THE STRUCTURE OF THE NORMED LATTICE GENERATED BY THE CLOSED, BOUNDED, CONVEX SUBSETS OF A NORMED SPACE

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Abstract. Let $\mathcal{C}(X)$ denote the set of all non-empty closed bounded convex subsets of a normed linear space X. In 1952 Hans Rådström described how $\mathcal{C}(X)$ equipped with the Hausdorff metric could be isometrically embedded in a normed lattice with the order an extension of set inclusion. We call this lattice the *Rådström* of X and denote it by $R(X)$. We:

(1) outline Rådström's construction,
(2) examine the structure and properties of $R(X)$ as a Banach space, including; completeness, density character, induced mappings, inherited subspace structure, reflexivity, and its dual space, and
(3) explore possible synergies with metric fixed point theory.

1. Introduction

Throughout $X \equiv (X, \|\cdot\|)$ denotes a real normed linear space. B_X and B_X° the closed and open balls of X respectively, and X^* the dual space of continuous linear functionals on X. $\mathcal{C}(X)$ denotes the set of non-empty, closed, bounded, convex subsets of X.

In 1952 Hans Vilhem Rådström [?] constructed a canonical isometric embedding of $\mathcal{C}(X)$ equipped with the Hausdorff metric into a normed lattice $R(X)$ with the order an extension of set inclusion in $\mathcal{C}(X)$. While numerous applications and extensions of Rådström's ideas are to be found in the literature, little seems known about the normed space structure of $R(X)$ or its relation to the structure of X. In this note we initiate such an investigation.

In the ensuing outline of Rådström's construction we follows the approach found in Coppel [?], where the reader is referred for more details. For those interested an alternative construction based on the support functional representation of convex sets may be found in Hörmander [?].

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We begin the construction of $R(X)$ by introducing an algebraic structure into $\mathcal{C}(X)$.

For any $A, B \in \mathcal{C}(X)$, define $\lambda A := \{\lambda a : a \in A\} \in \mathcal{C}(X)$ and their **Minkowski sum** $A + B$ by $A + B := \{a + b : a \in A, b \in B\}$.

While $A + B$ is non-empty, bounded, and convex. it may not be closed unless one of the sets is compact in, for example, the weak topology on X.

Example 1.1. Let $X = \ell_1$, $A = \phi^{-1}\{1\} \cap 2B_X$, where $\phi \in X^*$ is such that $(a_n) \mapsto \sum_{n=1}^{\infty}(1 - 2^{-n})a_n$, and let $B = B_X$.
Then $a_k := (1 - 2^{-k})^{-1}e_k$ and $b_k := -e_k$ are sequences of elements in A and B respectively, with $a_k + b_k \to 0$, so $0 \in \overline{A + B}$, however, calculation reveals that $0 \notin A + B$.

To overcome this, we introduce a new "addition" in $\mathcal{C}(X)$:

$$A \oplus B := \overline{A + B} \in \mathcal{C}(X).$$

Observe that: $A \oplus B = \overline{A} \oplus B$ for any $A, B \in \mathcal{P}(X)$, and so $\oplus$ is associative. In addition $\{0\}$ is an identity for $\oplus$. In particular then $(\mathcal{C}(X), \oplus)$ is a commutative monoid.

The following perhaps surprising result is a key feature in the construction.

Proposition 1.2 (Order Cancellation Law - Brunn, 1889 [?])**.** *If $A, B, C \in \mathcal{C}(X)$, and $A \oplus C \subseteq B \oplus C$, then $A \subseteq B$.*

In particular we have:

If $A \oplus C = B \oplus C$, then $A = B$.

Thus, $H := (\mathcal{C}(X), \oplus)$ is a commutative monoid with cancellation law.

Next we observe that H can be embed into an abelian group G (its Grothendieck group) as follows,

Define an equivalence relation $\sim$ on $H \times H$ by

$$(A, B) \sim (C, D) \iff A \oplus D = C \oplus B.$$

Denote by $[A, B]$ the equivalence class of the pair (A, B) and let G be the set of all such equivalence classes.

Then

$$[A, B] \oplus [C, D] := [A \oplus C, B \oplus D]$$

is a well-defined binary operation on G, with respect to which G is an abelian group; with identity $\mathbf{0} := [\{0\}, \{0\}]$ $(= [A, A])$ and inverses given by $\ominus[A, B] = [B, A]$.

Further,

$$\phi : H \to G : A \mapsto [A, \{0\}]$$

is an injective homomorphism, that is, G contains a copy of H, and $x \mapsto \phi(\{x\})$ provides an embedding of $(X, +)$ into G.

Remark 1.3. The construction of G from H mirrors the construction of $(\mathbb{Z}, +)$ from $(\mathbb{N} \cup \{0\}, +)$, and the construction of $(\mathbb{Q}\setminus\{0\}, \times)$ from $\mathbb{Z}\setminus\{0\}, \times)$ commonly found in introductory algebra courses.

As in these special cases, we avoid the cumbersome notation of pairs by using C to denote both a non-empty, closed, bounded, convex subset of X and its image $\phi(C) = [C, \{0\}]$ in G.

Moreover, if we define scalar multiplication by,

$$\lambda[A, B] := \begin{cases} [\lambda A, \lambda B] & : \lambda \geq 0 \\ [(-\lambda)B, (-\lambda)A] & : \lambda < 0 \end{cases}$$

then, after some tedious verification, we have:

Proposition 1.4. *G with the operations defined above is a real linear space.*

Remark 1.5. Since we identify $C \in \mathcal{C}(X)$ with $[C, \{0\}] \in G$, the expression λC may refer either to the scalar multiplication on $\mathcal{C}(X)$ or the scalar multiplication on G. For $\lambda \geq 0$ leads to no ambiguity as $\phi(\lambda C) = \lambda\phi(C)$, however, when $\lambda < 0$, $\phi(\lambda C) = [\lambda C, \{0\}] \neq [\{0\}, (\lambda)C] =: \lambda\phi(C)$.

Unless otherwise stated, λC will henceforth refer to the scalar multiplication on G.

This suggest defining,

$$\begin{aligned} A \ominus B : &= A \oplus (-1)B \\ &= [A, \{0\}] \oplus [\{0\}, B] \\ &= [A, B] \end{aligned}$$

Henceforth, we will mostly use the more suggestive notation $A \ominus_X B$ for the element $[A, B]$ of G.

Remark 1.6. In some circumstances the subscript X is necessary to identify the space in which the elements of the equivalence class reside.
For example, if we have Y, a closed, strict subspace of X, then for any $A, B \in \mathcal{C}(Y) \subset \mathcal{C}(X)$, the class $A \ominus_Y B$ is a strict subset of $A \ominus_X B$. However, when the space is clear from the context, we will simply write $A \ominus B$.

Due to the order cancellation law, the subset partial order on $\mathcal{C}(X)$ can be extended to G by defining,

$$A \ominus B \leq C \ominus D \iff A \oplus D \subseteq C \oplus B,$$

Proposition 1.7. *The relation $\leq$ on G is well-defined, and makes G a vector lattice.*

The positive cone is $G^+ = \{A \ominus B : A \supseteq B\}$.

Remark 1.8. Despite the fact that $G = \mathcal{C}(X) \ominus \mathcal{C}(X)$, its positive cone and $\mathcal{C}(X)$ do not coincide.

Further, G is a Kakutani space [**?**], indeed:

(1) $\mathbf{u} := B_X$ is an order unit for G, as $|A \ominus B| \leq n\mathbf{u}$, when n is any integer larger than $\max_{a \in A} \|a\| + \max_{b \in B} \|b\|$, and
(2) if $A \ominus B \leq \frac{1}{n}\mathbf{u}$ for all $n \in \mathbb{N}$, then $A \ominus B \leq \mathbf{0}$.

From these it follows that

$$\|A \ominus B\| := \inf \{\lambda \geq 0 : |A \ominus B| \leq \lambda \mathbf{u}\}$$

defines a lattice norm on G.

Calculation reveals that,

$$\|A \ominus B\| = \mathcal{H}(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\},$$

where $\mathcal{H}$ is the **Hausdorff metric** on $\mathcal{C}(X)$.

G with this norm is a normed linear lattice (for a succinct account of normed lattices the reader is referred to [**?**]).

We will henceforth refer to G with this structure as the **Rådström of X** and denote it by $\mathcal{R}(X)$. [$R(X)$ is also variously known as a Minkowski-Rådström-Hörmander (MRH) space, or a Pinsker-Minkowski-Rådström-Hörmander (PMRH) lattice].

ϕ provides a monotone isometric embedding of $(\mathcal{C}(X), \mathcal{H})$ into $\mathcal{R}(X)$ and $x \mapsto \phi(\{x\})$ is a linear isometry from X into $\mathcal{R}(X)$.

1.1. Representation in a $\mathbf{C}(K)$ space.

By the Krein-Kakutani theorem [**?**], there is a monotone linear isometry $\psi : \mathcal{R}(X) \to \mathbf{C}(K)$ with $\psi(\mathcal{R}(X)))$ a dense subspace of $\mathbf{C}(K)$ and $\psi(\mathbf{u})$ the constant function 1, where K is a compact, Hausdorff topological space.
Specifically, K comprises the extreme points of the set of positive linear functionals in $B_{\mathcal{R}(X)^*}$ equipped with the the weak* topology, and for all $\mathbf{x} \in \mathcal{R}(X)$, the action of $\psi(\mathbf{x})$ is evaluation at $\mathbf{x}$; $\psi(\mathbf{x}) = \hat{\mathbf{x}}|_K$.

1.2. Examples of Rådströms.

There are (precisely) two elementary examples of Rådström spaces, arising from the two simplest real normed linear spaces; the trivial space 0 and $\mathbb{R}$.

Example 1.9. $\mathcal{R}(\{0\}) = \{\mathbf{0}\}$.

Example 1.10. Due to the simplistic nature of convex sets in $\mathbb{R}$,

$\mathcal{R}((\mathbb{R}, |\cdot|))$ is isometric to ℓ^2_∞, that is $(\mathbb{R}^2, \|\cdot\|_\infty)$,
under the surjective, linear isometry:

$$\iota : \mathcal{R}(\mathbb{R}) \to \mathbb{R}^2 : [a,b] \ominus [c,d] \mapsto (a-c, b-d).$$

On the other hand we have:

Proposition 1.11. *If* dim $X \geq 2$*, then* $\mathcal{R}(X)$ *is infinite-dimensional.*

Proof. This is best derived as a consequence of stronger results discussed later in the paper. Most succinctly by an appeal to Corollary **??**. Alternatively one can first establish that $\mathcal{R}(\mathbb{R}^2)$ is infinite dimensional, then use Theorem **??** to embed $\mathcal{R}(\mathbb{R}^2)$ into $\mathcal{R}(X)$. Infinite dimensionality of $\mathcal{R}(\mathbb{R}^2)$ follows via a routine calculation to establish the linear independence of

$$\{\text{co}\{(0,0),(x,y)\} \subset \mathbb{R}^2 : \|(x,y)\| = 1,\ x,y \geq 0\} \subset \mathcal{R}(\mathbb{R}^2),$$

where co designates the convex hull. □

Further,

Proposition 1.12. $\mathcal{R}(X)$ *is separable if and only if* X *is finite-dimensional.*

Proof. If $X = \mathbb{R}^n$, then the set P of polytopes whose extreme points with rational vertices forms a countable dense subset of $\mathcal{C}(X)$. The set of equivalence classes of the form $A \ominus B$, where $A, B \in P$, is therefore countable and dense in $\mathcal{R}(X)$.

Otherwise, if X is infinite dimensional, by an appeal to Theorem **??** we may assume without loss of generality that X is separable. In which case X admits a Markushevich basis (e_n, e_n^*) with $\|e_n\| = 1$ and $\|e_n^*\| \leq 2$ [**?**]. For any nonempty subset I of $\mathbb{N}$, define

$$C_I = \overline{\text{co}}\{e_n : n \in I\},$$

here $\overline{\text{co}}$ designates the closed convex hull.
Calculation reveals that $\mathcal{H}(C_I, C_J) < 1/2$ if and only if $I = J$. From this and the observation that the subsets of $\mathbb{N}$ have cardinality that of the continuum it follows that $\mathrm{R}(X)$ is not separable. □

2. Rådströms as normed linear spaces

Extending proposition **??** we have;

Theorem 2.1. *If* dim $X \geq 2$*, then* $\mathcal{R}(X)$ *is incomplete.*

Proof. Suppose $\mathcal{R}(X)$ is complete. Define

$$\begin{aligned} \mathcal{C}_n(X) &:= \{C \in \mathcal{C}(X) : \|C\| \leq n\} \quad \text{and} \\ \mathcal{R}_n(X) &:= \mathcal{C}_n(X) \ominus \mathcal{C}_n(X). \end{aligned}$$

By the Blaschke Selection Principle [**?**], $\mathcal{C}_n(X)$, and hence $\mathcal{R}_n(X)$, are compact. Additionally,

$$\mathcal{R}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n(X).$$

By the Baire Category Theorem, some $\mathcal{R}_n(X)$ has a non-empty interior. Thus, $B_{\mathcal{R}(X)}$ is compact, so by Proposition **??**, dim $X < 2$. □

As a consequence, if dim $X \geq 2$, then $\mathcal{R}(X)$ is not reflexive and, from the Krein-Kakutani represention, neither is its completion. Loosely speaking this means that $\mathcal{R}(X)$ inherits few of the geometric properties that X may enjoy.

We have already seen that $\mathcal{R}(X)$ contains a subspace isometric to X, and this is indeed the only subspace wholly contained in $\mathcal{C}(X)$.

Other subspaces include,

$$\begin{aligned} \mathcal{R}_{FD}(X) &:= \{A \ominus B \in \mathcal{R}(X) : \text{span}(A), \text{span}(B) \text{ are finite-dimensional}\}, \\ \mathcal{R}_K(X) &:= \{A \ominus B \in \mathcal{R}(X) : A, B \text{ are compact}\}, \\ \mathcal{R}_{wK}(X) &:= \{A \ominus B \in \mathcal{R}(X) : A, B \text{ are weakly compact}\}, \\ \mathcal{R}_{w^*K}(X^*) &:= \{A \ominus B \in \mathcal{R}(X^*) : A, B \text{ are weak* compact}\}. \end{aligned}$$

$\mathcal{R}_{FD}(X)$ is dense in $\mathcal{R}_K(X)$, and

$$\mathcal{R}_{FD}(X) \subseteq \mathcal{R}_K(X) \subseteq \mathcal{R}_{wK}(X), \quad \text{also } \mathcal{R}_{wK}(X^*) \subseteq \mathcal{R}_{w^*K}(X^*).$$

Further, as might be expected, the subspace structure of X is mirrored in $\mathcal{R}(X)$, indeed;

Theorem 2.2. *Suppose Y is a subspace of X, not necessarily closed, then $\mathcal{R}(Y)$ is isometrically isomorphic to a closed subspace of $\mathcal{R}(X)$. Moreover, if Y is dense in X, then this subspace is the entirety of $\mathcal{R}(X)$.*

We delay the proof of theorem **??** until Section **??**.

3. Induced operators

For $T : X \to Y$ a Lipschitz continuous map between normed linear spaces X and Y the Lipschitz constant is

$$C_L(T) := \sup \left\{ \frac{d(Tx, Ty)}{d(x, y)} \, : \, x, y \in X \text{ and } x \neq y \right\}.$$

Define,

$$\rho_T : \mathcal{C}(X) \to \mathcal{C}(Y) : C \mapsto \overline{\text{co}}T(C),$$

then ρ_T is also Lipschitz continuous with $C_L(\rho_T) = C_L(T)$.

Further, if T is linear then, taking the convex hull is superfluous, ρ_T is additive and positive scalar-homogeneous, and we can extend it to a map from $\mathcal{R}(X)$ to $\mathcal{R}(Y)$ by defining

$$\rho_T(A \ominus B) := \rho_T(A) \ominus \rho_T(B).$$

The following proposition readily follows from the definitions.

Proposition 3.1. *Suppose X, Y and Z are normed spaces and suppose $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators, then:*

(1) ρ_T is a well defined linear operator,
(2) ρ_T is bounded with $\|\rho_T\| = \|T\|$.
(3) ρ_T is monotone.
(4) For any $k \in [0, \infty)$, $\rho_{kT} = k\rho_T$.
(5) $\rho_{I_X} = I_{\mathcal{R}(X)}$
(6) If T is an isomorphism, then $\rho_T^{-1} = \rho_{T^{-1}}$.
(7) If T is an isometry, then ρ_T is an isometry.
(8) $\rho_{ST} = \rho_S \rho_T$.

Armed with these notions we now return to the proof of **??**.

3.1. Proof of Theorem ??.

Proving that the embedding of $\mathcal{R}(Y)$ into $\mathcal{R}(X)$ is closed is surprisingly non-trivial. We first need a lemma:

Lemma 3.2. *Suppose $C \in \mathcal{C}(X)$ and $B_X \subseteq C$. Suppose further that Y is a subspace of X and $M > 0$ satisfies $C \cap Y \subseteq MB_Y$. Then the inequality:*

$$\inf_{y \in C \cap Y} \|x - y\| \leq M \inf_{c \in C} \|x - c\|$$

holds for all $x \in Y$.

Proof. Fix $x \in Y$. Assume without loss of generality that $x \in Y \setminus C$ and fix $c \in C$.

Consider the set $I = \{\lambda \in \mathbb{R} : \lambda x \in C\}$. Then $1/\|x\| \in I$ and I is bounded above by 1, since $x \notin C$. Define $\mu = \sup I \in (0, 1)$.

Consider the ray $\lambda\mu x + (1-\lambda)c : \lambda \geq 1$. Suppose for the sake of contradiction that this ray intersected with the open unit ball B_X° for some $\lambda \geq 1$. Since μx is in the boundary of C, we have $\lambda \neq 1$. But then μx lies on the interior of the line segment from $c \in C$ to $\lambda\mu x + (1-\lambda)c \in B_X^\circ \subseteq \text{int}\, C$, so $\mu x \in \text{int}\, C$, which is a contradiction. Therefore,

$$\|\lambda\mu x + (1-\lambda)c\| \geq 1$$

for all $\lambda \geq 1$. In particular, consider $\lambda = \frac{1}{1-\mu} > 1$, then

$$1 \leq \left\| \frac{1}{1-\mu}\mu x + \left(1 - \frac{1}{1-\mu}\right) c \right\| = \frac{\mu}{1-\mu}\|x - c\| \implies \|x - c\| \geq \frac{1-\mu}{\mu}.$$

Since $\mu x \in C \cap Y \subseteq MB_Y$, we have $\mu\|x\| \leq M$. Therefore,

$$\inf_{y \in C \cap Y} \|x - y\| \leq \|x - \mu x\| = \frac{1-\mu}{\mu}\mu\|x\| \leq M\|x - c\|.$$

Taking the infimum over $c \in C$ yields the result. □

Proof of Theorem **??**. Let $\phi : Y \to X$ be the inclusion map. By Proposition **??**, the induced operator $\rho_\phi : \mathcal{R}(Y) \to \mathcal{R}(X)$ is a linear isometry.

If Y is dense in X, it may be shown, for all $C \in \mathcal{C}(X)$, that

$$\rho_\phi((C \oplus B_X) \cap Y) = \overline{(C \oplus B_X) \cap Y} = C \oplus B_X,$$

and hence ρ_ϕ is surjective.

We proceed to show that $\rho_\phi(\mathcal{R}(Y))$ is closed in $\mathcal{R}(X)$. Suppose a sequence $\overline{A_n} \ominus_X \overline{B_n} = \rho_\phi(A_n \ominus_Y B_n) \in \rho_\phi(\mathcal{R}(Y))$ converges to $A \ominus_X B \in \mathcal{R}(X)$. By adding sufficiently large multiples of B_X to A and B, we may assume without loss of generality that $B_X \subseteq A, B$. Choose some M such that $A \cap Y, B \cap Y \subseteq MB_Y$.

By Lemma **??**,

$$\begin{aligned}
\sup_{x \in B_n \oplus (A \cap Y)} \inf_{y \in A_n \oplus (B \cap Y)} \|x - y\| &= \sup_{b \in B_n} \sup_{a' \in A \cap Y} \inf_{a \in A_n} \inf_{b' \in B \cap Y} \|b + a' - a - b'\| \\
&\leq M \sup_{b \in B_n} \sup_{a' \in A \cap Y} \inf_{a \in A_n} \inf_{b' \in B} \|b + a' - a - b'\| \\
&\leq M \sup_{b \in B_n} \sup_{a' \in A} \inf_{a \in A_n} \inf_{b' \in B} \|b + a' - a - b'\| \\
&= M \sup_{x \in B_n \oplus A} \inf_{y \in A_n \oplus B} \|x - y\| \\
&\leq M\|(\overline{A_n} \ominus_X \overline{B_n}) \ominus_X (A \ominus_X B)\|.
\end{aligned}$$

Similarly,

$$\sup_{x \in A_n \oplus (B \cap Y)} \inf_{y \in B_n \oplus (A \cap Y)} \|x - y\| \leq M\|(\overline{A_n} \ominus_X \overline{B_n}) \ominus_X (A \ominus_X B)\|.$$

Therefore,

$$\begin{aligned}
&\|(A_n \ominus_Y B_n) \ominus_Y ((A \cap Y) \ominus_Y (B \cap Y))\| \\
&\leq M\|(\overline{A_n} \ominus_X \overline{B_n}) \ominus_X (A \ominus_X B)\| \to 0.
\end{aligned}$$

Thus $\overline{A_n} \ominus_X \overline{B_n} \to \overline{A \cap Y} \ominus_X \overline{B \cap Y}$, proving $\rho_\phi(\mathcal{R}(Y))$ is closed in $\mathcal{R}(X)$. □

Remark 3.3. When Y is a closed, complemented subspace of X, with projection $P : X \to Y$, it is straightforward to show ρ_P is also a projection, and hence $\mathcal{R}(Y)$ is complemented in $\mathcal{R}(Y)$. This painlessly establishes a special case of Theorem **??**.

It is worth noting that, in general, $I_{\mathcal{R}(X)} - \rho_P \neq \rho_{(I_X - P)}$.

3.2. **The dual space of a Rådström.**

Each $f \in X^*$ induces a linear transformation $\rho_f : \mathcal{R}(X) \to \mathcal{R}(\mathbb{R}) = \ell^2_\infty$, so

$$\phi = \mathbf{v} \circ \rho_f \in \mathcal{R}(X)^*,$$

where $\mathbf{v} \in \ell^2_1 = \left(\ell^2_\infty\right)^*$.
We refer to ϕ as **a functional** (on $\mathcal{R}(X)$) **induced by** f.

In particular we have,

$$\begin{aligned}\alpha_f(A \ominus B) &:= \sup f(A) - \sup f(B), \quad \text{here } \mathbf{v} = (1,0),\\ \omega_f(A \ominus B) &:= \inf f(A) - \inf f(B), \quad \text{here } \mathbf{v} = (0,1),\\ &= -\,\alpha_{-f}(A \ominus B).\end{aligned}$$

$\alpha_f \in \mathcal{R}(X)^*_+$, and every functional induced by f is a linear combination of α_f and ω_f.

Theorem 3.4. *$\phi \in \mathcal{R}(X)^*_+$ is induced by $f \in X^*$ if and only if $\phi(B_{\ker(f)}) = 0$.*

Proof. The "only if" direction is clear.

Suppose $f \in X^*$ and $\phi \in \mathcal{R}(X)^*_+$ satisfies $\phi(B_{\ker(f)}) = 0$. Assume $f \neq 0$, otherwise the result is trivial. We will establish that

$$\ker\,\phi \supseteq \ker\,\alpha_f \cap \ker\,\omega_f$$

and hence ϕ is induced by f.

Suppose $A \ominus B \in \ker\,\alpha_f \cap \ker\,\omega_f$. This implies $\overline{f(A)} = \overline{f(B)}$.

Fix $x \in f^{-1}\{1\}$ and let $\pi : X \to \ker\, f$ be the bounded projection map $y \mapsto y - f(y)x$. Let $M = \sup\{\|a - b\| : a \in \rho_\pi(A), b \in \rho_\pi(B)\}$.

Suppose $a \in A$ and $n \in \mathbb{N}$. Since $\overline{f(A)} = \overline{f(B)}$, there exists some $b_n \in B$ such that $|f(a) - f(b_n)| < 1/n$. Consider,

$$\|b_n + \pi(a) - \pi(b_n) - a\| = |f(a) - f(b_n)|\|x\| \leq \|x\|/n \to 0.$$

Since $\|\pi(a) - \pi(b_n)\| \leq M$, it follows that

$$a = \lim_{n\to\infty} b_n + \pi(a) - \pi(b_n) \in B \oplus M B_{\ker(f)}.$$

Therefore $A \subseteq B \oplus M B_{\ker(f)}$. Since $\phi \geq 0$, we have

$$\phi(A \ominus B) \leq M\phi(B_{\ker(f)}) = 0.$$

Applying the above argument to $B \ominus A$ yields $A \ominus B \in \ker\,\phi$ as desired. □

Corollary 3.5. *If $\phi \in \mathcal{R}(X)^*$ satisfies $0 \leq \phi \leq \alpha_f$, then $\phi = a\alpha_f$ for some $a \in [0,1]$.*

Proof. If $f = 0$, the result is trivial. Otherwise, $0 \leq \phi(B_{\ker(f)}) \leq 0$. By Theorem **??**, there exist $a, b \in \mathbb{R}$ such that $\phi = a\alpha_f + b\omega_f$.

Fix $x \in f^{-1}\{-1\}$. Let $C = \mathrm{co}\{0, x\}$. Then,

$$0 \leq \phi(C) = a\alpha_f(C) + b\omega_f(C) = b \leq 0.$$

Thus $\phi = a\alpha_f(C)$. Since $0 \leq \phi \leq \alpha_f$, we have $a \in [0, 1]$. □

Corollary 3.6. *The set* $\{\alpha_f : f \in S_{X^*}\}$ *is a (lattice) orthogonal set.*

Proof. Suppose $f, g \in S_{X^*}$. By Corollary **??**, there exist $a, b \in [0, 1]$ such that

$$\alpha_f \wedge \alpha_g = a\alpha_f = b\alpha_g.$$

Therefore,

$$a = a\alpha_f(B_X) = b\alpha_f(B_X) = b.$$

This yields two possibilities: $a = b = 0$ and $\alpha_f \wedge \alpha_g = 0$, or $\alpha_f = \alpha_g$, and hence $f = g$. □

This yields an orthogonal, and hence linearly independent, subset of $\mathcal{R}(X)^*$ that is infinite when $\dim(X) > 1$, proving Proposition **??**.

The next result exposes the structure of $\{\alpha_f : f \in S_{X^*}\}$.

Theorem 3.7. $\{\alpha_f : f \in S_{X^*}\}$ *is a subset of the extreme points of* $\left(\mathcal{R}(X)^*_+ \cap S_{\mathcal{R}(X)^*}\right)$*, with equality if and only if* X *is finite-dimensional.*

Remark 3.8. As mentioned in Section **??**, $\mathcal{R}(X)$ embeds densely in the space of weak* continuous real functions on $\mathrm{Ext}(\mathcal{R}(X)^*_+ \cap S_{\mathcal{R}(X)^*})$. Theorem **??** characterises this set explicitly in the case where X is finite-dimensional. One can additionally show that, in this case, the map

$$S_{X^*} \to \mathrm{Ext}(\mathcal{R}(X)^*_+ \cap S_{\mathcal{R}(X)^*}) : f \mapsto \alpha_f$$

is a norm to weak* homeomorphism. Therefore, if X is finite-dimensional, $\mathcal{R}(X)$ embeds densely into $\mathbf{C}(S_{X^*})$.

We delay the proof of Theorem **??** until the end of the paper. Meanwhile, we introduce two subspaces of $\mathcal{R}(X)^*$:

$$\Sigma := \left\{ \alpha \in \mathcal{R}(X)^* : \alpha = \sum_{f \in S_{X^*}} c_f \alpha_f \right\},$$

where only countably many of the scalars, c_f, are non-zero,
and

$$\Sigma^\perp := \{\psi \in \mathcal{R}(X)^* : \psi \text{ is orthogonal to } \alpha_f \text{ for all } f \in S_{X^*}\}.$$

Theorem 3.9 (Structure theorem for functionals)**.**

$$\mathcal{R}(X)^* \ = \ \Sigma \ \oplus \ \Sigma^{\perp},$$

where $\oplus$ denotes direct sum.
Moreover, $\phi = \psi + \sum\limits_{f \in S_{X^*}} c_f \alpha_f \geq 0$ if and only if $\psi \geq 0$ and $c_f \geq 0$ for all $f \in S_{X^*}$.

Proof. We wish to express any $\phi \in \mathcal{R}(X)^*$ in the form $\phi = \psi + \sum_{f \in S_{X^*}} c_f \alpha_f$, where $\psi \in \Sigma^{\perp}$ and the coefficients c_f are 0 in all but countably many cases. By expressing $\phi = \phi^+ - \phi^-$, we may assume without loss of generality that $\phi \geq 0$.

For any $f \in S_{X^*}$, Corollary **??** guarantees there exists $c_f \in [0, \|\phi\|]$ such that $c_f \alpha_f = \phi \wedge (\|\phi\| \alpha_f)$. Note that c_f is the largest multiple of α_f that remains less than ϕ.

Since the collection $\{c_f \alpha_f : f \in S_{X^*}\}$ is orthogonal and each element is positive and less than ϕ, we have

$$\phi \geq \sum_{k=1}^{m} c_{f_k} \alpha_{f_k}, \text{ for all finite subcollections } \{f_1, \ldots, f_m\} \subseteq S_{X^*}.$$

By applying the functionals to B_X, we see that for any $n \in \mathbb{N}$, only finitely many c_k can exceed $1/n$. This implies that only countably many of the c_k are non-zero.

Consider the series $\sum_{f \in S_{X^*}} c_f \alpha_f$. Regardless of how it is ordered, the partial sums of the norms of its terms never exceeds $\|\phi\|$. Hence it is absolutely convergent, and well-defined.

Let $\psi = \phi - \sum_{f \in S_{X^*}} c_f \alpha_f \geq 0$ and fix $g \in S_{X^*}$. By Corollary **??**, there exists some $d \in [0, 1]$ such that $\psi \wedge \alpha_g = d\alpha_g$. Then

$$\left(\phi - \sum_{f \in S_{X^*} \setminus \{g\}} c_f \alpha_f \right) \wedge (\alpha_g + c_g \alpha_g) = \psi \wedge \alpha_g + c_g \alpha_g = (d + c_g)\alpha_g.$$

But then $(d + c_g)\alpha_g \leq \phi$, so by the construction of c_g, we must have $d = 0$. Thus $\psi \in \Sigma^{\perp}$, as required.

Uniqueness of the representation follows from Corollary **??**.

Lastly, note the final claim is a direct consequence of the construction of the unique representation of an arbitrary positive element. □

Proposition 3.10. *For $\phi \in \mathcal{R}(X)^*_+$,*

$$\phi\left(\mathcal{R}_{FD}(X)\right) = \{0\} \implies \phi \in \Sigma^{\perp}.$$

Proof. Fix $f \in S_{X^*}$. Let $x \in f^{-1}\{1\}$ and $L = \operatorname{co}\{0, x\}$. By Corollary **??**, there exists some $c \in [0,1]$ such that

$$\begin{aligned} \phi \wedge \alpha_f &= c\alpha_f \\ 0 \le c = c\alpha_f(L) &\le \phi(L) = 0. \end{aligned}$$

Thus, $\phi \perp \alpha_f$. □

However, the converse is demonstrably false; indeed, for $X = \mathbb{R}^n$, with $n \geq 2$, and μ Lebesuge measure,

$$\phi : \mathcal{R}(X) \to \mathbb{R} : A \ominus B \mapsto \int_{B_{X^*}} \alpha_f(A \ominus B)\ \mathrm{d}\mu(f),$$

is in $\Sigma_+^{\perp}$.

3.3. **Proof of Theorem ??.**

We say $A \ominus B \in \mathcal{R}(X)$ is **subcompact** if there exists some compact $K \in \mathcal{C}(X)$ such that $|A \ominus B| \leq K$. Note that any $C \in \mathcal{C}(X)$, considered as the pair $C \ominus \{0\}$, is subcompact if and only if C is compact. The set of subcompact pairs in $\mathcal{R}(X)$ forms a subspace, and is denoted by $\mathcal{R}_{sK}(X)$.

Our proof of theorem **??** will make use of the following proposition, a proof of which may be found in Coppel [**?**].

Proposition 3.11. *ϕ is an extreme point of mathcal$R(X)_+^* \cap S_{\mathcal{R}(X)^*}$ if and only if ϕ is a lattice homomorphism.*

Proof of Theorem **??**. It is straightforward to verify, using Theorem **??**, that α_f is an extreme point in $\mathcal{R}(X)_+^* \cap S_{\mathcal{R}(X)^*}$, for all $f \in S_{X^*}$. One can aslo verify that any other extreme points in $\mathcal{R}(X)_+^* \cap S_{\mathcal{R}(X)^*}$ must lie in $\Sigma^{\perp}$.

Suppose first that X is infinite-dimensional. Define Φ to be the subset of $\mathcal{R}(X)_+^* \cap S_{\mathcal{R}(X)^*}$ containing the annihilators of $\mathcal{R}_{sK}(X)$. By Proposition **??**, $\Phi \subseteq \Sigma^{\perp}$. It is straightforward to verify that Φ is a weak* compact face of $\mathcal{R}(X)_+^* \cap S_{\mathcal{R}(X)^*}$, and hence by the Krein-Milman theorem, we need only establish $\Phi \neq \emptyset$. However, this can be seen by noting that $\mathcal{R}_{sK}(X)_+$ is a face of $\mathcal{R}(X)_+$, and that int $\mathcal{R}_{sK}(X)_+ \neq \emptyset$, then applying the Hahn-Banach Separation Theorem.

Otherwise, suppose X is finite-dimensional and

$$\phi \in \Sigma^{\perp} \cap \operatorname{Ext}(\mathcal{R}(X)_+^* \cap S_{\mathcal{R}(X)^*}).$$

Fix any $f \in S_{X^*}$. Then

$$0 = (\phi \wedge \alpha_f)(B_X) = \inf_{\{0\} \le A \ominus B \le B_X} \phi(A \ominus B) + \alpha_f(B_X \ominus A \oplus B).$$

In particular, there exists $A_f \ominus B_f \in \mathcal{R}(X)$ such that

$$\phi(A_f \ominus B_f) < 1/2$$
$$\alpha_f(A_f \ominus B_f) > 1/2$$
$$\{0\} \leq A_f \ominus B_f \leq B_X.$$

Let $\mathcal{U}(f) = \{g \in S_{X^*} : \alpha_g(A_f \ominus B_f) > 1/2\}$. Note that for $C \in \mathcal{C}(X)$, the map $g \mapsto \alpha_g(C)$ is the support function on C. Therefore $g \mapsto \alpha_g(A_f \ominus B_f)$ is a difference convex function on a finite dimensional space, and hence is continuous. Thus $\mathcal{U}(f)$ is open, and the set of such sets forms an open cover of S_{X^*}. Using compactness, there exist $f_1, \ldots, f_m \in S_{X^*}$ such that

$$\bigcup_{k=1}^{m} \mathcal{U}(f_{n_k}) = S_{X^*}.$$

Let

$$A \ominus B = \bigvee_{k=1}^{m} \left(A_{f_{n_k}} \ominus B_{f_{n_k}}\right).$$

We claim that $A \ominus B \geq B_X/2$. Suppose this claim is false. Then there exists some point $x \in (B \oplus B_X/2) \setminus A$, and $g \in S_{X^*}$ such that $g(x) > g(a)$ for all $a \in A$.

Choose some n_k such that $g \in \mathcal{U}(f_{n_k})$. Then

$$\begin{aligned} &\alpha_g(A_{f_{n_k}} \ominus B_{f_{n_k}}) > 1/2 \\ \implies\ &\alpha_g(A \ominus B) > 1/2 \\ \implies\ &g(x) \geq \alpha_g(A) > \alpha_g(B \oplus B_X/2), \end{aligned}$$

which contradicts $x \in B \oplus B_X/2$. Thus $A \ominus B \geq B_X/2$ as required.

By Proposition **??**,

$$\phi(A \ominus B) = \max\{\phi(A_{f_{n_k}} \ominus B_{f_{n_k}}) : 1 \leq k \leq m\} < 1/2.$$

However, we also have

$$\phi(A \ominus B) \geq \phi(B_X/2) = \|\phi\|/2 = 1/2,$$

which is a contradiction. □

4. Some possible synergies with metric fixed point theory

For $C \in \mathcal{C}(X)$, we have seen how a nonexpansive map $T : C \to X$ induces a nonexpansive map

$$\rho_T : \mathcal{C}(C) \subset \mathcal{R}(X) \to \mathcal{R}(X),$$

where $\mathcal{C}(C) := \{A \in \mathcal{C}(X) : A \subseteq C\}$.

The fixed points of ρ_T are the invariant sets for T and the lattice minimal elements of $\text{Fix}(\rho_T)$ are the minimal invariant sets of T.

Thereby, opening the possibility of transferring:

(i) the structure of fixed point sets to the family of (minimal) invariant sets of T,
(ii) algorithms for approximating fixed points to ways of approximating invariant set,
(iii) results concerning approximate (ϵ-) fixed point sets [Bruck *et al*] to matching results for invariant sets.

Set valued mappings

A multifunction $\tau : C \in \mathcal{C}(X) \to 2^X$ taking nonempty closed bounded convex values can be regarded as a mapping

$$T : \mathbf{C} := \{\, \{x\} : x \in C\} \subset \mathcal{R}(X) \to \mathcal{R}(X) : \{x\} \mapsto \tau(x).$$

Further, if $H(\tau(x), \tau(y)) \leq \|x - y\|$ then T is nonexpansive, allowing us to exploit results from the theory of single valued mappings and suggesting the possibility of Leray-Schauder type results in this context.

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