THE LERAY-SCHAUDER ALTERNATIVE FOR NONEXPANSIVE MAPS FROM THE BALL CHARACTERIZE HILBERT SPACE

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Abstract: *We show that for a nonexpansive map from the unit ball of a Hilbert space into the space the existence of a fixed point and the Leray-Schauder alternative are mutually exclusive alternatives, and that this characterizes Hilbert space. The equivalence of several formulations of the Leray-Schauder alternative is also established.*

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For a real Banach space X we denote by B_X and S_X the unit ball and unit sphere respectively:

$$B_X := \{x \in X : \|x\| \leq 1\} \quad \text{and} \quad S_X := \{x \in X : \|x\| = 1\} = \text{bdry}(B_X).$$

When the space is a Hilbert space we will denote it by H and the inner-product by $\langle \cdot, \cdot \rangle$.

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We say that a mapping $T : B_X \longrightarrow X$ satisfies the *Leray-Schauder alternative principle* if either

(i) T has a fixed point in B_X; that is, $\mathrm{Fix}(T) := \{x : Tx = x\} \neq \emptyset$, or

(ii) (*The Leray-Schauder alternative*) there exists an $x_0 \in S_X$ and a scalar $\lambda > 1$ such that $Tx_0 = \lambda x_0$.

As indicated we will refer to the second possibility as the Leray- Schauder alternative for T.

Typically, the Leray-Schauder altrnative principle for a particular type of mapping is established via a homotopy argument. See, for example, Granas [G], where it is shown that if U is a nonempty open subset of a complete metric space (X, d), $T_t : \overline{U} \to X$ for $t \in [0,1]$ is a homotopic family of maps which are

(a) uniformly contractive; that is, $d(T_t x, T_t y) \leq k d(x, y)$, for all $t \in [0,1]$ and some $k < 1$,

satisfy

(b) $d(T_t x, T_s x) \leq M|t - s|$ for all $t, s \in [0,1]$, $x \in \overline{U}$ and some $M > 0$

and for which

(c) $\mathrm{Fix}(T_t) \cap \mathrm{bdry}(U) = \emptyset$, for all $t \in [0,1]$,

then, if T_0 has a fixed point in U so does T_t for each $t \in (0,1]$.

Applying this to the homotopic family tT, where $t \in [0,1]$ and $T : B_X \to X$ is a strict contration, we readily deduce the Leray-Schauder alternative principle for such a T.

Unfortunately, examples of Marlène Frigon [F] show that such a homotopy argument is not possible when T is only required to be *nonexpansive*; that is, $\|Tx - Ty\| \leq \|x - y\|$, even when T maps B_{ℓ_2} into ℓ_2. Never-the-less we shall see that it is relatively straightforward to show that such maps do indeed satisfy the Leray-Schauder alternative principle.

THEOREM 1: *Let C be a nonempty closed bounded convex subset of the Hilbert space H, and let $T : C \longrightarrow H$ be a nonexpansive mapping, then there exists x_0, necessarily in bdry(C), such that the following are equivalent.*

(i) $\mathrm{Fix}(T) = \emptyset$.

(ii) $0 < \|Tx_0 - x_0\| = \mathrm{dist}(Tx_0, C)$.

(iii) $C \subset \{x \in H : \langle Tx_0 - x_0, x - x_0 \rangle \leq 0\}$.

(iv) $Tx_0 \notin \bigcup_{c \in C} B[c, \|c - x_0\|]$.

Before proving the theorem we note the following two well know lemmas, proofs of which are included only for completeness. In both lemmas, C is a nonempty closed bounded convex subset of a Hilbert space H.

LEMMA 1: *The closest point map $Proj_C$ from H onto C is nonexpansive and characterized by $Proj_C(x) \in C$ and $(c - Proj_C(x), x - Proj_C(x)) \leq 0$ for all $x \in H$ and $c \in C$.*

PROOF: The characterization follows from the observation that $\mathrm{Proj}_C(x)$ is the closest point of C to x if and only if $\mathrm{Proj}_C(x) \in C$ and there is a hyperplane through $\mathrm{Proj}_C(x)$ which separates C from $B[x, \|x - \mathrm{Proj}_C(x)\|]$, and that this hyperplane is necessarily the unique hyperplane supporting $B[x, \|x - \mathrm{Proj}_C(x)\|]$ at $\mathrm{Proj}_C(x)$; namely,

$$\{y \in H : (x - \mathrm{Proj}_C(x), y) = (x - \mathrm{Proj}_C(x), \mathrm{Proj}_C(x))\}.$$

That Proj_C is nonexpansive now follows from the calculation:

For every $x, y \in H$,

$$\begin{aligned}\|x - y\|^2 = {} & \|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\|^2 \\ & + \|(I - \mathrm{Proj}_C)x - (I - \mathrm{Proj}_C)y\|^2 \\ & + 2(x - \mathrm{Proj}_C(x), \mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)) \\ & + 2(y - \mathrm{Proj}_C(y), \mathrm{Proj}_C(y) - \mathrm{Proj}_C(x)),\end{aligned}$$

and that both the last two terms are positive, so that $\|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\| \leq \|x - y\|$.

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The next lemma follows from more general results due to Browder, Göhde, and Kirk, see the book by Goebel and Kirk [G-K] for more details on this and metric fixed point theory in general. The proof we give essentially relies on Hilbert spaces enjoying the Opial property.

LEMMA 2: *If $T : C \longrightarrow C$ is nonexpansive, then T has a fixed point in C.*

PROOF: Choose $x_0 \in C$, then for each $n \in \mathbf{N}$ the mapping $T_n x := (1 - 1/n)Tx + (1/n)x_0$ is a strict contraction mapping C into C, and so by the Banach contraction mapping principle has a fixed point x_n. This gives a sequence (x_n) with $\|x_n - Tx_n\| \to 0$. By passing to a subsequence if necessary, we may also assume that (x_n) converges weakly to some point $x \in C$.

Now,

$$\begin{aligned}\|x_n - Tx\|^2 &= ((x_n - x) + (x - Tx), (x_n - x) + (x - Tx)) \\ &= \|x_n - x\|^2 + \|x - Tx\|^2 + 2(x_n - x, x - Tx),\end{aligned}$$

so,

$$\begin{aligned}
\|x - Tx\|^2 &= \|x_n - Tx\|^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\
&\le (\|x_n - Tx_n\| + \|Tx_n - Tx\|)^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\
&\le (\|x_n - Tx_n\| + \|x_n - x\|)^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\
&= \|x_n - Tx_n\|(\|x_n - Tx_n\| + 2\|x_n - x\|) - 2\langle x_n - x, x - Tx\rangle \\
&\longrightarrow 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, $Tx = x$, establishing the result.

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PROOF OF THEOREM 1: To see that (i) implies (ii) we first observe that the mapping $\mathrm{Proj}_C \circ T$ is nonexpansive, by lemma 1., and maps C into C. Thus, by lemma 2., $\mathrm{Proj}_C \circ T$ has a fixed point $x_0 \in C$, with $Tx_0 \notin C$, otherwise we would have $x_0 = \mathrm{Proj}_C \circ Tx_0 = Tx_0$ contradicting (i). It now follows, using the definition of Proj_C, that $0 < \|Tx_0 - x_0\| = \|Tx_0 - \mathrm{Proj}_C \circ Tx_0\| = \mathrm{dist}(Tx_0, C)$, establishing (ii).

That (ii) is equivalent to (iii) follows immediately from lemma 1. Thus, it only remains to prove that (iii) implies (iv) implies (i).

(iii) $\Longrightarrow$ (iv): Suppose (iv) is not true, then there exists $c \in C$ with $Tx_0 \in B[c, \|c - x_0\|]$, so both c and Tx_0 lie on the positive side of the support hyperplane to $B[c, \|c - x_0\|]$ at x_0; namely $\{x \in H : \langle c - x_0, x\rangle = \langle c - x_0, x_0\rangle\}$. That is, $\langle Tx_0 - x_0, c - x_0\rangle \ge 0$, contradicting (iii).

(iv) $\Longrightarrow$ (i): Suppose (i) is not true; that is, there exists $c_0 \in C$ with $Tc_0 = c_0$. Then, $\|Tx_0 - c_0\| = \|Tx_0 - Tc_0\| \le \|x_0 - c_0\|$, so $Tx_0 \in B[c_0, \|c_0 - x_0\|]$, and so certainly $Tx_0 \in \bigcup_{c \in C} B[c, \|c - x_0\|]$, contradicting (iv).

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REMARK 1: The equivalence of conditions (ii) and (iii) of theorem 1. and their relation to (i) were essentially studied by Williamson [W], where (iii) was introduced as a generalized Leray-Schauder alternative.

REMARK 2: Condition (ii) of theorem 1. was considered by Browder and Petryshyn [B-P] and the equivalence of (i) and (iii) represents a Ky Fan [Ky F] type result for nonexpansive maps on non-compact domains.

REMARK 3: Condition (iv) of theorem 1. seems new and like (ii) can be formulated in any Banach space where it may play the role of a generalized Leray-Schauder alternative. In particular one is led to ask: in which spaces X are the following two conditions equivalent for a nonexpansive map $T : B_X \to X$?

(a) $\mathrm{Fix}(T) \neq \emptyset$.

(b) For all $x \in B_X$ we have $Tx \in \bigcup_{p \in B_X} B[p, \|p - x\|]$.

Clearly we always have (a) implies (b).

REMARK 4: When $C = B_H$ it is clear that (ii) of theorem 1. is equivalent to the Leray-Schauder alternative (the closest point map onto the unit ball is radial retraction). This observation combined with the above theorem yields the following.

COROLLARY 1: *If $T : B_H \longrightarrow H$ is a nonexpansive mapping, then T satisfies the Leray-Schauder alternative principle and the two alternatives are mutually exclusive.*

We conclude by showing that this dichotomy between the two alternatives of the Leray-Schauder alternative principle for nonexpansive mappings of the unit ball is only possible when the space is a Hilbert space, and so characterizes Hilbert spaces among all Banach spaces.

THEOREM 2: *A Banach space X is a Hilbert space if and only if for all nonexpansive mappings $T : B_X \longrightarrow X$ the two possibilities below are mutually exclusive.*

(i) $\mathrm{Fix}(T) \neq \emptyset$.

(ii) The Leray-Schauder alternative holds.

Proof: Necessity has been established in corollary 1. Thus, we need only establish sufficiency. To this end, suppose X is not a Hilbert space. Then, there exists points x_0 and p_0 in S_X such that every closest point of the line $\mathbf{R}p_0 := \{\lambda p_0 : \lambda \in \mathbf{R}\}$ to x_0 lies outside B_X. This follows, for example, from characterization (13.8) of Amir's book [A], or see [H].

Let y_0 be a closest point of $\mathbf{R}p_0$ to x_0, then we have, $y_0 = \lambda p_0$ for some λ with $|\lambda| > 1$. Replacing p_0 by $-p_0$ if necessary, we therefore have,

$$\begin{aligned} &y_0 = \lambda p_0, \quad \text{where } \lambda > 1, \text{ and} \\ &\|x_0 - y_0\| < \|x_0 - p_0\|. \end{aligned}$$

Denote by $\mathcal{L}$ the line through x_0 and y_0, which we can identify with a copy of $\mathbf{R}$, and define $T : \{x_0, p_0\} \subset B_X \longrightarrow \mathcal{L}$ by

$$T(x_0) := x_0 \quad \text{and} \quad T(p_0) := y_0.$$

Then, T is nonexpansive and, since $\mathbf{R}$ is an *injective* metric space (see for example [A-P]), T has a nonexpansive extension $\tilde{T}$ from B_X into $\mathcal{L} \subset X$.

Thus, $\tilde{T} : B_X \longrightarrow X$ is a nonexpansive mapping which has a fixed point, $\tilde{T}(x_0) = x_0$, and for which the Leray-Schauder alternative holds, $\tilde{T}(p_0) = y_0 = \lambda p_0$, with $\lambda > 1$. These two conditions are therefore not mutually exclusive in X.

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