noncompact versions of the KKM principle and fixed point theorems in hyperconvex spaces

Motivated by the recent study of Kirk and Shin [15] we now establish noncompact versions of both the KKM principle and the fixed point theorem for set-valued mappings in hyperconvex spaces. These results are noncompact generalizations of the corresponding results given in Sections 2 and 3 above.

Let (M, d) be an arbitrary metric space and let μ denote the usual Kuratowksi measure of noncompactness on M. Thus for each nonempty bounded $A \subset M$:

$$\mu(A) = \inf \left\{ \varepsilon > 0\text{: there exists } n \text{ with } A \subset \bigcup_{i=1}^{n} A_i, \text{ where } diam(A_i) < \varepsilon \right\}.$$

(Recall that $diam(A_i) := \sup_{x,y \in A_i} d(x, y)$ for $i = 1, 2, \ldots, n$.)

We also need the following result which is Theorem 1 of Horvath [15, p. 403].

Lemma 4.1. *Let (M, d) be a complete metric space and let $\{F_i : i \in I\}$ be a family of nonempty closed subsets of M having the finite intersection property. If $\inf_{i \in I} \mu(F_i) = 0$, then $\bigcap_{i \in I} F_i$ is nonempty and compact.*

We now give the following noncompact version of the KKM principle for hyperconvex spaces which includes Theorem 2.5 above as a special case.

Theorem 4.2. *Let X be a nonempty set and M a hyperconvex metric space. Suppose $G: X \to 2^M \setminus \{\emptyset\}$ is transfer closed valued and $\inf_{x\in X} \mu(clG(x)) = 0$. Then $\bigcap_{x\in X} G(x)$ is nonempty if and only if the mapping clG is a generalized metric KKM mapping.*

Proof. If clG is a GMKKM mapping, it follows by Theorem 2.1 that the family $\{clG(x): x\in X\}$ has the finite intersection property. Note that $\inf_{x\in X} \mu(clG(x)) = 0$. This implies that $\bigcap_{x\in X} clG(x) \neq \emptyset$ by Lemma 4.1 and the same is true for the family $\{G(x): x\in X\}$ by Lemma 2.4. On the other hand, if $\bigcap_{x\in X} clG(x) \neq \emptyset$, it is, of course, a GMKKM mapping. □

As an application of Theorem 4.2 we have the following noncompact version of Fan's best approximation theorem for set-valued mappings in noncompact hyperconvex spaces. This extends Theorem 2.9.

Theorem 4.3. *Let M be a hyperconvex metric space and $X \in \mathscr{A}(M)$ be a nonempty subset of M. Suppose $F: X \to \mathscr{A}(M)$ is a set-valued continuous mapping such that*

$$\inf_{x\in X} \mu(\{y\in X: (y, F(y)) \leq (x, F(y))\}) = 0.$$

Then there exists $x_0 \in X$ such that $d(x_0, F(x_0)) = \inf_{x\in X} d(x, F(x_0))$.

Proof. By following the proof of Theorem 2.9, we know that the mapping $G: X \to 2^M \setminus \{\emptyset\}$ given by

$$G(x) := \{y\in X: d(y, F(y)) \leq d(x, F(y))\}$$

for each $x\in X$ is a GMKKM mapping with nonempty closed values as F is continuous. By the assumption we have $\inf_{x\in X} \mu(G(x)) = 0$. It follows by Theorem 4.2 that $\bigcap_{x\in X} G(x) \neq \emptyset$. Take any point $x_0 \in \bigcap_{x\in X} G(x)$. Then it is clear that $d(x_0, F(x_0)) \leq d(x, F(x_0))$, for all $x\in X$, which implies that $d(x_0, F(x_0)) = \inf_{x\in X} d(x, F(x_0))$. □

As an application of Theorem 4.3 we also have the following noncompact version of a fixed point theorem which extends Theorem 3.3 of Kirk and Shin [15].

Theorem 4.4. *Let $X \in \mathscr{F}(M)$ be a subset of a hyperconvex metric space M. Suppose $F: X \to \mathscr{A}(M)$ is a set-valued continuous mapping with nonempty closed values such that*
(1) $\inf_{x\in X} \mu(\{y\in X: (y, F(y)) \leq (x, F(y))\}) = 0$; *and*
(2) *for each $x\in X$ with $x \notin F(x)$, there exists $z\in X$ such that $d(z, F(x)) < d(x, F(x))$.*
Then F has a fixed point in X.

Proof. By Theorem 4.3, there exists $x_0 \in X$ such that

$$d(x_0, F(x_0)) = \inf_{x\in X} d(x, F(x_0)).$$

Then x_0 is a fixed point of F by the same proof of Theorem 3.5. □

5. Applications to saddle points and Nash equilibria in game theory

In this section we use our generalized metric KKM principle in hyperconvex spaces to give some applications to the existence for saddle points, intersection theorems and the existence of Nash equilibria in game theory.

Theorem 5.1. *Let $X \in \mathscr{A}(M)$ be a nonempty compact subset of a hyperconvex metric space M. Suppose $\psi : X \times X \to (-\infty, +\infty)$ satisfies*

(1) *for each fixed $y \in X$, $\psi(x,y)$ is lower semicontinuous in x; and for each fixed $x \in X$, $\psi(x,y)$ is hyper 0-generalized quasi-concave in y;*

(2) *for each fixed $x \in X$, $\psi(x,y)$ is upper semicontinuous in y; and for each fixed $x \in X$, $\psi(x,y)$ is hyper 0-generalized quasi-convex in x.*

Then ψ has at least one saddle point in $X \times X$, i.e., there exists $(x_0, y_0) \in X \times X$ such that

$$\max_{y \in X} \inf_{x \in X} \psi(x,y) = \psi(x_0, y_0) = \min_{x \in X} \sup_{y \in X} \psi(x,y) = 0.$$

Proof. By our assumptions, ψ satisfies all the hypotheses of Theorem 2.8 and it follows that there exists $x_0 \in X$ such that

$$\sup_{y \in X} \psi(x_0, y) \le 0. \tag{5.1}$$

Let $\lambda(x,y) = -\psi(x,y)$ for each $(x,y) \in X \times X$, then $\lambda : X \times X \to (-\infty, +\infty)$ is a mapping which also satisfies all hypotheses of Theorem 2.8. By Theorem 2.8 there exists $y_0 \in X$ such that

$$\sup_{x \in X} \lambda(x, y_0) \le 0. \tag{5.2}$$

By combining Eqs. (5.1) and (5.2)

$$\max_{y \in X} \min_{x \in X} \psi(x,y) = \psi(x_0, y_0) = \min_{x \in X} \sup_{y \in X} \psi(x,y) = 0,$$

thus completing the proof. □

In what follows, we establish an intersection theorem and the existence of Nash equilibria in hyperconvex metric spaces. We use the following notation: Given a Cartesian product $X = \prod_{i=1}^{n} X_i$ of topological spaces, let $X^j = \prod_{i \ne j} X_i$ and let $p_i : X \to X_i$ and $p^i : X \to X^i$ denote the natural projections; write $p_i(x) = x_i$ and $p^i(x) = x^i$. Given $x, y \in X$, we let $(y_i, x^i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

The next theorem is a hyperconvex version of Fan's intersection theorem in [9].

Theorem 5.2. *Let $X_1, \ldots, X_n$ be non-empty compact sets in hyperconvex metric spaces M_i such that $X_i \in \mathscr{A}(M_i)$ for $i = 1, \ldots, n$ and let $A_1, A_2, \ldots, A_n$ be n subsets of X (where, $X = \prod_{i=1}^{n} X_i$) such that:*

(1) *for each fixed* $x \in X$ *and for each* $i = 1, \ldots, n$, *the set*

$$A_i(x) := \{y \in X : (y_i, x^i) \in A_i\}$$

is a nonempty and admissible set;

(2) *for each fixed* $y \in X$ *and each* $i = 1, \ldots, n$, *the set*

$$A^i(y) := \{x \in X : (y_i, x^i) \in A_i\}$$

is open.

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define $G : X \to 2^X$ by

$$G(x) := X \setminus \bigcap_{i=1}^n A^i(x)$$

for each $x \in X$. Then G is not a metric KKM mapping. Indeed, by condition (1), for each $x \in X$ there exists $y \in X$ such that $(y_i, x^i) \in A_i$ for all $i = 1, 2, \ldots, n$. This implies $x \in \bigcap_{i=1}^n A^i(y)$, which means $X \subset \bigcup_{y \in X}(\bigcap_{i=1}^n A^i(y))$. Therefore $\bigcap_{y \in X} G(y) = \emptyset$ and by Corollary 2.3, G cannot be a metric KKM mapping. Thus there exists a nonempty finite subset $\{x(1), x(2), \ldots, x(m)\} \subset X$ for which there exists $w \in co\{x(1), \ldots, x(m)\}$ with the property that $w \notin G(x(j))$, i.e., $w \in \bigcap_{i=1}^n A^i(x(j))$ for all $j = 1, \ldots, m$. Note that the set $A_i(w)$ is admissible and $x(j) \in A_i(w)$ for all $j = 1, 2, \ldots, m$. It follows that $w \in co\{x(1), x(2), \ldots, x(m)\} \subset A_i(w)$. Thus we have $w \in A_i(w)$ for all $i = 1, 2, \ldots, n$, i.e., $w \in \bigcap_{i=1}^n A_i \neq \emptyset$. □

As an immediate corollary of Theorem 5.2 we have the following existence of Nash equilibria in hyperconvex spaces (see also [14, 26, 29, 36] and references therein for details of the study in topological vector spaces).

Theorem 5.3. *Let* $X_1, \ldots, X_n \in \mathscr{A}(M)$ *be nonempty compact sets in a hyperconvex metric space* M. *Let* $f_1, \ldots, f_n$ *be* n *real-valued continuous functions defined on* $X = \prod_{i=1}^n X_i$ *such that for each* $y \in X$ *and for each* $i = 1, \ldots, n$, *the function* $x_i \mapsto f_i(x_i, y^i)$ *is hyper quasi-concave on* X_i. *Then there exists a point* $y_0 \in X$ *such that* $f_i(y_0) = \max_{x_i \in X_i} f_i(x_i, y_0^i)$ *for* $i = 1, \ldots, n$.

Proof. For any given $\varepsilon > 0$, define for each $i = 1, 2, \ldots, n$,

$$A_i^\varepsilon = \left\{ z \in X : f_i(z) > \max_{x_i \in X_i} f_i(x_i, y^i) - \varepsilon \right\}.$$

It follows that for each fixed $x \in X$, there exists $y \in X$ such that $(y_i, x^i) \in A_i^\varepsilon$ for all $i = 1, \ldots, n$ and the set A_i^ε is nonempty and admissible, by the hyper quasi-concavity of the mapping $x_i \mapsto f_i(x_i, y^i)$ for each $y \in X$. Also, for each $y \in X$ and for each $i = 1, \ldots, n$, we note that $x \in A_i^\varepsilon(y)$ if and only if $(y_i, x^i) \in A_i^\varepsilon$. Further A_i^ε is open by the upper semicontinuity of the mapping $y^i \mapsto f_i(x_i, y^i)$ for each fixed $x \in X$. Thus all the hypotheses of Theorem 5.2 are satisfied and so there exists $x(\varepsilon)\in X$ such that $x(\varepsilon)\in\bigcap_{i=1}^{n}A_i^{\varepsilon}$. As X is compact, without loss of the generality, we may assume that $x(\varepsilon)\mapsto y_0\in X$ as $\varepsilon\mapsto 0^+$. The conclusion then follows by the continuity of f_i for $i=1,2,\dots,n$, and the proof is completed. □

Remark 5.1. *By following methodology and arguments similar to those used by Fan* [12] (*see also* [29] *and references therein for a more recent study*), *many variations and generalizations of the results given in this paper can be established. Since the ideas are basically the same as those used in Fan* [12], *we will not include further discussion here. Finally, we point out that by using the notion of noncompactness measure given in Section* 4, *we can also obtain noncompact versions of results in this section. As the arguments are not particularly difficult we will omit the details.*

Remark 5.2. *After this paper was written the authors became aware that in his forthcoming paper "Fixed point theorems in hyperconvex metric spaces" Professor Sehie Park obtains a theorem equivalent to our Theorem* 3.1 *using a different approach. Also, an idea in Park's paper can be expanded on to easily show that if* x_0 *in Theorem* 2.9 *is not a fixed point of* F *then it must be a boundary point of* X. *Similarly in Theorem* 4.2. *In Theorem* 3.5, *if* F *takes on compact values the conclusion still holds if for each* $x\in Bd(X)$ *with* $x\notin F(x)$ *there exists* $z\in X$ *for which* $d(z,F(x))<d(x,F(x))$. *An analogous conclusion holds in Theorem* 4.4.

References

[1] N. Aronszajn, P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956) 405–439.

[2] J.B. Baillon, Nonexpansive mapping and hyperconvex spaces, in: R.F. Brown (Ed.), Fixed Point Theory and Its Applications, vol. 72, Contemp. Maths., Amer. Math. Soc. Providence, RI 1988, pp. 11–19.

[3] C. Bardaro, R. Ceppitelli, Some further generalizations of Knaster–Kuratowski–Mazurkiewicz theorem and minimax inequalities, J. Math. Anal. Appl. 132 (1988) 484–490.

[4] F.E. Browder, The fixed point theory of multivalued mappings in topological vector spaces, Math. Ann. 177 (1968) 283–301.

[5] S.S. Chang, Y. Zhang, Generalized KKM theorem and variational inequalities, J. Math. Anal. Appl. 159 (1991) 208–223.

[6] X.P. Ding, K.-K. Tan, A minimax inequality with applications to existence of equilibrium points and fixed point theorems, Coll. Math. 63 (1992) 233–247.

[7] J. Dugundji, A. Granas, Fixed Point Theory, vol. 1, Warszawa, 1982.

[8] K. Fan, A generalization of Tychonoff's fixed point theorem, Math. Ann. 142 (1961) 305–310.

[9] K. Fan Sur un théorème minimax, C. R. Acad. Sci. Paris, Groupe I 259 (1964) 3925–3928.

[10] K. Fan, A minimax inequality and applications, in: O. Shisha (Ed.), Inequalities, vol. III, Academic Press, New York, 1972, pp. 103–113.

[11] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z. 112 (1969) 234–240.

[12] K. Fan, Some properties of convex sets related to fixed point theorems, Math. Ann. 266 (1964) 519–537.

[13] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[14] A. Granas, KKM-maps and their applications to nonlinear problems, in: R. Daniel Mauldin (Ed.), The Scottish Book: Mathematics from the Scottish Café, Birkhäuser, Boston, 1981, pp. 45–61.

[15] C. Horvath, Measure of non-compactness and multivalued mappings in complete metric topological vector spaces, J. Math. Anal. Appl. 108 (1985) 403–408.
[16] J.R. Isbell, Six theorems about injective metric spaces, Comm. Math. Helvetici 39 (1964) 65–76.
[17] J.L. Kelley, Banach spaces with the extension property, Trans. Amer. Math. Soc. 72 (1952) 323–326.
[18] M.A. Khamsi, KKM and Ky Fan theorems in hyperconvex metric spaces, J. Math. Anal. Appl. 204 (1996) 298–306.
[19] M.A. Khamsi, M. Lin, R.C. Sine, On the fixed points of commuting nonexpansive maps in hyperconvex spaces, J. Math. Anal. Appl. 168 (1992) 372–380.
[20] M.A. Khamsi, S. Reich, Nonexpansive mappings and semigroups in hyperconvex spaces, Math. Japon. 35 (1990) 467–471.
[21] B. Knaster, C. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensional simplexe, Fund. Math. 14 (1929) 132–137.
[22] W.A. Kirk, Fixed point theory for nonexpansive mappings, in: R.C. Sine (Ed.), II, Fixed Points and Nonexpansive Mappings vol. 18, Contemp. Math., Amer. Math. Soc., Providence, RI, 1983, pp. 121–140.
[23] W.A. Kirk, Continuous mappings in compact hyperconvex metric spaces, Numer. Funct. Anal. Optimiz. 17 (1996) 599–603.
[24] W.A. Kirk, S.S. Shin, Fixed point theorems in hyperconvex spaces, Houston J. Math. 23 (1997) 175–187.
[25] H.E. Lacey, The Isometric Theory of classical Banach Spaces, vol. 208, Springer, New York, 1974.
[26] M. Lassonde, On the use of KKM multifunctions in fixed point theory and related topics, J. Math. Anal. Appl. 97 (1983) 151–201.
[27] M. Lin, R.C. Sine, Retractions on the fixed point set of semigroups of nonexpansive maps in hyperconvex spaces, Nonlinear Anal. T.M.A. 15 (1990) 943–954.
[28] L. Nachbin, A theorem of the Hahn-Banach type for linear transformations, Trans. Amer. Math. Soc. 68 (1950) 28–46.
[29] S. Park, Some coincidence theorems on acyclic multifunctions and applications to KKM theory, in: K.K. Tan (Ed.), Fixed Point Theory and Applications, World Scientific, Singapore, 1992, 248–278.
[30] R.C. Sinc, On nonlinear contraction semigroups in Sup-norm spaces, Nonlinear Anal. T.M.A. 3 (1979) 885–890.
[31] R.C. Sine, Hyperconvexity and nonexpansive multifunctions, Trans. Amer. Math. Soc. 315 (1989) 755–767.
[32] R.C. Sine, Hyperconvexity and approximate fixed points, Nonlinear Anal. T.M.A. 13 (1989) 863–869.
[33] P.M. Soardi, Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73 (1979) 25–29.
[34] K.K. Tan, X.Z. Yuan, A minimax inequality with applications to existence of equilibrium points, Bull. Austral. Math. Soc. 47 (1993) 483–503.
[35] E. Tarafdar, A fixed point theorem equivalent to the Fan–Knaster–Kuratowski–Mazurkiewicz theorem, J. Math. Anal. Appl. 128 (1987) 475–479.
[36] X.Z. Yuan, The Knaster–Kuratowski and Mazurkiewicz theorem, Ky Fan minimax inequalities and fixed point theorems, Nonlinear World 2 (1995) 131–169.
[37] E. Zeidler, Nonlinear Functional Analysis and Its Applications, I: Fixed Point Theorems, Springer, New York, 1986.
[38] J.X. Zhou, G. Chen, Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, J. Math. Anal. Appl. 132 (1988) 213–225.