UNIFORM NORMAL STRUCTURE IS EQUIVALENT TO THE JAGGI* UNIFORM FIXED POINT PROPERTY

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Abstract. Jaggi and Kassay proved that for reflexive Banach spaces X, normal structure is equivalent to the Jaggi fixed point property (i.e. all Jaggi-nonexpansive maps on closed, bounded, convex sets in X have a fixed point); which we note is equivalent to a natural variation: the Jaggi* fixed point property.

In the spirit of this result, we prove that for all Banach spaces X, uniform normal structure is equivalent to the Jaggi* uniform fixed point property: i.e. there exists a constant $\gamma_0 \in (1,\infty)$ such that for all $\gamma \in [1,\gamma_0)$, every Jaggi* γ-uniformly Lipschitzian map T on a closed, bounded, convex subset K of X has a fixed point.

Here, T is Jaggi* γ-uniformly Lipschitzian if for all T-invariant subsets G of K, for all $x \in \overline{\mathrm{co}}(G)$, for all $n \in \mathbb{N}$

$$\sup_{z\in G} \|T^n x - T^n z\| \leq \gamma \sup_{z\in G} \|x - z\| \ .$$

1. Introduction

In 1965 W.A. Kirk [?] proved that in every reflexive Banach space, normal structure implies the fixed point property: i.e., every nonexpansive map on a non-empty, closed, bounded, convex (c.b.c.) set $C \subseteq X$ has a fixed point. (See the preliminaries for the definition of *normal structure* and *nonexpansive map.*)

Building on these ideas, D.S. Jaggi [?] and G. Kassay [?] proved that a *reflexive* Banach space $(X, \|\cdot\|)$ has *normal structure* if and only if $(X, \|\cdot\|)$ has *the Jaggi fixed point property*: i.e., all Jaggi nonexpansive maps T on c.b.c. sets C in X have a fixed point. (Jaggi proved necessity, and later Kassay proved sufficiency.) Here, T is Jaggi nonexpansive precisely if for all T-invariant c.b.c. subsets E of C, for every $x \in E$,

$$\sup_{y\in E} \|Tx - Ty\| \leq \sup_{y\in E} \|x - y\| \ .$$

Further, we note below that these properties are equivalent to a natural variation: $(X, \|\cdot\|)$ has the Jaggi* fixed point property: i.e. all Jaggi* nonexpansive maps T on c.b.c. sets C in X have a fixed point. Here, T is Jaggi* nonexpansive precisely

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if for all T-invariant subsets G of C, for every $x \in \overline{\mathrm{co}}(G)$,

$$\sup_{z \in G} \|Tx - Tz\| \leq \sup_{z \in G} \|x - z\| \ .$$

Theorem 1.1 (D.S. Jaggi [?], G. Kassay [?]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space. The the following are equivalent.*

(1) *The space $(X, \|\cdot\|)$ has normal structure.*
(2) *The space $(X, \|\cdot\|)$ has the Jaggi fixed point property.*
(3) *The space $(X, \|\cdot\|)$ has the Jaggi* fixed point property.*

Proof. [(1) $\Longrightarrow$ (2).] This is due to Jaggi [?].
[(2) $\Longrightarrow$ (3).] Let T be a *Jaggi* nonexpansive map* on a c.b.c. non-empty subset C of X. Since T is necessarily also a *Jaggi nonexpansive map*, T has a fixed point.
[(3) $\Longrightarrow$ (1).] This is due to Kassay [?]. Kassay assumes that $(X, \|\cdot\|)$ fails to have normal structure and builds a c.b.c. set $C \subseteq X$ and a *Jaggi nonexpansive map* $T : C \longrightarrow C$ that is fixed point free. It is easy to check that Kassay's map is also *Jaggi* nonexpansive.* □

In the spirit of this result, we prove that for all Banach spaces $(X, \|\cdot\|)$, uniform normal structure is equivalent to the Jaggi* uniform fixed point property: i.e. there exists a constant $\gamma_0 \in (1, \infty)$ such that for all $\gamma \in [1, \gamma_0)$, every Jaggi* γ-uniformly Lipschitzian map T on a closed, bounded, convex subset K of X has a fixed point.

Here, T is Jaggi* γ-uniformly Lipschitzian if for all T-invariant subsets G of K, for all $x \in \overline{\mathrm{co}}(G)$, for all $n \in \mathbb{N}$,

$$\sup_{z \in G} \|T^n x - T^n z\| \leq \gamma \sup_{z \in G} \|x - z\| \ .$$

2. Preliminaries

As usual, we will denote the set of all positive integers by $\mathbb{N}$, while $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Also, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{I}$ denote the *real numbers*, *rational numbers* and *irrational numbers*, respectively.

Definition 2.1. We say that a Banach space $(X, \|\cdot\|)$ has *normal structure* if for all c.b.c. subsets C of X with $\mathrm{diam}(C) > 0$, we have that

$$\mathrm{rad}(C) < \mathrm{diam}(C) \ .$$

The *radius* of C, $\mathrm{rad}(C)$, is defined by

$$\mathrm{rad}(C) := \inf_{x \in C} \sup_{y \in C} \|x - y\| \ ,$$

and the *diameter* of C, is given by

$$\mathrm{diam}(C) := \sup_{x, y \in C} \|x - y\| \ .$$

Definition 2.2. We say that a Banach space $(X, \|\cdot\|)$ has *the fixed point property* if for all non-empty c.b.c. subsets C of X, for every *nonexpansive* mapping $T : C \longrightarrow C$, T has a fixed point.

The map T is *nonexpansive* precisely if for all $x, y \in X$,

$$\|T\,x - T\,y\| \leq \|x - y\| \ .$$

In the introduction, we gave the definitions of *the Jaggi fixed point property* and the *the Jaggi* fixed point property*, and the corresponding notions of a Jaggi (respectively, Jaggi*) nonexpansive map T on a closed, bounded, convex subset K of X.

Let's consider the example in Remark 2.3 of Kassay [**?**] of a Jaggi nonexpansive mapping that is *not* everywhere continuous, and thus is *not nonexpansive*. Let us now show that this example is also a *Jaggi* nonexpansive mapping*.

Example 2.3. Kassay [**?**] gave the following example of a Jaggi nonexpansive mapping. Let $(X, \|\cdot\|) := (\mathbb{R}, |\cdot|)$. Let $K := [0,1]$. This interval is a c.b.c. subset of $(X, \|\cdot\|)$. Define $S : K \longrightarrow K$ by

$$\begin{aligned} S\,x &:= 0 \ , \text{ if } x \in \mathbb{Q} \cap [0,1] \ ; \text{ and} \\ S\,x &:= \frac{x}{2} \ , \text{ if } x \in \mathbb{I} \cap [0,1] \ . \end{aligned}$$

Fix a non-empty set $G \subseteq K$ with $S(G) \subseteq G$. There exists $r \in G$. If $r \in \mathbb{Q}$, then $0 = S\,r \in G$. If $r \in \mathbb{I}$, then $r/2^k \in G$, for all $k \in \mathbb{N}_0$. Thus, in all cases, $[0,r] \subseteq \overline{\text{co}}(G)$; and so

$$\overline{\text{co}}(G) = [0,s] \ , \text{ where } s := \sup(G) \ .$$

Fix $x \in \overline{\text{co}}(G)$. Next fix $z \in G$.

$$|S\,x - S\,z| \leq \max\{S\,x, S\,z\} \leq \frac{s}{2} \ ,$$

and therefore

$$\sup_{z \in G} |S\,x - S\,z| \leq \frac{s}{2} \ .$$

On the other hand,

$$\sup_{z \in G} |x - z| = \sup_{z \in \overline{\text{co}}(G)} |x - z| \geq \text{rad}(\overline{\text{co}}(G)) = \text{rad}([0,s]) = \frac{s}{2} \ .$$

Thus, S is a *Jaggi* nonexpansive mapping*.

3. Uniform Normal Structure

Definition 3.1. Let $(X, \|\cdot\|)$ be a Banach space. We say that $(X, \|\cdot\|)$ has *uniform normal structure* if there exists $k \in (0,1)$ such that for all $C \subseteq X$ with C *closed, bounded and convex* (c.b.c.),

$$\text{rad}(C) \leq k \,\text{diam}(C) \ .$$

Definition 3.2. In any Banach space $(X, \|\cdot\|)$, we define $N(X) = N(X, \|\cdot\|)$ by

$$N(X) := \sup \left\{ \frac{\text{rad}(C)}{\text{diam}(C)} : C \text{ is a c.b.c. subset of } X \text{ with } \text{diam}(C) > 0 \right\} \ .$$

We see that a Banach space $(X, \|\cdot\|)$ has *uniform normal structure* if and only if $N(X) < 1$.

On the other hand, a Banach space $(X, \|\cdot\|)$ *fails* to have *uniform normal structure* if and only if for all $k \in (0,1)$, there exists a c.b.c. subset $C = C(k)$ of X such that

$$\text{rad}(C) > k \,\text{diam}(C) \ .$$

Consequently, a Banach space $(X, \|\cdot\|)$ *fails uniform normal structure* if and only if for every $j \in \mathbb{N}$, there exists a c.b.c. subset C_j of X such that

$$\mathrm{rad}\,(C_j) > \frac{j}{j+1}\,\mathrm{diam}\,(C_j) \ .$$

In 1948 M.S. Brodskiĭ and D.P. Mil'man [**?**] proved that a Banach space $(X, \|\cdot\|)$ *fails* to have *normal structure* (Definition ...) if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\Delta := \mathrm{diam}\{x_n : n \in \mathbb{N}\} > 0$, and for all $n \in \mathbb{N}$,

$$\mathrm{dist}(x_{n+1}, \mathrm{co}\{x_1, \ldots, x_n\}) \geq \Delta \left(1 - \frac{1}{2\,n}\right) \ .$$

G. Kassay [**?**] used this theorem to prove that if a Banach space $(X, \|\cdot\|)$ has *the Jaggi fixed point property*, then $(X, \|\cdot\|)$ has *normal structure*.

The following proposition is an analogue of the necessity part of Brodskiĭ and Mil'man's theorem, that will help us prove a "uniform" anologue of Kassay's theorem (Theorem 4.3 below).

Proposition 3.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let C be a closed, bounded, convex subset of X with $d_0 := \mathrm{diam}(C) > 0$. Further suppose that $k \in (0, 1)$ is such that*

$$\mathrm{rad}(C) > k\,d_0 \ .$$

Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements in C such that for the set

$$D := \overline{\mathrm{co}}\{z_n : n \in \mathbb{N}\} \ ,$$

we have that for all $x \in D$, for every $t \in \mathbb{N}$,

$$\sup_{y\in\{z_n : n\geq t\}} \|x - y\| > k\,\mathrm{diam}(D) \ .$$

Note that in the above proposition, $\mathrm{diam}(D)$ is necessarily positive.

Proof. Our set $C \neq \emptyset$. Choose $x_1 \in C$.

$$\mathrm{rad}(x_1, C) \geq \mathrm{rad}(C) > k\,d_0 \ .$$

Thus, there exists $x_2 \in C$ with

$$\|x_2 - x_1\| > k\,d_0 \ .$$

Consider $D_2 := \mathrm{co}\{x_1, x_2\} \subseteq C$. The set D_2 is totally bounded. Let $\varepsilon > 0$ be defined by

$$\varepsilon := \frac{\mathrm{rad}(C) - k\,d_0}{4} \ ,$$

and let $\{u_1^{(2)}, \ldots, u_{\nu_2}^{(2)}\} \subseteq D_2$ be an ε-net for D_2; i.e., for all $x \in D_2$, there exists $j \in \{1, \ldots, \nu_2\}$ such that

$$\|x - u_j^{(2)}\| < \varepsilon \ .$$

Fix an arbitrary $j \in \{1, \ldots, \nu_2\}$.

$$\mathrm{rad}(u_j^{(2)}, C) \geq \mathrm{rad}(C) > k\,d_0 \ ,$$

and so there exists $x_j^{(2)} \in C$ with

$$\|x_j^{(2)} - u_j^{(2)}\| > \frac{k\,d_0 + \mathrm{rad}(C)}{2} =: \delta \ .$$

Fix an arbitrary $x \in D_2$. Then there exists $j \in \{1, \dots, \nu_2\}$ such that

$$\|x - u_j^{(2)}\| < \varepsilon \ ,$$

and consequently,

$$\begin{aligned} \|x_j^{(2)} - x\| &= \|x_j^{(2)} - u_j^{(2)} + u_j^{(2)} - x\| \\ &\geq \|x_j^{(2)} - u_j^{(2)}\| - \|u_j^{(2)} - x\| \\ &> \delta - \varepsilon = \frac{k\,d_0 + \mathrm{rad}(C)}{2} - \frac{\mathrm{rad}(C) - k\,d_0}{4} \\ &> \frac{k\,d_0 + \mathrm{rad}(C)}{2} - \frac{\mathrm{rad}(C) - k\,d_0}{2} \\ &= k\,d_0 \ . \end{aligned}$$

Thus, for all $x \in D_2 := \mathrm{co}\{x_1, x_2\}$, there exists $j \in \{1, \dots, \nu_2\}$ such that

$$\|x_j^{(2)} - x\| > \delta - \varepsilon > k\,d_0 \ .$$

At this stage, we delete repeated elements in the set $\left\{x_1^{(2)}, \dots, x_{\nu_2}^{(2)}\right\}$, if necessary. Next, we let $D_3 := \mathrm{co}\left\{x_1, x_2, x_1^{(2)}, \dots, x_{\nu_2}^{(2)}\right\} \subseteq C$. The set D_3 is totally bounded. Repeating the above argument, we can construct a finite sequence $\left(x_1^{(3)}, \dots, x_{\nu_3}^{(3)}\right)$ of distinct elements in C, such that for all $x \in D_3$, there exists $j \in \{1, \dots, \nu_3\}$ satisfying

$$\|x_j^{(3)} - x\| > \delta - \varepsilon > k\,d_0 \ .$$

Let $x_1^{(1)} := x_2$ and $\nu_1 := 1$. Continuing inductively, we produce a sequence $(\nu_m)_{m\in\mathbb{N}}$ of positive integers and a sequence

$$(z_n)_{n\in\mathbb{N}} = \left(x_1, x_2, x_1^{(2)}, \dots, x_{\nu_2}^{(2)}, x_1^{(3)}, \dots, x_{\nu_3}^{(3)}, \dots, x_1^{(m)}, \dots, x_{\nu_m}^{(m)}, \dots\right)$$

of distinct elements in C such that the following holds.

(♠) [For all $m \in \mathbb{N}\{1\}$, for every element x of

$$D_m := \mathrm{co}\left\{x_1, x_2, x_1^{(2)}, \dots, x_{\nu_2}^{(2)}, \dots, x_1^{(m-1)}, \dots, x_{\nu_{m-1}}^{(m-1)}\right\} \ ,$$

there exists $j \in \{1, \dots, \nu_m\}$ satisfying

$$\|x_j^{(m)} - x\| > \delta - \varepsilon > k\,d_0 \ .]$$

Define

$$D := \overline{\mathrm{co}}\{z_n : n \in \mathbb{N}\} \subseteq C \ .$$

Fix $x \in D$ and $t \in \mathbb{N}$, arbitrary. Next fix $\eta > 0$. Clearly, there exists $k \in \mathbb{N}\backslash\{1\}$, with $2 + \nu_2 + \cdots + \nu_k > t$, and there exists

$$w \in D_{k+1} := \mathrm{co}\left\{x_1, x_2, x_1^{(2)}, \dots, x_{\nu_2}^{(2)}, \dots, x_1^{(k)}, \dots, x_{\nu_k}^{(k)}\right\}$$

such that

$$\|x - w\| < \eta \ .$$

Hence,

$$\begin{aligned}\sup_{y\in\{z_n:n\geq t\}} \|x-y\| &\geq \sup_{y\in\{z_n:n\geq t\}} (\|w-y\|-\|x-w\|) \\ &\geq \sup_{y\in\{z_n:n\geq t\}} (\|w-y\|-\eta) \\ &= \left(\sup_{y\in\{z_n:n\geq t\}} \|w-y\|\right)-\eta\ .\end{aligned}$$

By condition (♠) above, there exists $j \in \{1, \ldots, \nu_{k+1}\}$ such that

$$\|x_j^{(k+1)} - w\| > \delta - \varepsilon > k\, d_0\ .$$

Moreover, since $2 + \nu_2 + \cdots + \nu_k > t$, we see that $x_j^{(k+1)} \in \{z_n : n \geq t\}$. Thus,

$$\sup_{y\in\{z_n:n\geq t\}} \|x-y\| \geq \|w - x_j^{(k+1)}\| - \eta > \delta - \varepsilon - \eta\ .$$

But $\eta > 0$ is arbitrary. Therefore,

$$\sup_{y\in\{z_n:n\geq t\}} \|x-y\| \geq \delta - \varepsilon > k\, d_0 = k\, \mathrm{diam}(C) \geq k\, \mathrm{diam}(D)\ .$$

In summary, we have proven that there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements in C such that for the set

$$D := \overline{\mathrm{co}}\{z_n : n \in \mathbb{N}\}\ ,$$

we have that for all $x \in D$, for every $t \in \mathbb{N}$,

$$\sup_{y\in\{z_n:n\geq t\}} \|x-y\| > k\, \mathrm{diam}(D)\ .$$

□

4. The Jaggi* Uniform Fixed Point Property implies Uniform Normal Structure

Definition 4.1. We say that a Banach space $(X, \|\cdot\|)$ has *the uniform fixed point property* if there exists a constant $\gamma_0 \in (1, \infty)$ such that for all $\gamma \in [1, \gamma_0)$, every γ-uniformly Lipschitzian map T on a closed, bounded, convex subset K of X has a fixed point.

Here, T is γ-uniformly Lipschitzian if for all $x, z \in K$, for all $n \in \mathbb{N}$

$$\|T^n x - T^n z\| \leq \gamma\, \|x - z\|\ .$$

We note that there is another property in the literature, not closely related to this one, also called *the uniform fixed point property.* See, for example, U. Kohlenbach and L. Leuştean [**?**].

Let us now introduce a stronger property than that in Definition 4.1 above.

Definition 4.2. We say that a Banach space $(X, \|\cdot\|)$ has *the Jaggi* uniform fixed point property* if there exists a constant $\gamma_0 \in (1, \infty)$ such that for all $\gamma \in [1, \gamma_0)$, every Jaggi* γ-uniformly Lipschitzian map T on a closed, bounded, convex subset K of X has a fixed point.

Here, T is Jaggi* γ-uniformly Lipschitzian if for all T-invariant subsets G of K, for all $x \in \overline{\mathrm{co}}(G)$, for all $n \in \mathbb{N}$

$$\sup_{z\in G} \|T^n x - T^n z\| \leq \gamma \sup_{z\in G} \|x - z\|\ .$$

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a Banach space with the Jaggi* uniform fixed point property. Then $(X, \|\cdot\|)$ has uniform normal structure.*

Proof. Suppose $(X, \|\cdot\|)$ is a Banach space that *fails* to have *uniform normal structure.* So, there exists a sequence of c.b.c. subsets of X, $(C_j)_{j\in\mathbb{N}}$, such that for all $j \in \mathbb{N}$,

$$\operatorname{rad}(C_j) > \frac{j}{j+1} \operatorname{diam}(C_j) \ .$$

By Proposition 3.3, for all $j \in \mathbb{N}$, there exists a sequence $\left(z_n^{(j)}\right)_{n\in\mathbb{N}}$ of distinct elements in C_j such that for

$$D_j := \overline{\operatorname{co}}\left\{z_n^{(j)} : n \in \mathbb{N}\right\} \ ,$$

we have that for all $x \in D_j$, for every $t \in \mathbb{N}$,

$$\sup_{y\in\{z_n^{(j)}:n\geq t\}} \|x - y\| > \frac{j}{j+1} \operatorname{diam}(D_j) \ .$$

We wish to show that for all $\gamma \in (1, \infty)$, there exists $\delta = \delta_\gamma \in [1, \gamma)$, there exists a c.b.c. non-empty subset $K = K_\gamma$ of X and there exists a Jaggi* δ-uniformly Lipschitzian map $T = T_\gamma$ on K, such that T is *fixed point free.*

Equivalently, we aim to show that for all $j \in \mathbb{N}$, there exists a c.b.c. non-empty set $E_j \subseteq X$ and there exists a Jaggi* $((j+1)/j)$-uniformly Lipschitzian map $U_j : E_j \longrightarrow E_j$, such that U_j is *fixed point free.*

Indeed, fix an arbitrary $j \in \mathbb{N}$. We define the c.b.c. non-empty subset E_j of X by

$$E_j := D_j := \overline{\operatorname{co}}\left\{z_n^{(j)} : n \in \mathbb{N}\right\} \ .$$

Note that, by construction, $z_n^{(j)} \neq z_m^{(j)}$, for all $n \neq m$.

Let's now define the map $U_j : D_j \longrightarrow D_j$ by

$$\begin{aligned} U_j\left(z_n^{(j)}\right) &:= z_{n+1}^{(j)} \ , \text{ for all } n \in \mathbb{N} \ ; \text{ and} \\ U_j(u) &:= z_1^{(j)} \ , \text{ for all } u \in D_j \backslash \left\{z_n^{(j)} : n \in \mathbb{N}\right\} \ . \end{aligned}$$

Clearly, U_j is fixed point free. It only remains to show that U_j is a Jaggi* $((j+1)/j)$-uniformly Lipschitzian map.

Fix an arbitrary U_j-invariant subset G of D_j. Since $G \neq \emptyset$, there exists some $g \in G$. If $g = z_t^{(j)}$, for some $t \in \mathbb{N}$, then $\left\{z_n^{(j)} : n \geq t\right\} \subseteq G$. On the other hand, if $g \in D_j \backslash \left\{z_n^{(j)} : n \in \mathbb{N}\right\}$, then $\left\{z_n^{(j)} : n \in \mathbb{N}\right\} \subseteq G$. Consequently, there exists $s \in \mathbb{N}$ such that

$$\left\{z_n^{(j)} : n \geq s\right\} \subseteq G \ .$$

Fix an arbitrary $x \in \overline{\operatorname{co}}(G) \subseteq D_j$ and fix $N \in \mathbb{N}$. Then

$$\begin{aligned} \sup_{y\in G} \|U_j^N x - U_j^N y\| &\leq \operatorname{diam}(D_j) \\ &< \frac{j+1}{j} \sup_{y\in\{z_n^{(j)}:n\geq s\}} \|x - y\| \\ &\leq \frac{j+1}{j} \sup_{y\in G} \|x - y\| \ . \end{aligned}$$

□

5. Uniform Normal Structure implies the Jaggi* Uniform Fixed Point Property

The converse of Theorem 4.3 is also true. Let's first state a lemma that we will use to prove this.

Lemma 5.1 (E. Casini and E. Maluta [?], Lemma 3.1)**.** *Let $(X, \|\cdot\|)$ be a Banach space with uniform normal structure.*

For all bounded sequences $\overrightarrow{x} = (x_n)_{n\in\mathbb{N}}$ in X, there exists

$$z \in \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\{x_n : n \geq m\} \ ,$$

satisfying

1. $\mathrm{arad}(\overrightarrow{x}, z) \leq N(X)\ \mathrm{adiam}(\overrightarrow{x})$*; and*
2. *for all $y \in X$, $\|z - y\| \leq\ \mathrm{arad}(\overrightarrow{x}, y)$.*

In this lemma (and henceforth in our paper), *the asymptotic radius* of $\overrightarrow{x}$ *about* y, $\mathrm{arad}(\overrightarrow{x}, y)$, is defined by

$$\mathrm{arad}(\overrightarrow{x}, y) := \limsup_{n\in\mathbb{N}}\ \|x_n - y\| \ ,$$

and *the asymptotic diameter* of $\overrightarrow{x}$, $\mathrm{adiam}(\overrightarrow{x})$, is given by

$$\mathrm{adiam}(\overrightarrow{x}) := \lim_{k\in\mathbb{N}}\ \sup\ \{\|x_n - x_m\| : n, m \geq k\} \ .$$

We remark that in the statement of Lemma 3.1 in [?], it is only stated that $z \in \overline{\mathrm{co}}\{x_n : n \geq 1\}$. From the proof, however, we see that the stronger statement

$$z \in \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\{x_n : n \geq m\}$$

is true. We will use this stronger statement below. We further remark that in the proof of the above lemma, an important ingredient is the fact that every Banach space with *uniform normal structure* is necessarily *reflexive*; which was proven by E. Maluta [?].

The proof of our converse to Theorem 4.3 below (Theorem 5.2) is a variation on the theme of the proof of Theorem 3.1 of E. Casini and E. Maluta [?], who prove that every Banach space with *uniform normal structure* has *the uniform fixed point property* (Definition 4.1).

Theorem 5.2. *Let $(X, \|\cdot\|)$ be a Banach space with uniform normal structure. Then $(X, \|\cdot\|)$ has the Jaggi* uniform fixed point property, with constant*

$$\gamma_0 := \frac{1}{\sqrt{N(X)}} \ .$$

Proof. Fix $\gamma \in [1, \gamma_0)$. Next, fix a c.b.c. non-empty subset K of $X$and a Jaggi* γ-uniformly Lipschitzian map $T : K \longrightarrow K$. We will show that there exists $p \in K$ satisfying $T\,p = p$.

For all $w \in K$, we define

$$f(w) := \mathrm{rad}\left((T^n\, w)_{n\in\mathbb{N}_0}, w\right) := \sup_{n\in\mathbb{N}_0}\ \|T^n\, w - w\| \ .$$

Fix $u \in K$. For the sequence $\overrightarrow{x} := (x_n := T^n u)_{n\in\mathbb{N}_0}$ in X, choose $z(u)$ to be a point $z \in \cap_{m\in\mathbb{N}_0} \overline{\text{co}}\{x_n : n \geq m\}$ satisfying the conclusions of Lemma 5.1. Consider the T-invariant subset G of K defined by

$$G := \left\{T^j u : j \geq 0\right\} ,$$

and note that $u \in G$. By part (1) of Lemma 5.1 and the fact that T is Jaggi* γ-uniformly Lipschitzian,

$$\begin{aligned}
\text{arad}\left((T^n u)_{n\in\mathbb{N}_0}, z(u)\right) &\leq N(X)\ \text{adiam}\left((T^n u)_{n\in\mathbb{N}_0}\right) \\
&\leq N(X) \sup_{n\geq m\geq 0} \|T^n u - T^m u\| \\
&= N(X) \sup_{m\in\mathbb{N}_0} \sup_{j\geq 0} \left\|T^m(T^j u) - T^m u\right\| \\
&= N(X) \sup_{m\in\mathbb{N}_0} \sup_{y\in G} \|T^m y - T^m u\| \\
&\leq N(X)\, \gamma \sup_{y\in G} \|y - u\| \\
&= N(X)\, \gamma \sup_{j\in\mathbb{N}_0} \left\|T^j u - u\right\| \\
&= N(X)\, \gamma\, f(u) .
\end{aligned}$$

Further, fix $\nu \in \mathbb{N}$. For all $m \in \mathbb{N}_0$ we define the T-invariant subset G_m of K by

$$G_m := \left\{T^j u : j \geq m\right\} .$$

We note that for all $m \in \mathbb{N}_0$, $z := z(u) \in \overline{\text{co}}(G_m)$, by Lemma 5.1 above. Moreover, since T is Jaggi* γ-uniformly Lipschitzian, it follows that

$$\begin{aligned}
\text{arad}\left((T^n u)_{n\in\mathbb{N}_0}, T^\nu z\right) &:= \limsup_{n\in\mathbb{N}_0} \|T^n u - T^\nu z\| \\
&= \limsup_{n\geq\nu} \|T^n u - T^\nu z\| \\
&= \inf_{n\geq\nu} \sup_{k\geq n} \left\|T^k u - T^\nu z\right\| \\
&= \inf_{n\geq\nu} \sup_{j\geq n-\nu} \left\|T^\nu(T^j u) - T^\nu z\right\| \\
&= \inf_{n\geq\nu} \sup_{y\in G_{n-\nu}} \|T^\nu y - T^\nu z\| \\
&\leq \inf_{n\geq\nu} \gamma \sup_{y\in G_{n-\nu}} \|y - z\| \\
&= \gamma \inf_{j\geq 0} \sup_{y\in G_j} \|y - z\| \\
&= \gamma \inf_{j\geq 0} \sup_{k\geq j} \left\|T^k u - z\right\| \\
&= \gamma \limsup_{j\in\mathbb{N}_0} \left\|T^j u - z\right\| \\
&= \gamma\ \text{arad}\left((T^n u)_{n\in\mathbb{N}_0}, z(u)\right) .
\end{aligned}$$

Consequently, for $z := z(u)$,

$$\begin{aligned}
f(z) &:= \operatorname{rad}\left((T^n z)_{n\in\mathbb{N}_0}, z\right) := \sup_{\nu\in\mathbb{N}_0} \|T^\nu z - z\| \\
&= \sup_{\nu\in\mathbb{N}_0} \|z - T^\nu z\| \\
&\leq \sup_{\nu\in\mathbb{N}_0} \operatorname{arad}\left((T^n u)_{n\in\mathbb{N}_0}, T^\nu z\right) \text{ [by Lemma 5.1, part (2)]} \\
&\leq \sup_{\nu\in\mathbb{N}_0} \gamma \operatorname{arad}\left((T^n u)_{n\in\mathbb{N}_0}, z\right) \text{ [from the previous string of inequalities]} \\
&= \gamma \operatorname{arad}\left((T^n u)_{n\in\mathbb{N}_0}, z\right) \\
&\leq \gamma\, N(X)\ \gamma f(u) \text{ [from the first sequence of inequalities]} \\
&= \gamma^2\, N(X)\ f(u)\ .
\end{aligned}$$

Define the constant $\eta := \gamma^2 N(X) < \gamma_0^2 N(X) = 1$, by hypothesis. So, in summary, we see that there exists $\eta \in (0,1)$ such that for every $u \in K$

$$f(z(u)) \leq \eta\, f(u)\ .$$

Now let's define the sequence $(w_n)_{n\in\mathbb{N}}$ in K in the following way. Choose any $w_1 \in K$. For each $n \in \mathbb{N}$, define

$$w_{n+1} := z(w_n)\ .$$

Fix $n \in \mathbb{N}$. Then fix $k \in \mathbb{N}_0$. We have that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \|w_{n+1} - T^k w_n\| + \|T^k w_n - w_n\| \\
&\leq \|w_{n+1} - T^k w_n\| + f(w_n)\ .
\end{aligned}$$

Letting $k \longrightarrow \infty$, we see that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \limsup_{k\in\mathbb{N}_0} \|w_{n+1} - T^k w_n\| + f(w_n) \\
&= \operatorname{arad}\left(\left(T^k w_n\right)_{k\in\mathbb{N}_0}, w_{n+1}\right) + f(w_n) \\
&\leq N(X)\ \gamma\, f(w_n) + f(w_n) = (N(X)\ \gamma + 1)\, f(w_n) \\
&\leq (N(X)\ \gamma + 1)\, \eta\, f(w_{n-1}) \\
&\leq (N(X)\ \gamma + 1)\, \eta^{n-1}\, f(w_1)\ .
\end{aligned}$$

Since $0 < \eta < 1$, it follows that $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(X, \|\cdot\|)$. Thus, there exists $p \in K$ such that

$$\lim_{n\in\mathbb{N}} \|p - w_n\| = 0\ .$$

We finally will show that $T\,p = p$. Of course, if T is also assumed to be norm continuous on K, this follows because, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\|p - T\,p\| &\leq \|p - w_n\| + \|w_n - T\,w_n\| + \|T\,w_n - T\,p\| \\
&\leq \|p - w_n\| + f(w_n) + \|T\,w_n - T\,p\| \\
&\leq \|p - w_n\| + \eta^{n-1}\, f(w_1) + \|T\,w_n - T\,p\| \\
&\underset{n}{\longrightarrow} 0\ .
\end{aligned}$$

But T may not be norm continuous, as we saw in Example 2.3 above. What shall we do in this case? Well, firstly, note that

$$\|p - T\,p\| = \lim_{n\in\mathbb{N}} \|w_{n+1} - T\,p\|\ .$$

Fix $n \in \mathbb{N}_0$. By Lemma 5.1, part (2),

$$\begin{aligned} \|w_{n+1} - T\,p\| &\leq \operatorname{arad}\left((T^s\,w_n)_{s\in\mathbb{N}_0}, T\,p\right) \\ &:= \limsup_{s\in\mathbb{N}_0} \|T^s\,w_n - T\,p\| \\ &= \limsup_{s\geq 1} \|T^s\,w_n - T\,p\| \\ &= \inf_{s\geq 1} \sup_{t\geq s} \|T^t\,w_n - T\,p\| \\ &= \inf_{s\geq 1} \sup_{j\geq s-1} \left\|T\left(T^j\,w_n\right) - T\,p\right\| \\ &= \inf_{s\geq 1} \sup_{y\in Q_{n,s}} \|T\,y - T\,p\| \ , \end{aligned}$$

where $Q_{n,s} := \left\{T^j\,w_n : j \geq s-1\right\}$. We also define the set $H_n \subseteq K$ by

$$H_n := \left\{T^l\,w_m : l \in \mathbb{N}_0 \text{ and } m \geq n\right\} \ .$$

Note that H_n is a T-*invariant set* and $\{w_m : m \geq n\} \subseteq H_n$. Thus,

$$p \in \overline{H_n}^{\text{norm}} \subseteq \overline{\text{co}}(H_n) \ .$$

Moreover, for all $s \geq 1$, $Q_{n,s} \subseteq H_n$. Consequently, since T is a *Jaggi* γ-uniformly Lipschitzian mapping* on K, we have that

$$\begin{aligned} \|w_{n+1} - T\,p\| &\leq \inf_{s\geq 1} \sup_{y\in Q_{n,s}} \|T\,y - T\,p\| \\ &\leq \inf_{s\geq 1} \sup_{y\in H_n} \|T\,y - T\,p\| \\ &= \sup_{y\in H_n} \|T\,y - T\,p\| \\ &\leq \gamma \sup_{y\in H_n} \|y - p\| \\ &= \gamma \sup_{m\geq n} \sup_{l\in\mathbb{N}_0} \|T^l\,w_m - p\| \ . \end{aligned}$$

Fix $m \geq n$ and then fix $l \in \mathbb{N}_0$. We see that

$$\begin{aligned} \|T^l\,w_m - p\| &\leq \|T^l\,w_m - w_m\| + \|w_m - p\| \\ &\leq f(w_m) + \|w_m - p\| \ . \end{aligned}$$

So,

$$\begin{aligned} \|w_{n+1} - T\,p\| &\leq \gamma \sup_{m\geq n} \left(f(w_m) + \|w_m - p\|\right) \\ &\leq \gamma \sup_{m\geq n} \left(\eta^{m-1}\,f(w_1) + \|w_m - p\|\right) \\ &\xrightarrow[n]{} 0 \ ; \end{aligned}$$

because $\eta^{k-1} \xrightarrow[k]{} 0$ and $\|w_k - p\| \xrightarrow[k]{} 0$. Hence,

$$\|p - T\,p\| = \lim_{n\in\mathbb{N}} \|w_{n+1} - T\,p\| = 0 \ ;$$

and therefore $T\,p = p$. □

As an immediate consequence of Theorems 4.3 and 5.2, we have the following summarizing theorem.

Theorem 5.3. *A Banach space $(X, \|\cdot\|)$ has uniform normal structure if and only if $(X, \|\cdot\|)$ has the Jaggi* uniform fixed point property.*

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