PERGAMON

Nonlinear Analysis 50 (2002) 1085–1091

Nonlinear Analysis

www.elsevier.com/locate/na

# Fixed point theorems for mappings of asymptotically nonexpansive type

Gang Li[a,1], Brailey Sims[b,*]

[a] Department of Mathematics, Yangzhou University, Yangzhou 225002, People's Republic of China
[b] Department of Mathematics, University of Newcastle, NSW 2308, Australia

Received 20 October 2000; accepted 22 January 2001

*Keywords:* Fixed point; Asymptotically nonexpansive type mapping; Uniform normal structure

## 1. Introduction

Let $C$ be a nonempty subset of a Banach space $X$ and let $T: C \to C$ be a mapping. Then $T$ is said to be asymptotically nonexpansive [6] if there exists a sequence $(k_n)$ of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\| \quad \text{for } x, y \text{ in } C \text{ and } n = 1, 2, \ldots.$$

If this is valid for $n = 1$ with $k_1 = 1$ (and hence $k_n = 1$ for all $n$) then $T$ is said to be nonexpansive. If for each $x$ in $C$, we have

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leqslant 0,$$

then $T$ is said to be of asymptotically nonexpansive type [8].

In 1965, Kirk [7] proved that if $C$ is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping $T$ of $C$ has a fixed point. A nonempty convex subset $C$ of a normed linear space is said to have normal structure if each bounded convex subset $K$ of $C$ consisting of more than one point contains a nondiametral point. That is, a point $x \in K$ such that $\sup\{\|x - y\|: y \in K\} < \sup\{\|u - v\|: u, v \in K\} = \operatorname{diam} K$. Seven years later, in 1972, Goebel and Kirk [6] proved that if the space $X$ is assumed to be uniformly convex, then every

* Corresponding author. Tel.: +61-49-21-5526.
*E-mail addresses:* yzlgang@pub.yz.jsinfo.net (G. Li), bsims@maths.newcastle.au (B. Sims).

[1] This work is supported by National Natural Science Foundation of China.

0362-546X/02/$ - see front matter 
PII: S0362-546X(01)00744-1

asymptotically nonexpansive self-mapping $T$ of $C$ has a fixed point. This was extended to mappings of asymptotically nonexpansive type by Kirk in [8]. More recently these results have been extended to wider classes of spaces, see for example [2,5,9,11,12]. In particular, Lim and Xu [12] and Kim and Xu [9] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform normal structure, see also [4] for some related results. However, whether normal structure implies the existence of fixed points for mappings of asymptotically nonexpansive type is a natural question that remains open.

The present paper answers a question raised by Kim and Xu in [9]. It extends results in their paper and [12] to mappings of asymptotically nonexpansive type and so represents a further step toward a resolution of the question raised above.

## 2. Main theorems

In this section, let $X$ be a Banach space, let $C$ be a nonempty bounded subset of $X$ and let $T: C \to C$ be a mapping of asymptotically nonexpansive type. For each $x \in C$ and $n \geqslant 1$, put

$$r_n(x) = \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

Then for each $x \in C$,

$$\lim_{n \to \infty} r_n(x) = 0. \tag{1}$$

Let $E$ be a nonempty bounded closed convex subset of a Banach space $X$ and let $d(E) = \sup\{\|x - y\|: x, y \in E\}$ be the diameter of $E$. For each $x \in E$, let $r(x,E) = \sup\{\|x - y\|: y \in E\}$ and let $r(E) = \inf\{r(x,E): x \in E\}$, the Chebyshev radius of $E$ relative to itself. The normal structure coefficient of $X$ is defined to be

$$\tilde{N}(X) = \sup\{r(E)/d(E): E \text{ is a bounded closed convex subset of } X \text{ with } d(E) > 0\}.$$

Note, the normal structure coefficient $\tilde{N}(X)$, introduced by Maluta [13], is the reciprocal of $N(X)$ defined by Bynum in [3]. A space $X$ for which $\tilde{N}(X) < 1$ is said to have uniform normal structure. It is known that a space with uniform normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure.

**Theorem 2.1.** *Suppose $X$ is a Banach space with uniform normal structure, $C$ is a nonempty bounded subset of $X$, and $T: C \to C$ is an asymptotically nonexpansive type mapping such that $T$ is continuous on $C$. Further, suppose that there exists a nonempty closed convex subset $E$ of $C$ with the following property $(P)$:*

$$x \in E \quad \textit{implies } \omega_w(x) \subset E,$$

*where* $\omega_w(x)$ *is the weak* $\omega$*-limit set of* $T$ *at* $x$; *that is, the set*

$$\left\{y \in X\colon\ y = weak - \lim_i T^{n_i}x \ for\ some\ n_i \uparrow \infty\right\}.$$

*Then* $T$ *has a fixed point in* $E$.

To prove the theorem we use the following lemma from [14].

**Lemma 1.** *Let* $C$ *be a nonempty subset of a Banach space* $X$ *and let* $T$ *be a mapping of asymptotically nonexpansive type* $C$. *Suppose there exists a nonempty bounded closed convex subset* $E$ *of* $C$ *with the property* $(P)$. *Then there is a closed convex nonempty subset* $K$ *of* $C$ *and a* $\rho \geqslant 0$ *such that*:
(i) *if* $x \in K$, *then every weak limit point of* $(T^n x)$ *is contained in* $K$;
(ii) $\rho_x(y) = \rho$ *for all* $x, y \in K$, *where* $\rho_x$ *is the functional defined by*

$$\rho_x(y) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in X.$$

**Proof of Theorem 1.** Let $K$, $\rho_x$ and $\rho$ be as in Lemma 1. Let $x$ be any element in $K$ and let $G$ be a sub-semigroup of $\mathbb{N}$. That is, $G = \{in_0\colon i \in \mathbb{N}\}$ for some $n_0 \in \mathbb{N}$. For each $i \in G$, consider the sequence $(T^j x)_{i \leqslant j \in G}$. From the definition of $\tilde{N}(X)$, we have a $y_i \in \overline{co}\{T^j x\colon i \leqslant j \in G\}$ (here, $\overline{co}$ denotes the closed convex hull) such that

$$\limsup_{j\in G} \|T^j x - y_i\| \leqslant \tilde{N}(X) A((T^j x)_{i\leqslant j \in G}), \tag{2}$$

where $A(z_n)$ is the asymptotic diameter of the sequence $(z_n)$; that is, the number

$$\lim_n (\sup\{\|z_i - z_j\|\colon i, j \geqslant n\}).$$

Since $X$ is reflexive, $(y_i)$ admits a subsequence $(y_{i'})$ converging weakly to some $x^* \in X$. From 2 and the w-l.s.c. of the functional $\limsup_{j\in G} \|T^j x - y\|$, it follows that

$$\limsup_{j\in G} \|T^j x - x^*\| \leqslant \tilde{N}(X) A((T^j x)_{j\in G}). \tag{3}$$

It is easily seen that $x^* \in \bigcap_{i\in G} \overline{co}\{T^j x\colon i \leqslant j \in G\}$ and that

$$\|z - x^*\| \leqslant \limsup_{j\in G} \|z - T^j x\| \quad \text{for all } z \in X. \tag{4}$$

Using property $(P)$ and the fact that $\bigcap_{i\in G} \overline{co}\{T^j x\colon i \leqslant j \in G\} = \overline{co}\, \omega_w\{T^j x\colon j \in G\}$, which is easy to prove by using the Separation Theorem (cf. [1]), we get that $x^*$ actually lies in $K$. We claim that:

> there exists $x \in K$ such that $\omega(x) \neq \emptyset$, where $\omega(x)$ is the strong $\omega$-limit set of $T$ at $x$, and $\rho = 0$.

To derive a contradiction, we suppose that (1) is not true. In particular then, for any sub-semigroup $G$ of $\mathbb{N}$ and for any $x, y \in K$, we have that $D = \limsup_{j\in G} \|T^j x - y\|$ is strictly greater than zero. Let $r_0$ be a positive number chosen so that $r = (2r_0 + 1)\tilde{N}(X) < 1$, this is possible since by assumption $\tilde{N}(X) < 1$.

Now, take any $x_0$ in $K$ and put $G_0 = \mathbb{N}$, then from 3 and 4 there exists $x_1 \in K$ with

$$0 < D_0 = \limsup_{j \in G_0} \|T^j x_0 - x_1\| \leqslant \tilde{N}(X) A((T^j x_0)_{j \in G_0})$$

and

$$\|z - x_1\| \leqslant \limsup_{j \in G_0} \|z - T^j x_0\| \quad \text{for all } z \in X.$$

It then follows from 1 that there exists $n_0 \in \mathbb{N}$ such that

$$r_n(x_1) < r_0 D_0 \quad \text{for all } n \geqslant n_0.$$

Put $G_1 = \{in_0 \colon i \in \mathbb{N}\}$, it is a sub-semigroup of $\mathbb{N}$. It follows that there exists $x_2 \in K$ such that

$$0 < D_1 = \limsup_{j \in G_1} \|T^j x_1 - x_2\| \leqslant \tilde{N}(X) A((T^j x_1)_{j \in G_1})$$

and

$$\|z - x_2\| \leqslant \limsup_{j \in G_1} \|z - T^j x_1\| \quad \text{for all } z \in X.$$

By 1 again, there exists $n_1 \in G_1$ such that

$$r_n(x_2) < r_0 D_1 \quad \text{for all } n \geqslant n_1.$$

Put $G_2 = \{in_1 \colon i \in \mathbb{N}\}$, it is a sub-semigroup of $G_1$. It follows that there exists $x_3 \in K$ such that

$$0 < D_2 = \limsup_{j \in G_2} \|T^j x_2 - x_3\| \leqslant \tilde{N}(X) A((T^j x_2)_{j \in G_2})$$

and

$$\|z - x_3\| \leqslant \limsup_{j \in G_2} \|z - T^j x_2\| \quad \text{for all } z \in X.$$

We can repeat the above process to obtain a sequence $(x_n)_{n=1}^{\infty}$ in $K$ and a series of semigroups $\{G_n\}_1^{\infty}$ with the properties:

(i) $\mathbb{N} = G_0 \sqsupset G_1 \sqsupset G_2 \sqsupset \cdots$.;
(ii) $D_n = \limsup_{i \in G_n} \|T^i x_n - x_{n+1}\| \leqslant \tilde{N}(X) A((T^i x_n)_{i \in G_n})$;
(iii) $\|z - x_{n+1}\| \leqslant \limsup_{i \in G_n} \|z - T^i x_n\|$ for all $z \in X$;
(iv) $r_i(x_{n+1}) \leqslant r_0 D_n$ for all $i \in G_{n+1}$.

Now for $i, j \in G_n$ with $i > j$, we have that $i - j \in G_n \subset G_{n-1}$ and from (iii)–(iv) that

$$\begin{aligned} \|T^i x_n - T^j x_n\| &\leqslant r_j(x_n) + \|T^{i-j} x_n - x_n\| \\ &\leqslant r_j(x_n) + \limsup_{m \in G_{n-1}} \|T^{i-j} x_n - T^m x_{n-1}\| \\ &\leqslant r_j(x_n) + r_{i-j}(x_n) + \limsup_{m \in G_{n-1}} \|x_n - T^m x_{n-1}\| \\ &\leqslant (2r_0 + 1) D_{n-1}. \end{aligned}$$

It follows from (ii) that

$$D_n \leqslant \tilde{N}(X)(2r_0 + 1) D_{n-1} = r D_{n-1} \leqslant \cdots \leqslant r^{n-1} D_1.$$

Therefore, for each $i \in G_n$ and $n \geqslant 2$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant \|x_{n+1} - T^i x_n\| + \|T^i x_n - x_n\| \\ &\leqslant \|x_{n+1} - T^i x_n\| + \limsup_{m \in G_{n-1}} \|T^i x_n - T^m x_{n-1}\| \\ &\leqslant \|x_{n+1} - T^i x_n\| + r_i(x_n) + \limsup_{m \in G_{n-1}} \|x_n - T^m x_{n-1}\|.\end{aligned}$$

Consequently,

$$\|x_{n+1} - x_n\| \leqslant D_n + D_{n-1} \leqslant (r^{n-1} + r^{n-2})D_1.$$

That is, $(x_n)$ is a Cauchy sequence and there is $x \in K$ such that $x_n \to x$ strongly as $n \to \infty$. Since

$$\begin{aligned}\|T^j x - x\| &\leqslant \|T^j x - T^j x_n\| + \|T^j x_n - x_{n+1}\| + \|x_{n+1} - x\| \\ &\leqslant r_j(x) + \|x - x_n\| + \|x - x_{n+1}\| + \|T^j x_n - x_{n+1}\|,\end{aligned}$$

we have

$$\begin{aligned}\liminf_{j \to \infty} \|T^j x - x\| &\leqslant \|x - x_n\| + \|x - x_{n+1}\| + \liminf_{j \to \infty} \|T^j x_n - x_{n+1}\| \\ &\leqslant \|x - x_n\| + \|x - x_{n+1}\| + \limsup_{j \in G_n} \|T^j x_n - x_{n+1}\| \\ &= \|x - x_n\| + \|x - x_{n+1}\| + D_n.\end{aligned}$$

Taking $n \to \infty$, we get

$$\liminf_{j \to \infty} \|T^j x - x\| = 0.$$

This is a contradiction.

To prove (2), by Lemma 1, it is enough to show that if $\rho > 0$ then there exist $z, y \in K$ such that $\rho_y(z) = \limsup_{n \to \infty} \|z - T^n y\| < \rho$. To this end, by (1), there exists $x \in K$ such that $\omega(x) \neq \emptyset$. Let $y = \lim_i T^{n_i} x$ for some $n_i \uparrow \infty$. It is easily seen that $\{T^n y\colon n \geqslant 1\} \subset K$. Put

$$\rho_0 = \operatorname{diam}(\overline{co}\{T^n y\colon n \geqslant 1\}) = \operatorname{diam}(\{T^n y\colon n \geqslant 1\}).$$

Since

$$\begin{aligned}\|T^n y - T^m y\| &= \lim_{i \to \infty} \|T^{n+n_i} x - T^m y\| \\ &\leqslant \limsup_{i \to \infty} \|T^i x - T^m y\| \\ &= \rho,\end{aligned}$$

we have $\rho_0 \leqslant \rho$. Since $K$ has normal structure, there exists $z \in \overline{co}\{T^n y\colon n \geqslant 1\}$ such that

$$\sup_{n \geqslant 1} \|z - T^n y\| < \operatorname{diam}(\overline{co}\{T^n y\colon n \geqslant 1\}) \leqslant \rho.$$

This proves (2).

By (2), $K = \{x\}$ and $T^n x \to x$ strongly as $n \to \infty$. Therefore, $Tx = x$ by the continuity of $T$.

**Corollary 1.** *Let $C$ and $X$ be as in Theorem* 1 *and let $T: C \to C$ be an asymptotically nonexpansive mappings. Suppose there exists a nonempty bounded closed convex subset $E$ of $C$ with the property $(P)$. Then $T$ has a fixed point.*

**Proof.** This follows since an asymptotically nonexpansive mapping is of asymptotically nonexpansive type.

From Theorem 1 we readily capture the following result announced by Taehwa Kim, who also gives an alternative proof [10].

**Corollary 2.** *Let $X$ be a Banach space with uniform normal structure, let $C$ be a bounded closed convex subset of $X$, and suppose $T: C \to C$ is a continuous mapping of asymptotically nonexpansive type. Then $T$ has a fixed point.*

We conclude the paper by stating the semigroup version of Theorem 1. The proof is similar to that of Theorem 1 and is therefore omitted.

**Theorem 2.2.** *Suppose $X$ is a Banach space with uniform normal structure, $C$ is a nonempty bounded subset of $X$, and $\mathfrak{J} = \{T(t)\colon t \geqslant 0\}$ is a semigroup of asymptotically nonexpansive type mappings on $C$ such that $T(t)$ is continuous on $C$ for each $t \geqslant 0$. Suppose also that there exists a nonempty bounded closed convex subset $E$ of $C$ with the following property $(P)$:*

$$x \in E \quad \textit{implies } \omega_w(x) \subset E,$$

*where $\omega_w(x)$ is the weak $\omega$-limit set of $\{T(t)x\}$, i.e. the set*

$$\left\{ y \in X\colon\ y = \text{weak} - \lim_i T(t_i)x \text{ for some } t_i \uparrow \infty \right\}.$$

*Then $\mathfrak{J}$ has a common fixed point in $E$, i.e. there exists a $z \in E$ for which $T(t)z = z$ for all $t \geqslant 0$.*

## References


[1] R.E. Bruck, On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak-limit set, Israel J. Math. 29 (1978) 1–16.

[2] R. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically nonexpansive mapping in Banach spaces with the uniform Opial property, Nonlinear Anal. 21 (1995) 926–946.

[3] W.L. Bynum, Normal structure coefficients for Banach space, Pacif. J. Math. 86 (1980) 427–436.

[4] E. Casini, E. Maluta, Fixed points of uniformly Lipschitzian mappings in space with uniformly normal structure, Nonlinear Anal. TMA 9 (1985) 103–108.

[5] J. Garcia-Falset, B. Sims, M.A. Smyth, The demiclosedness principle for mappings of asymptotically nonexpansive type, Houston J. Math. 22 (1996) 101–108.

[6] K. Goebel, W. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171–174.

[7] W. Kirk, A fixed point theorem for mappings which do not increase distances, Proc. Amer. Math. Soc. 72 (1965) 1004–1006.

[8] W. Kirk, A fixed point theorem of non-Lipschitzian mappings of asymptotically nonexpansive type, Israel J. Math. 17 (1974) 339–346.

[9] T.H. Kim, H.K. Xu, Remarks on asymptotically nonexpansive mappings, Nonlinear Anal. TMA 41 (2000) 405–415.
[10] Taehwa Kim, Fixed point theorems for non-Lipschitzian self-mappings and geometric properties of Banach spaces, preprint.
[11] P.K. Lin, K.K. Tan, H.K. Xu, Demiclosedness principle and asymptotic behaviour for asymptotically nonexpansive mappings, Nonlinear Anal. TMA 24 (1995) 929–946.
[12] T.C. Lim, H.K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal. TMA 22 (1994) 1345–1355.
[13] E. Maluta, Uniformly normal structure and related coefficients, Pacific J. Math. 111 (1984) 357–369.
[14] H.K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, Nonlinear Anal. TMA 16 (1991) 1139–1146.