Projection methods in geodesic metric spaces

Brailey Sims

Computer Assisted Research Mathematics & Applications
University of Newcastle
http://carma.newcastle.edu.au/

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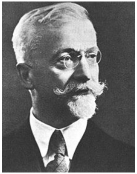
Introduction

Metric spaces in which every pair of points can be joined by an arc isometric to a compact interval of the real line (*geodesic metric spaces*) and every triangle satisfies the "$CAT(\kappa)$ inequality" (*CAT(κ) spaces*)are of considerable interest because not only does this inequality capture the concept of non-positive curvature, but spaces satisfying this condition have much of the geometry inherent in Euclidean space (a CAT(0) space).

The acronym CAT is derived from the names Elie Joseph Cartan $(1869 - 1951)$, Alexander Danilovich Alexandrov $(1912 - 1999)$ and Victor Andreevich Toponogov $(1930 - 2004)$ in recognition of their pioneering work.

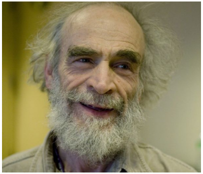


Cartan

Alexandrov

Toponogov

Mikhael Gromov (b 1943) gave prominence to one of Alexandrov's definitions for what it might mean for a metric space to have a curvature bounded above by a real number κ which he called the $CAT(\kappa)$ inequality.



Mikhail Gromov

Comparison spaces

Curvature bounds on the space could be defined by comparing triangles in that space to triangles in appropriate *comparison spaces* M^2_κ where,

$$M^2_\kappa = \begin{cases} S^2_\kappa, & \text{if } \kappa > 0; \\ E^2, & \text{if } \kappa = 0; \\ H^2_\kappa, & \text{if } \kappa < 0. \end{cases}$$

Here:

S^2_κ is classical two dimensional spherical space of constant positive curvature κ; namely, the two sphere,

$$\{x = (x_1, x_2, x_3) \in \mathbb{E}^3 : ||x|| := \sqrt{\langle x|x\rangle} = 1\}$$

where $\langle x|y\rangle := x_1y_1 + x_2y_2 + x_3y_3$, equipped with the metric, $d(x, y) := \frac{1}{\sqrt{\kappa}} \cos^{-1}(\langle x|y\rangle)$,

$\mathbb{E}^2$ is two dimensional Euclidean space and

Comparison spaces

Curvature bounds on the space could be defined by comparing triangles in that space to triangles in appropriate *comparison spaces* M_κ^2 where,

$$M_\kappa^2 = \begin{cases} S_\kappa^2, & \text{if } \kappa > 0; \\ E^2, & \text{if } \kappa = 0; \\ H_\kappa^2, & \text{if } \kappa < 0. \end{cases}$$

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H^2_κ is the hyperbolic two manifold of constant negative curvature κ; namely,

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle x|x\rangle = -1,\ x_3 > 0\}$$

where here $\langle x|y\rangle := x_1y_1 + x_2y_2 - x_3y_3$ and the distance between x and y is $d(x,y) := \frac{1}{\sqrt{-\kappa}}\cosh^{-1}(-\langle x|y\rangle)$.

Comparison spaces

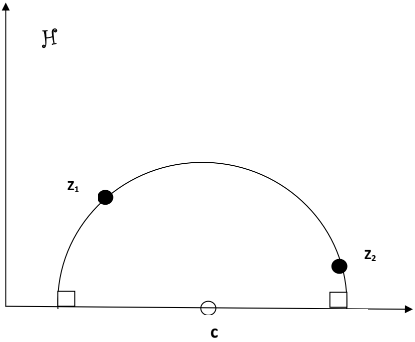
Equivalently, H^2_κ may be identified with the Poincaré upper half-plane $\{z \in \mathbb{C} : \Im z > 0\}$ equipped with the metric $\frac{1}{\sqrt{-\kappa}} d_P$ where

$$d_P(z_1, z_2) \;=\; \int_{z_1}^{z_2} \frac{|dz|}{\Im z} \;=\; \cosh^{-1}\left(1 + \frac{|z_1 - z_2|^2}{2\Im z_1 \Im z_2}\right).$$

In which case, geodesics are semicircles with centres on the extended real axis.

Henri Poincaré 1854–1912

Comparison spaces



The geodesic through the points $\mathbf{z_1}$ and $\mathbf{z_2}$ in H^2_{-1}

Geodesic spaces

Let (X, d) be a metric space.

Definition

For x, $y \in X$ with $x \neq y$ a *geodesic segment* from x to y in X is an isometry $\gamma_{xy} : [0, d(x, y)] \mapsto X$ with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x, y)) = y$.

Definition

A *geodesic ray* from $\gamma(0)$ is an isometry $\gamma : [0, \infty) \to X$

Definition

A *geodesic line* is an isometry $\gamma : \mathbb{R} \to X$.

Definition

The metric space (X, d) is called a *geodesic metric space* if for every pair of points $x, y \in X$ there is a geodesic segment from x to y.

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Geodesic spaces

Definition

We say (X, d) is *uniquely geodesic* if for all $x, y \in X$ there is exactly one geodesic segment from x to y in which case we use $[x, y]$ to denote the unique segment from x to y.

If the space is complete being a geodesic metric space is equivalent to being Menger (or metrically) convex; that is, for every pair of points $x, y \in X$ and $t \in [0, 1]$ there exists a point z_t such that $d(x, z_t) = (1 - t)d(x, y)$ and $d(z_t, y) = td(x, y)$. When the space is uniquely geodesic we denote z_t by $(1 - t)x \oplus ty$.

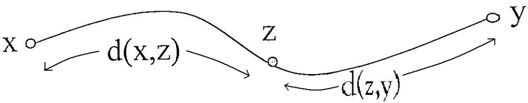

Figure: Metric (or Menger) convexity.

Karl Menger 1902–1985

Definition

We say *geodesics are extendable* or the space has the *geodesic extension property* if every geodesic segment is contained within a geodesic line.

NOTE: In a complete space this is equivalent to geodesics being "locally extendable".

Karl Menger 1902–1985

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Convex sets in geodesic metric spaces

Definition

A subset C of a geodesic metric space is *convex* if whenever x, y are in C every metric segment from x to y also lies in C.

Equivalently, for a complete uniquely geodesic space, $(1-t)x \oplus ty \in C$ whenever $x, y \in C$ and $t \in [0, 1]$.

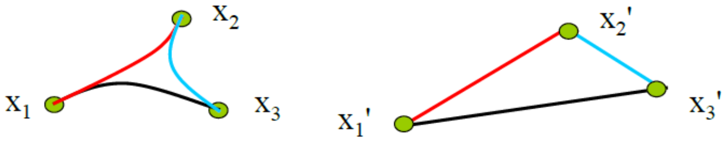
Geodesic and comparison triangles

Definition

A *geodesic triangle* $\triangle$ in a geodesic metric space consists of three points $x, y, z \in X$ and three geodesic segments γ_{xy}, γ_{yz} and γ_{zx}.

Definition

A **comparison triangle** for $\triangle$ is a geodesic triangle $\bar{\triangle}$ in M^2_κ with vertices $\bar{x}, \bar{y}, \bar{z}$ such that $d(\bar{x}, \bar{y}) = d(x, y)$, $d(\bar{y}, \bar{z}) = d(y, z)$ and $d(\bar{z}, \bar{x}) = d(z, x)$. For $p \in [x, y]$ the comparison point in $\bar{\triangle}$ is the point $\bar{p} \in [\bar{x}, \bar{y}]$ with $d(\bar{x}, \bar{p}) = d(x, p)$. Comparison points for points in $[y, z]$ and $[z, x]$ are defined in a similar way.



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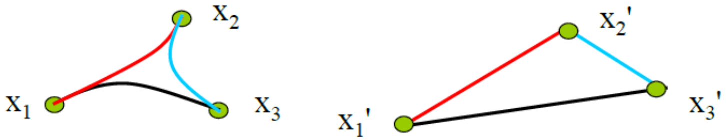

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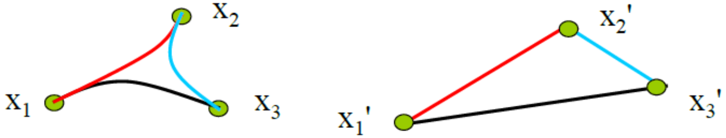

$CAT(\kappa)$ spaces

A geodesic triangle satisfies the $CAT(\kappa)$ *inequality* if for every pair of points p and q on it we have $d(p,q) \leq d(\bar{p},\bar{q})$ where $\bar{p}$ and $\bar{q}$ are the respective corresponding points of the comparison triangle in M^2_κ.

Definition

For a given κ, a $CAT(\kappa)$ space is a geodesic metric space in which every geodesic triangle satisfies the $CAT(\kappa)$ inequality.

Such spaces enjoy a rich geometric structure. We will be especially interested in spaces which are $CAT(0)$.

Spaces with *curvature bounded below*; that is, spaces X for which $\inf\{\kappa : X \text{ is a } CAT(\kappa) \text{ space}\} > -\infty$, have non-bifurcating geodesics and are important to us because if geodesics are extendable their extensions are unique.

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Angles in a geodesic metric spaces

For any three points x, y, z in a (geodesic) metric space, in the comparison triangle $\bar{\triangle}(x, y, z)$ the interior angle at $\bar{x}$ is called the comparison angle subtended at x by y and z and we denote it by $\bar{\angle}_x(y, z)$.

NOTE: $\bar{\angle}_x(y, z)$ is the unique angle $\{\theta : 0 \leq \theta \leq \pi\}$ such that

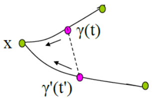
$$d(x, y)^2 = d(x, z)^2 + d(z, y)^2 + 2d(x, z)d(z, y)\cos\theta.$$

Alexandrov angles

Definition

Let $\gamma : [0, a] \mapsto X$ and $\gamma' : [0, a'] \mapsto X$ be two geodesics with $\gamma(0) = \gamma'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$ we consider the comparison triangle $\bar{\Delta}(\gamma(0), \gamma(t), \gamma'(t'))$ and the comparison angle $\bar{\angle}_{\gamma(0)}(\gamma(t), \gamma'(t'))$. The *Alexandrov angle* (or *upper angle*) between the geodesics γ and γ' at $\gamma(0)$ is

$$\angle(\gamma, \gamma') = \limsup_{t,t' \to 0} \bar{\angle}_{\gamma(0)}(\gamma(t), \gamma'(t'))$$



NOTE: In $CAT(0)$ spaces
$\angle(\gamma, \gamma') = 2 \lim_{t \to 0} \arcsin(d(\gamma(t), \gamma'(t))/2)$.

Hadamard spaces

The definition of a $CAT(0)$ spaces did not require that X be complete.However, in our work we will assume that our spaces are complete and we recall that **complete** $CAT(0)$ **spaces are often called Hadamard spaces**.

Jacques Salomon Hadamard 1865–1963

Basic Facts for $CAT(0)$ spaces

If (X, d) is a $CAT(0)$ space then

(1) (X, d) is a hyperbolic metric space in the sense of Gromov;

(2) Geodesics are unique;

(3) Geodesics vary continuously with their endpoints;

(4) Approximate mid-points are near to mid-points;

(5) Every closed ball $B[x, r] := \{y; d(y, x) \leq r\}, r > 0$ is metrically convex, and is contractible to a point;

(6) The metric d is a convex function.

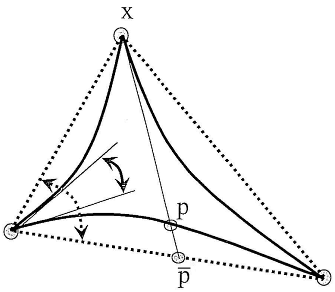
Basic Facts for $CAT(0)$ spaces

For a geodesic metric space (X, d) the following are equivalent:

(1) (X, d) is a $CAT(0$ space;

(2) If x is any vertex of a geodesic triangle and p is any point on the opposite side then $d(x, p) \leq d_2(\bar{x}, \bar{p})$;

(3) The Alexandrov angle at any vertex of a geodesic triangle is less than or equal to the angle at the corresponding vertex of the comparison triangle.



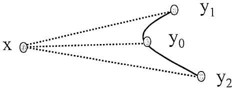
This is equivalent to the law of cosines; for a geodesic triangle with sides of length a,b and c we have,

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma$$

where γ is the Alexandrov angle between the sides of length a and b.

(4) The CN (Courbure Négative)-inequality of Bruhat and Titts holds. That is for any three points $x, y_1, y_2 \in X$, and noting that y_0 is the metric midpoint of y_1 and y_2 if $d(y_1, y_2) = d(y_1, y_0) + d(y_0, y_2)$ and $d(y_1, y_0) = d(y_0, y_2)$ then

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$



FRANCOIS BRUHAT

JACQUES TITS

Basic Facts for $CAT(0)$ spaces

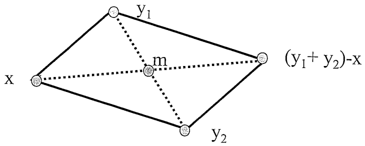
We recall that the parallelogram law for a translated parallelogram with vertices at x, y_1, y_2 and $(y_1 + y_2) - x$ in a Hilbert space is

$$||(y_1 + y_2) - 2x||^2 + ||y_1 - y_2||^2 = 2(||y_1 - x||^2 + ||y_2 - x||^2).$$

By rearranging and dividing by 4 we get

$$||\frac{1}{2}(y_1 + y_2) - x||^2 = \frac{1}{2}(||y_1 - x||^2 + \frac{1}{2}||y_2 - x||^2) - \frac{1}{4}||y_1 - y_2||^2.$$

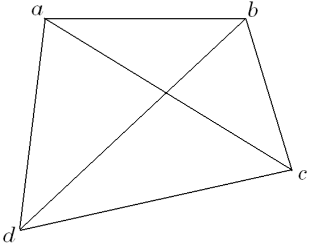
After noting that $m = \frac{1}{2}(y_1 + y_2)$ is the (metric) midpoint of y_1 and y_2, we recognize this as the case of equality in the CN-inequality for points in a Hilbert space.



Basic Facts for $CAT(0)$ spaces

Fortunately the CN-inequality goes the right way to allow many Hilbert space arguments that depend on the parallelogram law to continue to work in $CAT(0)$ spaces.

(5) The 4-point condition holds. That is for any 4 points a, b, c, d let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the corresponding comparison points in R^2 such that $d(a,b) = d(\bar{a},\bar{b})$, $d(b,c) = d(\bar{b},\bar{c})$, $d(c,d) = d(\bar{c},\bar{d})$ and $d(d,a) = d(\bar{d},\bar{a})$ then $d(a,c) \leq d(\bar{a},\bar{c})$ and $d(b,d) \leq d(\bar{b},\bar{d})$.



Nearest Point Projections

The following proposition ensures the existence of the nearest point projection, $P_C : X \to C$, onto a closed convex subset C of a Hadamard space.

Proposition

Let X be a $CAT(0)$ space and C be a convex subset which is complete in the induced metric. Then,

(1) *for every $x \in X$ there exists a unique point $P_c(x) \in X$ such that $d(x, P_C(x)) = d(x, C) := \inf_{y \in C} d(x, y)$;*

(2) *if y belongs to the geodesic segment $[x, P_C(x)]$ we have $P_C(y) = P_C(X)$;*

(3) *for any $x \in X \backslash C$ and $y \in C \backslash P_C(x)$ we have $\angle_{P_C(x)}(x, y) \geq \frac{\pi}{2}$*

(4) *P_C is a nonexpansive retraction onto C; the map $H : X \times [0, 1] \to X$ sending (x, t) to the point a distance $td(x, P_C(x))$ from x on the geodesic segment $[x, P_C(x)]$ is a continuous homotopy from the identity map I to P_C.*

Corollary

For X a $CAT(0)$ space, C a complete convex subset in X and d_C the distance function to C, that is $d_C(x) = d(x, C)$ then it follows that

(1) d_C *is a convex function;*

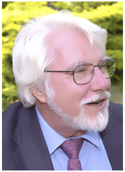
(2) $|d_C(x) - d_C(y)| \leq d(x, y)$*;*

(3) *the restriction of d_C to a sphere with centre x and radius $r \leq d_C(x)$ means it attains its infimum at a unique point y and $d_C(x) = d_C(y) + r$.*

An analogue of weak sequential convergence in $CAT(0)$ spaces

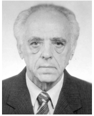
One of the difficulties in extending results from Hilbert spaces into $CAT(0)$ spaces is the seeming lack of a dual space and hence of a weak topology and of weak-compactness.

However, as noted by Art Kirk and others, many arguments involving weak-compactness can be replaced by asymptotic centre arguments.



William A. (Art) Kirk

In 1976, T. C. Lim introduced a concept of convergence in a general metric space setting which he called Δ-convergence. Kuczumow introduced an identical notion of convergence in Banach spaces which he called 'almost convergence'. Kirk and B. Panyanak adapted Lim's concept to CAT(0) spaces. They also showed that many Banach space results involving weak convergence have precise analogues in CAT(0) spaces; for example, Opial's property, the Kadec-Klee property and the demiclosedness principle for nonexpansive mappings.

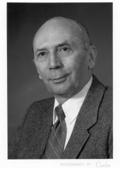


Mikhail I. Kadets (1923–2011) **Victor Klee** (1925–2007)

Let X be a complete $CAT(0)$ space. For any bounded sequence (x_n) its asymptotic radius about x is $r(x,(x_n)) := \limsup_n d(x,x_n)$; its asymptotic radius is $r((x_n)) : \inf\{r(x,(x_n)) : x \in X\}$; and, its *asymptotic centre* is

$$A((x_n)) := \{x : r(x,(x_n)) = r((x_n)\}.$$

The notion of asymptotic centre of a sequence seems first to have been introduced by Michael Edelstein (1917–2003) [**?**]



Michael Edelstein

$\triangle$-convergence

Definition

A sequence (x_n) in X is said to $\triangle$-*converge to* x, $(x_n \rightharpoonup x)$, if x is the unique asymptotic centre of every subsequence (x_{n_k}) of (x_n).

A bounded sequence (x_n) in X is said to be *regular* if $r((x_n)) = r((x_{n_k}))$ for every subsequence (x_{n_k}) of (x_n). It is known that every bounded sequence has a regular subsequence. Since every regular sequence $\triangle$-converges, the following proposition follows immediately;

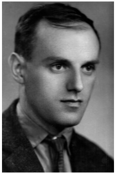
Proposition

Every bounded sequence of a complete $CAT(0)$ space has a $\triangle$-convergent subsequence.

$\triangle$-convergence

We note at this stage that the very definition of asymptotic centre shows that $CAT(0)$ spaces enjoy a property that has become known as Opial's property: that is for $(x_n) \subset X$ such that (x_n) $\triangle$ - converges to x and for $y \in X$ with $y \neq x$ we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$



Zdzislaw Opial (1930–1974)

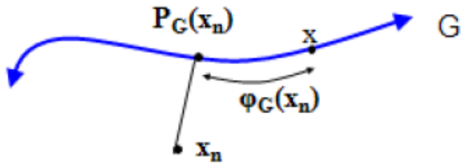
We also note that in a Hilbert space, $\triangle$-convergence and weak convergence coincide. This is a simple consequence of closed balls being weak compact and Opial's property.

Accordingly, in the context of CAT(0) spaces, we will and henceforth refer to $\triangle$-convergence as simply *weak convergence* and write $x_n \rightharpoonup x$.

A reformulation of weak-convergence in CAT(0) spaces

Let X be a $CAT(0)$ space, for $x \in X$ and G, any geodesic through x, we define the function $\phi_G : X \to R$ by

$$\phi_G(x_n) := d(x, P_G(x_n)).$$



A specialization of Sosov's notion of $\phi-$convergence .

A reformulation of weak-convergence in CAT(0) spaces II

Definition

A bounded sequence $(x_n) \subseteq X$ is said to $\phi-$converge to a point $x \in X$ if

$$\lim_{n\to\infty} \phi_{_G}(x_n) = 0,$$

for every geodesic G containing x.

Proposition

A sequence $(x_n) \subseteq X$ is weak-convergent to $x \in X$ if and only if it is $\phi-$convergent to $x \in X$.

Basic facts about Weak-convergence in CAT(0) spaces

(1) Closed convex sets are weakly closed.

(2) Closed bounded convex sets are weakly sequentially compact.

(3) For a convex set C the distance function d_C is weak lower semi-continuous.

(4) [Demiclosedness principle] If T is a nonexpansive self mapping of a closed convex set, $d(x_n, Tx_n) \to 0$ and $x_n \rightharpoonup x$, then $x = Tx$.

As a corollary we obtain Kirk's theorem: *any nonexpansive self mapping of a non-empty closed bounded convex subset of a CAT(0) space has a fixed point.*

New CAT(0) spaces from old

It is easily verified that:

(1) Convex subspaces of CAT(0) spaces are themselves CAT(0) spaces.

(2) If X and Y are CAT(0) spaces then so too is their product, $X \times Y$, equipped with the metric

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2)^2 \ + \ d_Y(y_1, y_2)^2}$$

(2) Closed bounded convex sets are weakly sequentially compact.

(3) Ultraproducts of CAT(0) spaces are CAT(0) spaces.

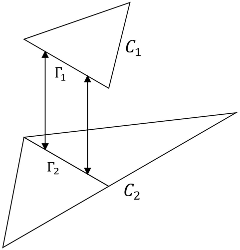
Another important way of building new CAT(0) spaces is:

Reshetnyak Gluings

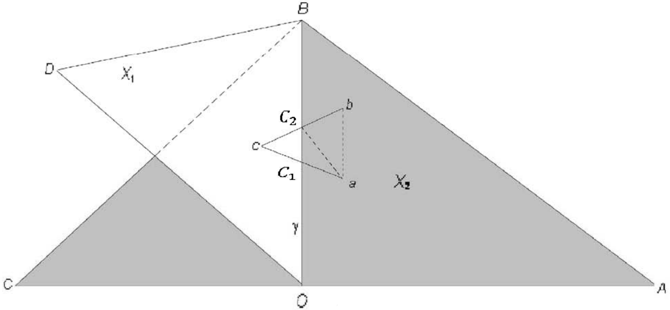
Theorem

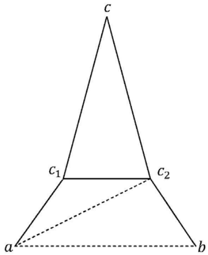
(Reshetnyak's Gluing Theorem.) Let $\{(X_i, d_i)\}, i = 1, 2$ be two complete spaces of curvature $\leq k$. Suppose that there are convex sets $C_i \in X_i$ and an isometry $f : C_1 \to C_2$. Attach these spaces together along the isometry f.
Then the resulting space (X, d) is a space of curvature $\leq k$.

We illustrate with an example.



By identifying Γ_1 with Γ_2 we can "glue" C_1 to C_2 along Γ_2 to produce the space shown.





Projection Algorithms in CAT(0) spaces

The convex feasibility problem associated with the nonempty closed convex sets A, B is to

$$\text{"find some } x \in A \cap B\text{"}.$$

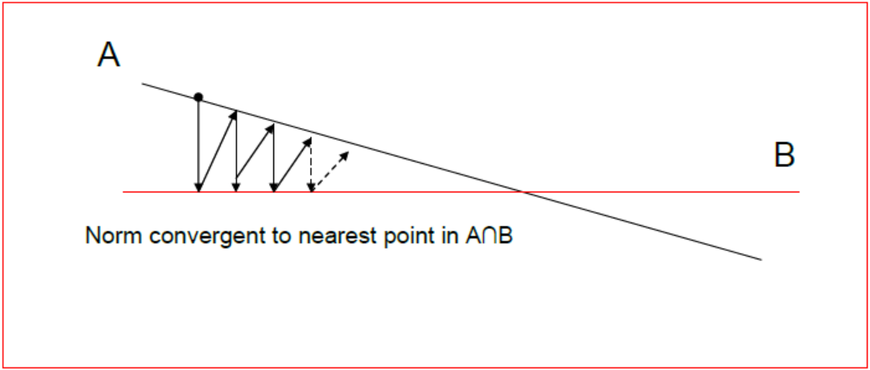
Projection algorithms in general aim to compute such a point. We consider two such algorithms in the context of CAT(0) spaces.

This allows us to treat feasability problems where the sets are metrically, but not necessarily algebraically, convex. For example star shaped sets in E^2.

These projection algorithms play key roles in optimization and have many applications outside mathematics - for example in medical imaging. An extension into CAT(0) spaces allows their use in a much more general setting where there may be no natural linear structure, for example tree spaces, state spaces, phylogenomics and configuration spaces in robotics.

The method of **alternating projection** into convex sets (sometimes known as "project, project") emerged from initial work by John von Neumann (1903 – 1957) who, in the $1930s$, proved that when A and B were closed affine manifolds of a Hilbert space the iterative scheme $x_{n+1} = P_B P_A x_n$ converged in norm for any initial starting point x_0 to $P_{A\cap B}x_0$. .

John von Neumann



von Neumann's alternating projection method

Alternating Projection method in CAT(0) spaces

In 1965, weak convergence was established by L. M. Bregman when $A, B \in H$ are closed convex sets in a Hilbert space with $A \cap B \neq \emptyset$. Examples show that norm convergence need not occur.

The Hilbert space proof can be adapted to obtain an analogous result in CAT(0) spaces [Bacak, Searston & S].

Theorem

Let X be a complete CAT(0) space and $A, B \subset X$ convex, closed subsets such that $A \cap B \neq \emptyset$. Let $x_0 \in X$ be a starting point and (x_n) be the sequence generated by alternating projections. Then (x_n) weakly converges to a point $x \in A \cap B$.
Strong convergence pertains when A and B satisfy certain "regularity" conditions and various estimates on the rate of convergence are possible.

The **Douglas-Rachford algorithm**, also known as *reflect-reflect-average* first appeared in a 1956 paper by Jim Douglas Jr and Henry. H. Rachford Jr.

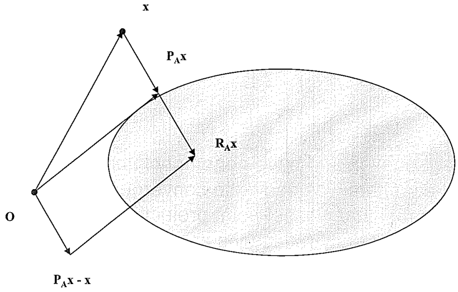
Jim Douglas **Henry H. Rachford**

Douglas-Rachford method

In a Hilbert space H we define *reflection* in a closed convex set A as

$$R_A x = P_A x + (P_A x - x) = (2P_A - I)x$$

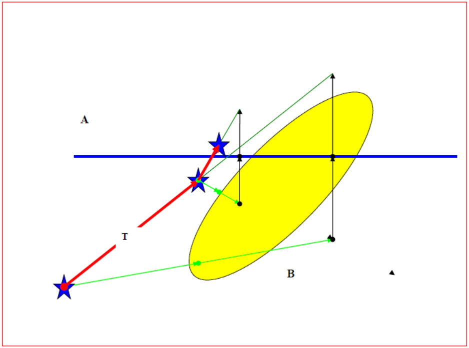
where P_A is the closest point projection onto A.



Douglas-Rachford method

Starting with any initial point x_0, the Douglas-Rachford algorithm is the iterative scheme

$$x_{n+1} := T(x_n) \text{ where, } T = \frac{1}{2}(R_A R_B + I).$$



Douglas-Rachford method II

Provided A and B are convex and have a non empty intersection the Douglas-Rachford algorithm was shown to converge weakly to a point x with $P_B x \in A \cap B$, by P-L. Lions and B. Mercier in 1979.

Pierre-Louis Lions **Bertrand Mercier**

Impediments to extending Douglas-Rachford into CAT(0) spaces.:

How to define reflection?

How to show convergence?

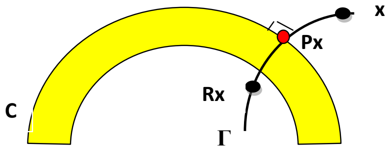
To discuss reflections in $CAT(0)$ spaces we require geodesics to be extendable. We also require that the extension is unique which happens if and only if the curvature is bounded below.

Reflections in $CAT(0)$ spaces

With the above conditions we can define the reflection of a point x in a closed convex subset C of X, a $CAT(0)$ space, to be a point $R_C(x)$ on a geodesic which is an extension of the segment $[x, P_C(x)]$ such that

$$d(R_C(x), P_C(x)) = d(x, P_C(x)),$$

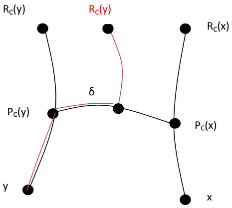
where $P_C x$ is the projection of x onto the set C.



Non-expansivity of Reflections in $CAT(0)$ spaces

It is well known that reflections in Hilbert space are non-expansive; this follows since the closest point projection is firmly nonexpansive, something which is also true in an appropriate sense in CAT(0) spaces [David Ariza].

R-trees are also an example of a $CAT(0)$ space in which reflections are non-expansive.



Proposition (Fernández-Leon – Nicolae, 2012)

For $k \in R$ and $n \in N$. Suppose C is a nonempty closed and convex subset of M_k^n and $x, y \in M_k^n$ such that $dist(x, C), dist(y, C) < D_k/2$. Then,

$$d(R_C x, R_C y) \leq d(x, y).$$

Using this they go on to establish weak convergence of Douglas-Rachford in such spaces of constant negative curvature.

However, in general reflections in CAT(0) spaces need not be nonexpansive.

To conclude we construct and investigate a special instance of a CAT(0) space of non-constant curvature.

We begin with the CAT(0) space Φ consisting of the geodesic (convex subset) $|z| = 1$ in the Poincar/'e upper half-plane which we may identify with

$$\Phi = \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), d_P\right)$$

where d_P is the restriction of the"Poincaré metric" given by

$$d_P(\phi_1, \phi_2) \;=\; \int_{\phi_1}^{\phi_2} \frac{d\phi}{\cos(\phi)} \;=\; [\ln(\sec(\phi) + \tan(\phi))]_{\phi_1}^{\phi_2}.$$

Since the function $\ln(\sec(\phi) + \tan(\phi))$ occurs frequently in what follows we will denote it by $H(\phi)$

$X := \Phi \otimes_2 |\mathbb{E}^1|$

$X := \Phi \otimes_2 |\mathbb{E}^1|$ - the ℓ_2^2 - direct product of Φ with $|\mathbb{E}^1|$, the positive cone in 1-dimensional, Euclidean space, and metric given by,

$$\begin{aligned} d_X((\phi_1, h_1), (\phi_2, h_2)) &= \sqrt{(d_P(\phi_1, \phi_2))^2 + (h_1 - h_2)^2} \\ &= \sqrt{(H(\phi_2) - H(\phi_1)^2 + (h_1 - h_2)^2}, \end{aligned}$$

for $h_1, h_2 > 0 \text{ and } -\pi/2 < \phi_1, \phi_2 < \pi/2$.

NOTE: X is flat (curvature 0).

Geodesics in X

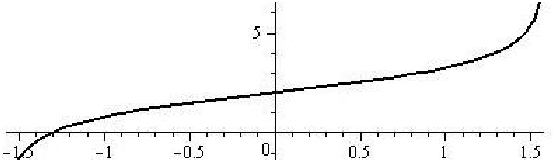
The unique **geodesic** Γ in X passing through two distinct points $P_1 : \ (\phi_1, h_1)$ and $P_2 : (\phi_2, h_2)$ is:

the'vertical' half line $\{(\phi_1, h) : h > 0\}$ if $\phi_1 = \phi_2$,
otherwise, using the Euler-Lagrange equation to minimize the length of a curve from P_1 to P_2, we find Γ has equation,

$$h(\phi) = AH(\phi) + B,$$

where the constants A and B are uniquely determined from the condition that $P_1, \ P_2 \in \Gamma$, in particular $A = \dfrac{h_1 - h_2}{H(\phi_1) - H(\phi_2)}$.

A geodesic in X



A geodesic in $X := \Phi \otimes_2 \mathbb{E}^1$

Calculations in X

The **length of the segment** of Γ between P_1 and P_2 is,

$$l(\Gamma_{P_1 P_2}) = \frac{\sqrt{1 + A^2}}{A} |h_1 - h_2|,$$

when $\phi_1 \neq \phi_2$, and

$$l(\Gamma_{P_1 P_2}) = |h_1 - h_2|,$$

when the geodesic is vertical.

The **midpoint**, (ϕ_m, h_m), of $\Gamma_{P_1 P_2}$ has, $h_m = (h_1 + h_2)/2$ and

$$\phi_m = 2 \tan^{-1}\Big(\frac{e^{H_m} - 1}{e^{H_m} + 1}\Big),$$

where $H_m = (h_m - B)/A$.

In the upper half plane model of the hyperbolic space H^2_{-1} let $Y = \{z : \Im z > 0, |z| \leq 1\}$ equipped with the metric d_P inherited from H^2_{-1}.

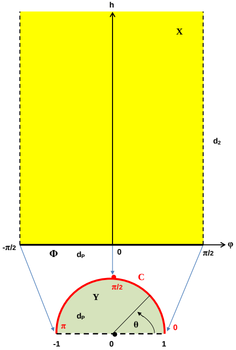
Y is a closed, convex subset of H^2_{-1} and hence a $CAT(0)$ space of constant curvature -1.

Let C be the geodesic in Y given by $C = \{e^{i\theta} : 0 < \theta < \pi\}$. Then, C is also a closed, convex subset of Y and under the mapping $\phi \mapsto \; e^{i(\frac{\pi}{2}-\phi})$, Φ **is isometric to** C.

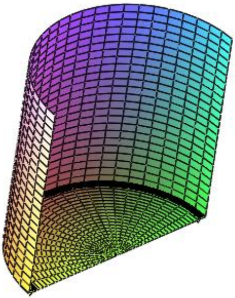
The space Z

Z is obtained by gluing X to Y under the identification of Φ with C which by Reshetnyak's gluing theorem is a CAT(0) space of non-constant curvature, bounded below by -1.

Geodesics in Z are uniquely extendable, and so **reflection in closed convex sets of Z is well defined**.

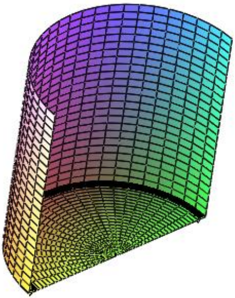


3-d model of Z



A model for Z as a submanifold of $\mathbb{E}^3$

More usefully, however,



A model for Z as a submanifold of $\mathbb{E}^3$

More usefully, however,

An upper-half plane model for Z

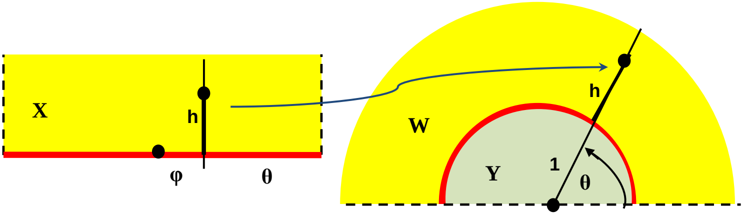
We model Y in the upper half-plane as above and identify points in X with points in $W := \{\rho e^{i\theta} : \rho \geq 1, 0 < \theta < \pi\}$ under the mapping

$$(\phi, h) \mapsto (1+h)e^{i(\frac{\pi}{2}-\phi)}.$$

This naturally identifies Θ with C and is an isometry when W is equipped with the metric,

$$d_W(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2} = \sqrt{(R(\theta_2) - R(\theta_1))^2 \; + \; (\rho_2 - \rho_1)^2},$$

where $R(\theta) := H(\frac{\pi}{2} - \theta) = \ln(\text{cosec}(\theta) + \cot(\theta))$.



Geodesics in the upper half-plane model for Z

Geodesics in W are radial rays: $z = \rho e^{i\theta}$, where $\rho \geq 1$ and θ is constant; or curves of the form,

$$z = (AR(\theta) + b)e^{i\theta}.$$

The geodesic segment joining $P_1 \in W$ to $P_2 \in Y$ is $[P_1, Q] \cup [Q, P_2]$ where $Q = e^{i\theta_0} \in C$ is chosen so that $d_W(P_1, Q) + d_P(Q, P_2)$ is a minimum.

The extension of a geodesic Γ in W into Y (or vice versa) may be determined from the requirement that $\dfrac{d\rho}{d\theta}$ be continuous at the point where Γ meets C.

All these calculations have been implemented in Maple.

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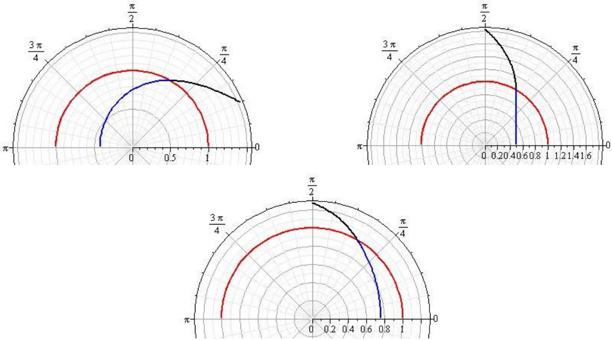
Some geodesics in Z



Some geodesics in the upper half-plane model of Z

Reflections in C

To exploit the structure of Z we consider reflections in C (identified with Φ).

The projection of a point z onto C is determined by the condition that the geodesic segment $[z, P_C(z)]$ meet C orthogonally.

The reflection of z is the point $R_C(z)$ on the extension of $[z, P_C(z)]$ such that $P_C(z)$ is the midpoint of $[z, R_C(z)]$.

From the uniqueness of geodesics it follows that,

$$R_C|_Y : Y \to W \quad \text{and } R_C|_W : W \to Y,$$

are inverses.

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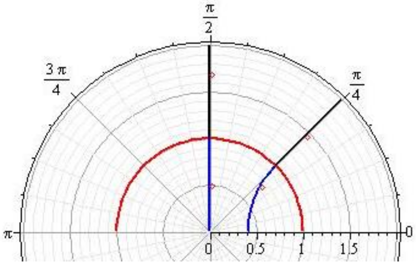
View of reflections in C



reflection in C

Reflection need not be nonexpansiive

In general $R_C|_Y$ is nonexpansive, but $R_C|_W = (R_C|_Y)^{-1}$ need not be.

For instance, as illustrated in the previous slide, the points $P_1 = i/2$ and $P_2 = 0.5439 + 0.4925i$ in Y
have
$Q_1 := R_C(P_1) = 1.6931i$ and $Q_2 := R_C(P_2) = 1.453e^{\pi/4}$
and

$$d_W(Q_1, Q_2) = 0.9135 \ < \ d_Y(P_1, P_2) = 1.0476$$

and so,

$$d_Z(R_C(Q_1), R_C(Q_2)) = d_Y(P_1, P_2) > d_Z(Q_1, Q_2)\,!$$

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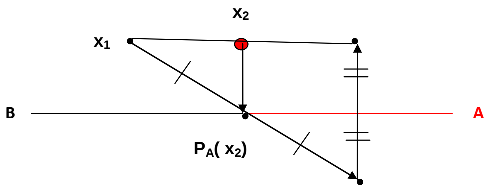
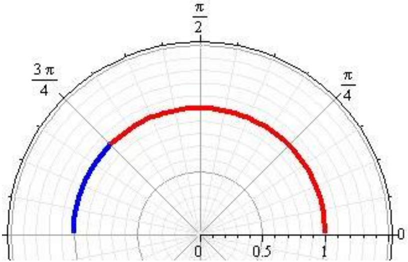
We consider in Z an analogue of the following instance of Douglas-Rachford in $\mathbb{E}^2$.



An instance of Douglas-Rachford in $\mathbb{E}^2$

Douglas-Rachford in Z II

We take as our two convex sets in Z the closed half-rays $A = \{e^{i\theta} : 3\pi/4 \leq \theta \leq \pi\}$ and $B = \{e^{i\theta} : 0 \leq \theta \leq 3\pi/4\}$ of C, so $A \cap B = \{e^{3\pi i/4}\} = \{-0.7071, 0.7071)\}$.



The following table shows three iteration of Douglas-Rachford, starting from $x_1 = (0.5439, 0.4925)$; points z in Y are specified by $(\Re z, \Im z)$ and those in X by (θ, h).

	$n=1$	$n=2$	$n=3$
x_n	$(0.5439, 0.4925)$	$(-0.607669, 0.625647)$	$(-0.624135, 0.613204)$
$P_B x_n$	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-0.690260, 0.723561)$	$(-0.707009, 0.707205)$
$R_B x_n$	$(\frac{\pi}{4}, 0.4530)$	$(-0.761849, 0.190098)$	$(-0.785260, 0.190012)$
$P_A R_B x_n$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$R_A R_B x_n$	$(-0.895023, 0.122895)$	$(-0.639941, 0.600583)$	$(-0.624326, 0.613055)$
x_{n+1}	$(-0.607669, 0.625647)$	$(-0.624135, 0.613204)$	$(-0.624231, 0.613130)$

The iterates appear to be rapidly stabilizing with $P_B x_n$ converging to the feasible point.

The iterates from the alternative starting point $y_1 = (0, 0.5)$ behave similarly.

	$n=1$	$n=2$	$n=3$
y_n	$(0, 0.5)$	$(-0.509291, 0.485280)$	$(-0.533052, 0.474962)$
$P_B y_n$	$(0, 1)$	$(-0.681383, 0.731927)$	$(-0.706154, 0.708058)$
$R_B y_n$	$(0, 0.6931)$	$(-0.749650, 0.492872)$	$(-0.784052, 0.495575)$
$P_A R_B y_n$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$R_A R_B y_n$	$(-0.744747, 0.231161)$	$(-0.555758, 0.463751)$	$(-0.534775, 0.474034)$
y_{n+1}	$(-0.509291, 0.485280)$	$(-0.533052, 0.474962)$	$(-0.533914, 0.474499)$

These two tables also show that in these instances the iterated map $T := \frac{1}{2}(I + R_A R_B)$ is nonexpansive, even though some intermediary steps are not.

Specifically;

$$d_Y(Tx_1, Ty_1) = 0.3098 \le d_Y(x_1, y_1) = 1.0476$$
$$d_Y(Tx_2, Ty_2) = 0.3056 \le d_Y(x_2, y_2) = 0.3098$$
$$d_Y(Tx_3, Ty_3) = 0.3056 \le d_Y(x_3, y_3) = 0.3056$$

While $d(R_A R_B x_1, R_A R_B y_1) = 1.0500 > d(x_1, y_1) = 1.0476$ **!**

Thus, while reflections in CAT(0) spaces of non-constant curvature need not be nonexpansive, it appears that the averaging process in Douglas-Rachford iteration may compensate for this.
This seems deserving of further investigation.

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