Quaestiones Mathematicae
18(1995), 9–19

ON NON–UNIFORM CONDITIONS GIVING WEAK NORMAL STRUCTURE

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ABSTRACT. Several non-uniform conditions sufficient for weak normal structure have recently been introduced. We show that some of these are in fact equivalent and also utilize them in applications towards a 3-space property for weak normal structure, thereby improving on earlier results.

1991 *Mathematics Subject Classification.* 47H10.

1. Introduction. Throughout X is a Banach space which is assumed not to be Schur. That is, X has weakly convergent sequences that are not norm convergent. Recall that X has (weak) normal structure if whenever C is a (weak compact) bounded convex subset of X with $\operatorname{diam} C > 0$ then $\operatorname{rad} C < \operatorname{diam} C$ where

$$\operatorname{diam} C := \sup\{\|x-y\| : x, y \in C\} \quad \text{and} \quad \operatorname{rad} C := \inf_{x\in C} \sup\{\|x-y\| : y \in C\}.$$

It is well known that X fails weak normal structure if and only if there exists a sequence (x_n) in X with $x_n \xrightarrow{w} 0$ and $\operatorname{diam} \overline{\operatorname{co}}\{x_n\}_{n=1}^{\infty}$ $(= \operatorname{diam}\{x_n\}_{n=1}^{\infty}) = 1$ and $\operatorname{dist}(x_{n+1}, \overline{\operatorname{co}}\{x_k\}_{k=1}^{n}) \to 1$.

In particular $\operatorname{diam}_a(x_n)$, $\operatorname{rad}_a(x_n)$ and $\lim_n \|x_n\|$ are all equal to 1, where

$$\operatorname{diam}_a(x_n) := \lim_n \operatorname{diam}\{x_k\}_{k=n}^{\infty} \quad \text{and} \quad \operatorname{rad}_a(x_n) := \inf\{\limsup_n \|x - x_n\| : x \in \overline{\operatorname{co}}\{x_n\}_{n=1}^{\infty}\}$$

are, respectively, the asymptotic diameter of (x_n) and the asymptotic radius of (x_n) in $\overline{\mathrm{co}}\,\{x_n\}_{n=1}^{\infty}$.

See [3] for details and the relevance of weak normal structure to fixed point theory of nonexpansive mappings. We now review some uniform conditions.

As in Maluta [9] we define $\tilde{N}(X)$ by

$$\sup\left\{\frac{\operatorname{rad} C}{\operatorname{diam} C} : C \text{ is a bounded convex non-singleton non-empty subset of } X\right\}.$$

This is the reciprocal of Bynum's normal structure coefficient, $N(X)$, defined in [1]. X is said to have uniform normal structure if $\tilde{N}(X) < 1$.

Also, put

$$\text{w.c.s. } (X) := \sup\left\{\frac{\operatorname{rad}_a(x_n)}{\operatorname{diam}_a(x_n)} : x_n \xrightarrow{w} 0, x_n \not\to 0\right\}.$$

This is the reciprocal of the weak convergent sequence coefficient defined in [1]. It can be checked that diam_a can be replaced with diam in the definition.

Since $\operatorname{rad}_a(y_n) \le \operatorname{rad}\{y_n\}_{n=1}^{\infty}$ for any sequence (y_n) it follows that w.c.s. $(X) \le \tilde{N}(X)$.

Of course if $\tilde{N}(X) < 1$ or w.c.s.$(X) < 1$ then X has weak normal structure.

Maluta [9] introduced

$$D(X) := \sup\left\{\frac{\limsup \operatorname{dist}\left(x_{n+1}, \overline{\mathrm{co}}\,\{x_k\}_{k=1}^{n}\right)}{\operatorname{diam}\{x_n\}_{n=1}^{\infty}} : (x_n) \text{ is a bounded nonconstant sequence in } X\right\}$$

She showed that diam can be replaced with diam_a (for nonconvergent sequences (x_n)), that $D(X) \le \tilde{N}(X)$, and also that $D(X) = 1$ whenever X is not reflexive, thus giving that uniform normal structure implies reflexivity.

Prus [11] showed that w.c.s.$(X) = D(X)$ if X is reflexive, the main argument being that in general w.c.s.$(X) \le D(X)$. He also obtained that w.c.s.(X) is the reciprocal of

$$\inf\left\{\operatorname{diam}_a(x_n) : x_n \xrightarrow{w} 0, \|x_n\| \to 1\right\}$$

Here also diam_a can be replaced with diam.

Recently several non-uniform conditions have been studied. Tan and Xu [12] introduced property P :

$$\liminf \|x_n - x\| < \mathrm{diam}\{x_n\}_{n=1}^{\infty} \quad \text{if} \quad x_n \xrightarrow{w} x \\ \text{and } (x_n) \text{ is nonconstant.}$$

By extracting appropriate subsequences this can be seen to be unaltered if lim sup is used instead of lim inf and, on normalizing, is equivalent to:

$$\text{If } \|x_n\| \to 1 \text{ and } x_n \xrightarrow{w} 0 \quad \text{then } \mathrm{diam}\{x_n\}_{n=1}^{\infty} > 1.$$

We say that X has asymptotic P if the above (again equivalent) conditions hold with diam replaced by diam_a (with the proviso that the sequence is nonconstant).

In section 2 the main results are that P is equivalent to a condition introduced in [13] by Tingley (subsequently known as WO), and that asymptotic P is the GGLD of [4]. We also give other equivalents of these conditions, some involving indices of noncompactness and others more closely related to the original definitions of w.c.s.(X) and $D(X)$.

Section 3 is concerned with problems of a 3-space nature:

Given $X = Y \oplus Z$, where Z is finite dimensional, what conditions on Y give weak normal structure for X? In [8] an example is given of a space X that has weak normal structure even though the direct ℓ_1^2 sum $X \oplus_{\ell_1} \mathbb{R}$ fails this property. We ask what properties of Y sufficient for weak normal structure are inherited by X. It is shown that asymptotic P, as well as P with appropriate conditions on the projections, are such conditions.

2. Some Banach Space Properties. We state below some Banach space properties which are then related to P and asymptotic P. Two of these properties involve indices of noncompactness; the others have appeared in the literature before and are discussed below.

In each case the inequality holds whenever (x_n) is a weak null sequence in X that is not norm convergent.

(1) $\liminf \|x_n\| < \sup_m \limsup_n \|x_m - x_n\|$.

(2) $\liminf \|x_n\| <$ Sep $(\{x_n\}_{n=1}^{\infty})$.

(3) $\liminf \|x_n\| < \limsup_m \limsup_n \|x_m - x_n\|$.

(4) $\liminf \|x_n\| < \alpha(\{x_n\}_{n=1}^{\infty})$.

Recall that if C is a bounded subset of X,

$$\text{Sep}\,(C) := \sup\{\inf_{n\neq m} \|y_n - y_m\| : (y_n) \text{ is a sequence in } C\} \quad \text{and}$$

$$\alpha(C) := \inf\{d : C \text{ has a finite cover of subsets of } X \text{ of diameter at most } d\},$$

which are, respectively, the separation and Kuratowski indices of non-compactness.

Of course the above properties can be restated replacing lim inf with lim sup and can also be normalized in a way similar to that for P.

Condition (1) is WO and represents a weakening of the Opial condition (see [10]):

$$\text{If} \quad x_n \xrightarrow{w} 0 \quad \text{and} \quad x \neq 0 \quad \text{then} \quad \limsup \|x_n\| < \limsup \|x - x_n\|.$$

As noted in [13], WO can be restated as follows: If $x_n \xrightarrow{w} 0$, (x_n) a non-constant sequence, then there exists $x \in \overline{\text{co}}\,\{x_n\}_{n=1}^{\infty}$ so that $\limsup \|x_n\| < \limsup \|x - x_n\|$.

Condition (3) is an asymptotic version of WO and is called GGLD in [4], where it is shown to be distinct from WO. Combining this with the following proposition establishes that asymptotic P and P are different Banach space properties. We also note that the space considered in [4] which separates P from asymptotic P has the Opial property, so that whereas Opial implies P, it doesn't imply asymptotic P. The example at the end of this section separates asymptotic P from w.c.s.$(X) < 1$.

PROPOSITION 2.1. *Condition (1) is equivalent to P and the other conditions are equivalent to asymptotic P.*

Proof. Clearly (1) $\Rightarrow$ *P*. To show the converse we use a technique due to Landes [7].

Suppose X has *P*, $x_n \in X$, $\|x_n\| \to 1$, $x_n \xrightarrow{w} 0$, but $\limsup_n \|x_m - x_n\| \leq 1$ for all m.

We construct a subsequence (y_n) of (x_n) as follows:

$y_1 = x_1$. If $y_1, \dots, y_k$ have been selected, then $y_{k+1} = x_m$ is chosen so that
$\|x_m - y_j\| \leq 1 + 1/k$ for all $j \leq k$ (possible by the condition on (x_n)).

Thus, for each k,

$$\|y_{k+1} - y_j\| \leq 1 + 1/k \quad \text{for all} \quad j \leq k.$$

Now put $z_k = \frac{k}{k+1} y_{k+1} + \frac{1}{k+1} y_1$. Clearly $z_k \xrightarrow{w} 0$ and $\|z_k\| \to 1$. Also, if $m > k$,

$$\begin{aligned}\|z_m - z_k\| &= \|\frac{m}{m+1} y_{m+1} - \frac{k}{k+1} y_{k+1} - (\frac{1}{k+1} - \frac{1}{m+1}) y_1\| \\ &= \|\frac{k}{k+1}(y_{m+1} - y_{k+1}) + (\frac{1}{k+1} - \frac{1}{m+1})(y_{m+1} - y_1)\| \\ &\leq (\frac{k}{k+1} + \frac{1}{k+1} - \frac{1}{m+1})(1 + \frac{1}{m}) \\ &= 1.\end{aligned}$$

Thus, diam $\{z_m\}_{m=1}^{\infty} \leq 1$, contradicting property P.

To establish the remainder of the proposition we first note that (2) $\Rightarrow$ (3) $\Rightarrow$ asymptotic P and (4) $\Rightarrow$ asymptotic P are clear. Also, (2) $\Rightarrow$ (4) follows from the fact that $\alpha(C) \geq$ Sep (C) for any set C. It remains to show that asymptotic $P \Rightarrow$ (2).

Suppose X has asymptotic P and $x_n \in X$, $\|x_n\| \to 1$, $x_n \xrightarrow{w} 0$. Suppose Sep $(\{x_n\}_{n=1}^{\infty}) \leq 1$ and $\epsilon > 0$.

Let $P_2(\{x_n\}_{n=1}^{\infty})$ denote the set of two element subsets of $\{x_n\}_{n=1}^{\infty}$. We define $A, B \subseteq P_2(\{x_n\}_{n=1}^{\infty})$ by

$$\begin{aligned}A &:= \{\{x_n, x_m\} \in P_2(\{x_n\}_{n=1}^{\infty}) : \|x_n - x_m\| \geq 1 + \epsilon\}, \\ B &:= \{\{x_n, x_m\} \in P_2(\{x_n\}_{n=1}^{\infty}) : \|x_n - x_m\| < 1 + \epsilon\}.\end{aligned}$$

Since $A \cup B = P_2(\{x_n\}_{n=1}^{\infty})$, Ramseys Theorem implies that there exists a subsequence (y_n) of (x_n) with

$$P_2(\{y_n\}_{n=1}^{\infty}) \subseteq A \quad \text{or} \quad P_2(\{y_n\}_{n=1}^{\infty}) \subseteq B.$$

But Sep $(\{x_n\}_{n=1}^{\infty}) \leq 1$, so $P_2(\{y_n\}_{n=1}^{\infty}) \subseteq B$ and diam $\{y_n\}_{n=1}^{\infty} \leq 1 + \epsilon$.

Repeated application of this process together with a diagonalization will produce a subsequence (z_n) of (x_n) with diam$_a$ $(z_n) \leq 1$, a contradiction. □

In [4] a uniform version of GGLD is also introduced. We relate this condition to w.c.s.(X) below.

With $D[(x_n)] := \limsup_n \limsup_m \|x_n - x_m\|$ [4] defines

$$\beta(X) := \inf\{D[(x_n)] : \|x_n\| \to 1, x_n \xrightarrow{w} 0\}.$$

By adapting the arguments used in the proof of the above proposition, we have the following.

PROPOSITION 2.2. *The following are equal:*

(1) $1/w.c.s.(X) = \inf\{diam_a(x_n) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$
(2) $\beta(X)$
(3) $\inf\{Sep(\{x_n\}_{n=1}^{\infty}) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$
(4) $\inf\{\alpha(\{x_n\}_{n=1}^{\infty}) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$

It is easily checked that the uniformization of WO is the same as the uniformization of GGLD. Thus the uniformization of all the properties considered in proposition 2.1 is w.c.s.$(X) < 1$.

It should be mentioned that that the equality of (1) and (3) was noted in [11].

We now recast P and asymptotic P in a manner similar to the original definitions of w.c.s.(X) and $D(X)$.

PROPOSITION 2.3. *The following is equivalent to P:*
If $x_n \xrightarrow{w} 0$, (x_n) *nonconstant, then*

$$\limsup_k rad\,(\overline{co}\,\{x_n\}_{n=k}^{\infty}) < diam\,\{x_n\}_{n=1}^{\infty}.$$

Similarly asymptotic P can be rewritten as:

$$\text{If } x_n \xrightarrow{w} 0,\ x_n \not\to 0, \text{ then } \limsup_k \left(\frac{\text{rad }(\overline{\text{co}}\,\{x_n\}_{n=k}^{\infty})}{\text{diam}\,\{x_n\}_{n=k}^{\infty}}\right) < 1.$$

We note that the left hand side of the last inequality is equal to

$$\frac{\limsup_k \text{rad }(\overline{\text{co}}\,\{x_n\}_{n=k}^{\infty})}{\text{diam}_a\,(x_n)}.$$

Proof of proposition 2.3. Suppose X has property P and $x_n \xrightarrow{w} 0$, $\text{diam}\,\{x_n\}_{n=1}^{\infty} = 1$ but $\limsup_k \text{rad }(\overline{\text{co}}\,\{x_n\}_{n=k}^{\infty}) = 1$. Now $0 \in \overline{\text{co}}\,\{x_n\}_{n=k}^{\infty}$ for all k and so $\sup_{k \geq m} \|x_k\| \geq \text{rad }(\overline{\text{co}}\,\{x_k\}_{k=m}^{\infty})$, giving $\limsup \|x_k\| = 1$, contradicting P.

Conversely, suppose X satisfies the first statement in the proposition and that $x_n \xrightarrow{w} 0$ and $\operatorname{diam}\{x_n\}_{n=1}^{\infty} = 1$. We now show that $\limsup \|x_n\| < 1$, and conclude that X has property P.

Indeed, we have $\limsup_k \operatorname{rad}(\overline{\operatorname{co}}\{x_n\}_{n=k}^{\infty}) < 1$. Thus there exists a sequence (y_k) with $y_k \in \overline{\operatorname{co}}\{x_n\}_{n=k}^{\infty}$ and $\epsilon > 0$ so that

$$\sup_{z \in \overline{\operatorname{co}}\{x_n\}_{n=k}^{\infty}} \|y_k - z\| = \sup_{n \geq k} \|y_k - x_n\| < 1 - \epsilon.$$

Thus $\limsup_n \|y_k - x_n\| < 1-\epsilon$ for all k. But $y_k \xrightarrow{w} 0$ and so by the weak lower semi-continuity of the mapping $x \mapsto \limsup_n \|x - x_n\|$, $\limsup \|x_n\|$

$\leq 1 - \epsilon < 1$, completing the proof.

The equivalence involving asymptotic P is proved in a similar way, using the observation made below the statement of the proposition. □

It may also be seen that the same method of proof yields that w.c.s.(X) is equal to

$$\sup\left\{\limsup_k \left(\frac{\operatorname{rad}(\overline{\operatorname{co}}\{x_n\}_{n=k}^{\infty})}{\operatorname{diam}\{x_n\}_{n=k}^{\infty}}\right) : x_n \xrightarrow{w} 0,\ x_n \not\to 0\right\},$$

making the connection with P and asymptotic P clear.

For completeness we also relate P and asymptotic P to $D(X)$, first isolating a result that is obtained in [11] with the following lemma.

LEMMA 2.4. *If $x_n \xrightarrow{w} 0$, $\|x_n\| \to 1$, then for $\epsilon > 0$ there exists a subsequence (y_n) of (x_n) so that*

$$\textit{dist}\ (y_{n+1}, \overline{\operatorname{co}}\{y_k\}_{k=1}^{n}) \geq 1 - \epsilon.$$

We note that by repeated application of Lemma 2.4 and a subsequent diagonalization, we can obtain a further subsequence (z_n) so that

$$\liminf_k \left(\inf_m \operatorname{dist}\ (z_{k+m+1}, \overline{\operatorname{co}}\{z_n\}_{n=k}^{k+m})\right) \geq 1.$$

PROPOSITION 2.5. *P is equivalent to the following: If $x_n \xrightarrow{w} 0$, (x_n) nonconstant, then*

$$\limsup_k \left(\limsup_m \; dist \; \left(x_{k+m+1}, \; \overline{\text{co}} \{x_n\}_{n=k}^{k+m}\right) \right) < diam \{x_n\}_{n=1}^{\infty}.$$

Asymtotic P is similarly characterized with diam replaced with $diam_a$ (assuming that $x_n \not\to 0$).

Proof. We consider the case of property P. That of asymptotic P is proved in the same way.

That the statement implies P follows from the observation preceding the proposition. Conversely suppose that $x_n \xrightarrow{w} 0$, (x_n) nonconstant, but

$$\limsup_k \left(\limsup_m \; \text{dist} \; \left(x_{k+m+1}, \; \overline{\text{co}} \{x_n\}_{n=k}^{k+m}\right) \right) = \text{diam} \{x_n\}_{n=1}^{\infty}.$$

We show that then $\limsup \|x_n\| = \text{diam} \{x_n\}_{n=1}^{\infty}$, contradicting P. Indeed, let $n \in \mathbb{N}$ and $\epsilon > 0$. Then the above implies that there exists a $k \geq n$ so that

$$\limsup_m \; \text{dist} \; \left(x_{k+m+1}, \; \overline{\text{co}} \{x_n\}_{n=k}^{k+m}\right) > \text{diam} \{x_n\}_{n=1}^{\infty} - \epsilon/2$$

Now, since $0 \in \overline{\text{co}} \{x_n\}_{n=k}^{\infty}$, there exists $p \in \mathbb{N}$ and $z \in \overline{\text{co}} \{x_n\}_{n=k}^{k+p}$ so that $\|z\| < \epsilon/2$. But the above will then give an $m \geq p$ so that $\|x_{k+m+1}\| > \text{diam} \{x_n\}_{n=1}^{\infty} - \epsilon$ and it follows that since n and ϵ where arbitrary, $\limsup \|x_n\| = \text{diam} \{x_n\}_{n=1}^{\infty}$, completing the proof. □

The following example separates w.c.s.$(X) < 1$ from asymptotic P.

EXAMPLE 2.6. Let $X = (\ell_2 \oplus \ell_3 \oplus \ell_4 \oplus \ldots)_2$. By considering the usual unit vector bases of ℓ_n , it is clear that w.c.s.$(X) = 1$. We now show that X has asymptotic P. We will use the well known fact that w.c.s.$(\ell_p) = 2^{-1/p}$.

So, suppose that (x^n) is a sequence in X with $\|x^n\| \to 1$, $x^n \xrightarrow{w} 0$.

We will denote the natural projection onto the subspace of X naturally identified with ℓ_m by P_m for $m > 1$.

Since the projections are weak continuous, $P_m(x^n) \xrightarrow{w} 0$ for fixed m.

Firstly, suppose that there exists $\epsilon > 0$, $m \in \mathbb{N}$ and a subsequence (x^{n_i}) of (x^n) so that

$$\|P_m(x^{n_i})\| > \epsilon \text{ for any } i.$$

Without loss of generality we assume that (x^{n_i}) is (x^n). Now

$$\|x^n\|^2 = \|P_m(x^n)\|^2 + \|(I-P_m)(x^n)\|^2$$
$$\|x^n - x^p\|^2 = \|P_m(x^n) - P_m(x^p)\|^2 + \|(I-P_m)(x^n) - (I-P_m)(x^p)\|^2.$$

Without loss of generality we suppose that $\|P_m(x^n)\|$ and $\|(I-P_m)(x^n)\|$ converge to, say, c_1 and c_2 respectively.

Suppose that $\delta > 0$. Since $(I-P_m)(x^n) \xrightarrow{w} 0$, and $\|(I-P_m)(x^n)\| \to c_2$ it follows that there exists a subsequence (x^{n_i}) of (x^n) so that

$$\|(I-P_m)(x^{n_i}) - (I-P_m)(x^{n_j})\| > c_2 - \delta \text{ for } i \neq j.$$

Again we assume that $(x^{n_i}) = (x^n)$. Now for $q \in \mathbb{N}$

$$\begin{aligned}\sup_{n,p>q} \|x^n - x^p\|^2 &= \sup_{n,p>q} \Big(\|P_m(x^n) - P_m(x^p)\|^2 \\ &\quad + \|(I-P_m)(x^n) - (I-P_m)(x^p)\|^2\Big) \\ &\geq \sup_{n,p>q} \|P_m(x^n) - P_m(x^p)\|^2 + (c_2 - \delta)^2.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{diam}_a(x^n) &\geq \left((\operatorname{diam}_a(P_m(x^n)))^2 + (c_2-\delta)^2\right)^{1/2} \\ &\geq \left(2^{2/m}c_1^2 + (c_2-\delta)^2\right)^{1/2} \quad \text{since w.c.s.}\,(\ell_m) = 2^{-1/m}.\end{aligned}$$

Thus, since δ was arbitrary,

$$\begin{aligned}\operatorname{diam}_a(x^n) &\geq \left(2^{2/m}c_1^2 + c_2^2\right)^{1/2} \\ &= \left((2^{2/m}-1)c_1^2 + c_1^2 + c_2^2\right)^{1/2} \\ &= \left((2^{2/m}-1)c_1^2 + 1\right)^{1/2} \\ &> 1 \quad (\text{since } c_1 \geq \epsilon > 0)\end{aligned}$$

and the result is obtained.

Otherwise, if our original supposition does not hold, $\|P_m(x^n)\| \to 0$ for any m.

Then for any n, $y^n := (\|P_m(x^n)\|)_{m=1}^{\infty} \in \ell_2$, $\|y^n\|_{\ell_2} \to 1$ and $y^n \xrightarrow{w} 0$.

Thus, since w.c.s.$(\ell_2) = 2^{-1/2}$, we have $\operatorname{diam}_a(y^n) \geq 2^{1/2}$.

But since $|\ \|P_m(x^n)\| - \|P_m(x^p)\|\ | \leq \|P_m(x^n) - P_m(x^p)\|$, it follows that $\|y^n - y^p\|_{\ell_2} \leq \|x^n - x^p\|_X$ and so $\operatorname{diam}_a(x^n) \geq 2^{1/2} > 1$, also giving the result.

3. On a 3-space Problem. As in the introduction, suppose Y is a closed subspace of X and $X = Y \oplus Z$ with Z finite dimensional. In [2] it is shown that if Y has uniform normal structure then X has normal structure (equivalent to weak normal structure in this case since X is reflexive).

PROPOSITION 3.1. *If Y has asymptotic P then so does X.*

Proof. Suppose $x_n = y_n + z_n$ is a sequence in X with $\|x_n\| \to 1$, $x_n \xrightarrow{w} 0$ and $y_n \in Y$, $z_n \in Z$. But then, since the linear projection onto Z is [weak]continuous, $z_n \xrightarrow{w} 0$. Thus $z_n \to 0$ since Z is finite dimensional. Then $\|y_n\| \to 1$ and $y_n \xrightarrow{w} 0$ giving $\operatorname{diam}_a(y_n) > 1$ since Y has asymptotic P. Hence $\operatorname{diam}_a(x_n) > 1$ since $z_n \to 0$. □

The same method used in the above proof will also establish that w.c.s.$(X) =$ w.c.s. (Y). Since w.c.s.$(X) \leq \tilde{N}(X)$, this strengthens the result of [2]. Clearly X is reflexive if and only if Y is. Combining this with the results of Maluta and Prus given in the introduction, we also have an alternative proof of Proposition 1 in [6] , that $D(X) = D(Y)$. It still appears unknown whether $\tilde{N}(Y) < 1$ gives uniform normal structure for X (although it is shown in [5] that super normal structure carries across).

PROPOSITION 3.2. *If Y has property P and the projection onto Y has norm 1 then X has P.*

Proof. Suppose that $\|x_n\| \to 1$, $x_n \xrightarrow{w} 0$ and $x_n = y_n + z_n$, with $y_n \in Y$, $z_n \in Z$. Then, as in the proof of proposition 3.1, $z_n \to 0$ with $\|y_n\| \to 1$, and $y_n \xrightarrow{w} 0$. Thus, $\operatorname{diam}\{y_n\}_{n=1}^{\infty} > 1$, since Y has property P. Now, since the projection on Y is of norm 1,

$$1 < \operatorname{diam}\{y_n\}_{n=1}^{\infty} \leq \operatorname{diam}\{x_n\}_{n=1}^{\infty}$$

and so X has property P. □

Since property P implies weak normal structure which implies the weak fixed point property (see [3]), the above proposition strenghens Theorem 2.3 of [12] (where it is shown that if Y has P and both projections have norm 1 then X has the weak fixed point property). We also note that the conclusions of the last two propositions remain valid if Z is a Schur space, a possibility which is considered in [2].

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Received 20 April 1994