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SUBSETS CHARACTERIZING THE CLOSURE OF THE NUMERICAL RANGE

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Abstract

For an operator on a Hilbert space, points in the *closure* of its numerical range are characterized as either extreme, non-extreme boundary, or interior in terms of various associated sets of bounded sequences of vectors. These generalize similar results due to Embry, for points in the numerical range.

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1. Introduction

Let T be an operator (that is, a bounded linear transformation) on a complex Hilbert space H with inner product $\langle\ ,\ \rangle$ and associated norm $\|\ \|$. It is well known that the numerical range

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1, x \in H\}$$

is a convex subset of the complex plane. Denote the closure of $W(T)$ by $W(T)^-$. Theorem 1 of M. R. Embry (1970) characterizes every point z of $W(T)$ as either an extreme point, a non-extreme boundary point or an interior point in terms of the subset $M_z(T)$ and its linear span, where

$$M_z(T) = \{x \in H: \langle Tx, x\rangle - z\|x\|^2 = 0\} \qquad (z \in W(T)).$$

This theorem, though very interesting, does not characterize the unattained boundary points of the numerical range. In this note we attempt to fill this gap by a generalization which can be applied to every point of $W(T)^-$. For any $z \in W(T)^-$, let

$$N_z(T) = \left\{(x_n) \in l_\infty(H) \colon \langle Tx_n, x_n\rangle - z\|x_n\|^2 \to 0\right\},$$

$$N_z'(T) = \left\{(x_n) \in l_\infty(H) \colon \langle Tx_n, x_n\rangle / \|x_n\|^2 \to z\right\},$$

$$N^L(T) = \bigcup_z \{N_z(T) \colon z \in L \cap W(T)^-\}$$

and

$$N_L(T) = \left\{(x_n) \in l_\infty(H) \colon \inf_{z \in L}\left|\langle Tx_n, x_n\rangle - z\|x_n\|^2\right| \to 0\right\}$$

where $l_\infty(H)$ is the set of all bounded sequences of vectors from H and L is a line of support for $W(T)^-$. Let $\gamma N_z(T)$ be the linear span of $N_z(T)$. Since $N_z(T)$ is homogeneous, $\gamma N_z(T) = N_z(T) + N_z(T)$. It is readily seen that $N_L(T)$ is a subspace (Majumdar and Sims (to appear)).

2. Basic lemmas

In order to establish our characterization for points of $W(T)^-$ we need the following two lemmas. The first, stated without proof, is an easy corollary to Lemma 3 of Majumdar and Sims (to appear).

LEMMA 1. *If b is an extreme point of $W(T)^-$ and L is a line of support for $W(T)$ passing though b, then* $\lim\langle (T-b)x_n, y_n\rangle = 0$ *and* $\lim\langle (T-b)y_n, x_n\rangle = 0$ *for all* $(x_n) \in N_b(T)$ *and* $(y_n) \in N_L(T)$.

LEMMA 2. *Let z be in the interior of a line segment lying in $W(T)^-$ with end points a and b. Then $N_a'(T) \subset \gamma N_z(T)$.*

PROOF. Without loss of generality we may take $a = 1$, $b = 0$ and $(x_n) \in N_1'(T)$ to have $\|x_n\| = 1$. Let $(y_n) \in N_0(T)$ be such that $\|y_n\| = 1$ and $\mathrm{Re}\langle \mathrm{Im}\, Tx_n, y_n\rangle = 0$. For any bounded sequence (r_n), let $h_n = r_n x_n + y$; then we have $\langle \mathrm{Im}\, Th_n, h_n\rangle \to 0$. We show the existence of two such distinct sequences (r_n) for which

$$\langle \mathrm{Re}\, Th_n, h_n\rangle - z\|h_n\|^2 = 0 \tag{1}$$

for all sufficiently large n. The equations in r_n given by (1) are equivalent to

$$r_n^2(1 - z + \varepsilon_n) + 2r_n \mathrm{Re}\langle(\mathrm{Re}\, T - z)x_n, y_n\rangle + (\varepsilon_n' - z) = 0$$

where $\varepsilon_n = \langle \mathrm{Re}\, Tx_n, x_n\rangle - 1$ and $\varepsilon_n' = \langle \mathrm{Re}\, Ty_n, y_n\rangle$, both of which tend to zero. Thus the equations in (1) are of the form $A_n r_n^2 + B_n r_n + C_n = 0$ where A_n, B_n, C_n are real numbers independent of r_n.

Let $D_n = B_n^2 - 4A_nC_n$, then

$$D_n = 4\left[\operatorname{Re}\langle \operatorname{Re} T - z)x_n, y_n\rangle\right]^2 + 4z(1-z) + \delta_n$$

where $\delta_n \to 0$. Hence there are positive constants α, β such that for all sufficiently large n, $\alpha \leqslant A_n$, $D_n \leqslant \beta$ and $|B_n| \leqslant \beta$. This shows the existence of two distinct sequences solving (1) both of which are bounded and whose differences $d_n = \sqrt{D_n}/A_n$ are eventually bounded away from zero. Thus we have for both these sequences that $h_n \in N_z(T)$. Subtraction and the fact that d_n is uniformly bounded away from zero gives $(x_n) \in \gamma N_z(T)$.

REMARK. A simplified version of the above argument applied to a pair of points a, b lying in a line segment in $W(T)$ shows the existence of a real number r and a vector y such that $a = \langle Tx, x\rangle$, $b = \langle Ty, y\rangle$, $\|x\| = \|y\| = 1$ and $\langle T(rx+y), rx + y\rangle/\|rx+y\|^2 = ta + (1-t)b$, $0 < t < 1$, yielding the convexity of $W(T)$. In contrast with the proof of convexity given by Halmos (1967), this argument gives two explicit values of r.

3. Characterization of $W(T)^-$

THEOREM 3. *Every element z of $W(T)^-$ can be characterized as follows.*

(i) *z is an extreme point of $W(T)^-$ if and only if $N_z(T)$ is a subspace.*

(ii) *If z is a nonextreme boundary point of $W(T)^-$ and L the line of support for $W(T)$ passing through z, then* (a) $\gamma N_z(T) = N_L(T) + N_z(T)$ *and* (b) $N_L(T) = l_\infty(H)$ *if and only if $W(T)^- \subset L$.*

(iii) *If $W(T)^-$ is not a straight line segment, then z is an interior point of $W(T)^-$ if and only if $N_a'(T) \subset \gamma N_z(T)$ for all $z \in W(T)^-$.*

PROOF. (i) See Das and Craven (1983) and also Majumdar and Sims (to appear). Also note that the result $N_z(T)$ is a subspace when z is an extreme point of $W(T)^-$ can be deduced as a corollary to Lemma 1. Homogeneity being obvious, we prove linearity. Let $(x_n^{(1)}), (x_n^{(2)}) \in N_z(T)$. Thus $(x_n^{(1)}), (x_n^{(2)}) \in N_L(T)$ where L is a line of support for $W(T)$ passing through z. But $N_L(T)$ is a subspace. So $(x_n^{(1)} + x_n^{(2)}) \in N_L(T)$. Now since $(x_n^{(1)}) \in N_z(T)$, $i = 1, 2$ and $(x_n^{(1)} + x_n^{(2)}) \in N_L(T)$, by Lemma 1 we have $\lim\langle (T-z)x_n^{(1)}, x_n^{(1)} + x_n^{(2)}\rangle = 0$ for $i = 1, 2$ and hence $\lim\langle (T-z)(x_n^{(1)} + x_n^{(2)}), x_n^{(1)} + x_n^{(2)}\rangle = 0$ as required.

(ii) (a) We first show $N_a(T) \subset \gamma N_z(T)$ for each $a \in L \cap W(T)^-$. Without loss of generality we may take L as the real axis and $\operatorname{Im} W(T) \geqslant 0$. Let $(x_n) \in N_a(T)$ and $(y_n) \in N_b(T)$, $\|y_n\| = 1$ where $b \in L$ is the extreme point of $W(T)^-$ such that $(a-z)/(z-b) \geqslant 0$. Then (y_n) can be chosen so that $\operatorname{Re}\langle y_n, x_n\rangle = 0$ and

Lemma 1 gives $\operatorname{Re}\langle Ty_n, x_n\rangle \to 0$. Also $\operatorname{Im} W(T) \geqslant 0$ implies $\operatorname{Im} Ty_n \to 0$. Let $r_n = [(a - z)/(z - b)]^{\frac{1}{2}}\|x_n\|$. Then easy calculations show that with our assumptions $\langle T(x_n \pm r_n y_n), x_n \pm r_n y_n\rangle - z\|x_n \pm r_n y\|^2 \to 0$. That is $(x_n \pm r_n y_n) \in N_z(T)$. As in the proof of Lemma 2, adding these two sequences and using the homogeneity of $N_z(T)$ we have $(x_n) \in \gamma N_z(T)$. Thus $N_a(T) \subset \gamma N_z(T)$ for all $a \in L \cap W(T)^-$ and so we have $N^L(T) \subset \gamma N_z(T)$. Since $N_z(T) \subset N^L(T) \subset \gamma N_z(T)$, by taking the vector sum of $N_z(T)$ with each of these subsets we obtain $\gamma N_z(T) = N^L(T) + N_z(T)$.

(b) As before, if we take L as the real axis, we have $N_L(T) = \{(x_n) \in l_\infty(H)\colon \operatorname{Im}\langle Tx_n, x_n\rangle \to 0\}$. Now if $W(T)^- \subset L$, $(x_n) \in l_\infty(H)$ implies $\operatorname{Im}\langle Tx_n, x_n\rangle = 0$ and so $(x_n) \in N_L(T)$. Hence $N_L(T) = l_\infty(H)$. Conversely if $W(T)^-$ is not a subset of L, there exists $(x_n) \in l_\infty(H)$, $\|x_n\| = 1$ such that $\operatorname{Im}\langle Tx_n, x_n\rangle$ does not tend to zero, or equivalently, $(x_n) \notin N_L(T)$. Hence $N_L(T) \neq l_\infty(H)$.

(iii) If z is an interior point of $W(T)^-$, by Lemma 2, $N_a'(T) \subset \gamma N_z(T)$ whenever $a \in W(T)^-$. On the other hand, if z is a boundary point of $W(T)^-$, without loss of generality we may take L, the line of support for $W(T)$ passing through z, as the real axis, in which case, $N_L(T) = \{(x_n) \in l_\infty(H)\colon \operatorname{Im}\langle Tx_n, x_n\rangle \to 0\}$. Thus $\gamma N_z(T) \subset N_L(T)$ since $N_L(T)$ is a subspace, but as $W(T)^-$ does not lie in L, there exists an $a \in W(T)$ such that $\operatorname{Im} a \neq 0$. Hence $N_a'(T)$ is not a subset of $\gamma N_z(T)$.

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