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Curve Sketching and Inequalities

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CURVE SKETCHING and INEQUALITIES

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The graph of a function is a powerful visual tool in analyzing and understanding the function. The adage 'a picture is worth a thousand words' is particularly relevant to the mathematical sciences. Sketching graphs allows you to *see* the features and behavior of functions and formulas.

The importance of graphs is not confined to mathematics. Graphs are used throughout the physical sciences; physics, chemistry, biology and geology, the social sciences, engineering, economics and business.

Sketching a function's graph should become second nature to you. It is a skill, like swimming or driving, which once mastered will come naturally for the rest of your life. Sketching a function's graph should not be seen as the culmination of a tedious sequence of evaluations, differentiations, and limits, but rather as the first step in any skirmish with the function.

In these notes I want to introduce and explore a systematic approach to the graphing of functions which is both powerful and efficient. If you were asked to differentiate the function

$$\frac{x}{x^2-1}$$

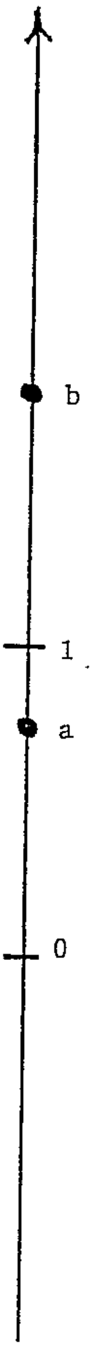
you would think of it as the quotient

$$\frac{f(x)}{g(x)},$$

of the two 'simpler' functions $f(x) = x$ and $g(x) = x^2 - 1$, and proceed to build its derivative using those for $f(x)$ and $g(x)$. A similar approach can be used when it comes to graphing such a function. We would first sketch graphs for $f(x)$ and $g(x)$ and then begin to build a graph for the quotient from them.

As a preliminary, illustrate the following points on the numberline given below:

$$a+b,\ a-b,\ ab,\ a/b,\ 1/a,\ \sqrt{a},\ \sqrt{b},\ a^2,\ b^2,\ b^3$$

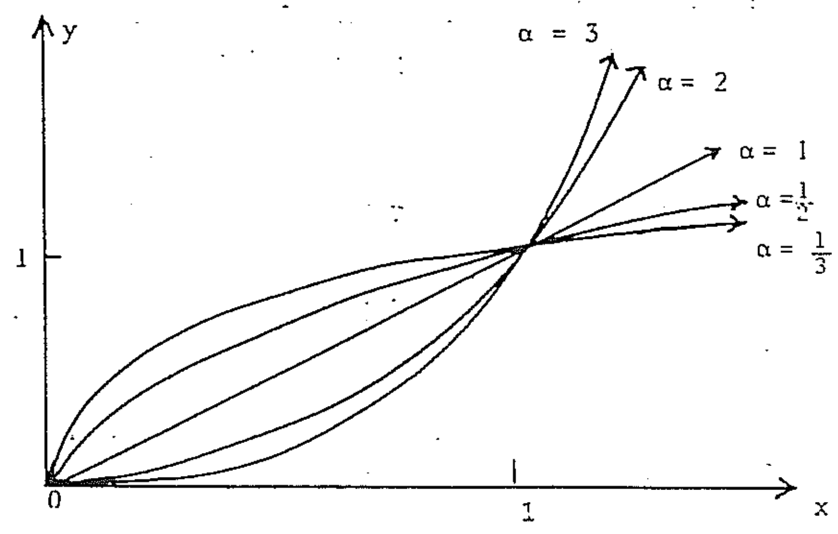


NOTE: The numberline has been drawn vertically because we will usually want to operate with ordinates of graphs.

POWER FUNCTIONS

An important class of functions are the *power functions* of the form $y = Ax^\alpha$.

Typical graphs of power functions for $x > 0$ and both A and α positive are sketched below.



$y = x^\alpha$ for $\alpha = 1/3$, $1/2$, 1, 2 and 3

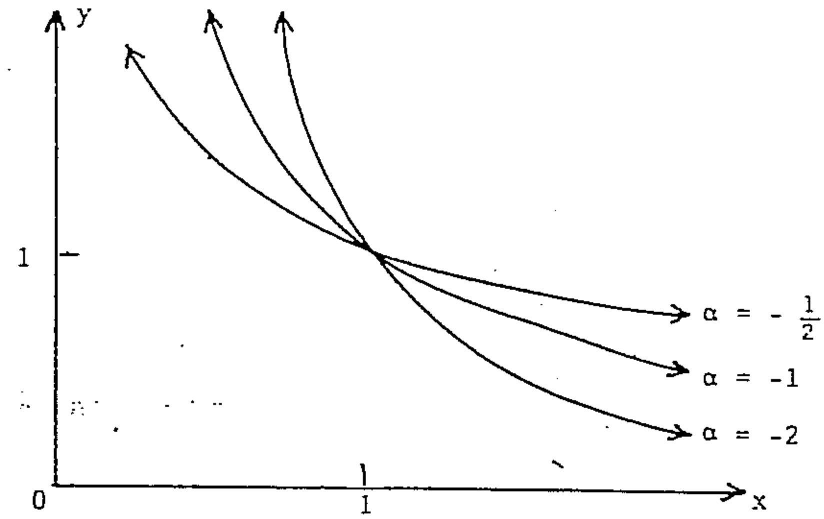
To understand these graphs observe:

(1) For all values of A and α the graph of $y = Ax^\alpha$ passes through the point $(1, A)$.

(2) $dy/dx = \alpha A x^{\alpha-1}$, so for A and α positive the functions is increasing. For α greater than one the slope increases as x does, while for α between zero and one it decreases.

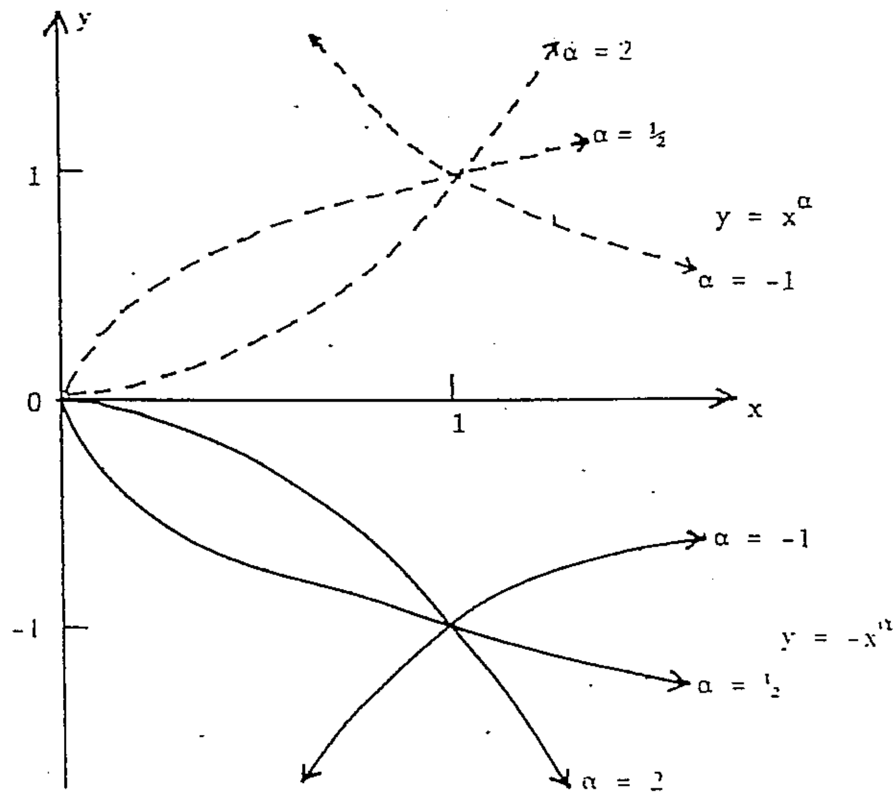
(3) In order to compare the graphs for different α note that for x greater than one the graphs for larger α lie above those with smaller α. For x between zero and one, the reverse is the case; that is, graphs for larger α lie below those with smaller α.

When A is positive and α is negative, the function $y = Ax^\alpha$ is decreasing. When $x = 0$ the function is undefined with the y-axis as a vertical *asymptote*. Some typical examples are sketched below:



$y = x^{\alpha}$ for $\alpha = -1/2$, -1 and -2

The graphs with A negative are the reflections in the x-axis of those for A positive:



$y = -x^{\alpha}$ for $\alpha = -1$, $1/2$ and 2

Exercises;

(1) Sketch the graph of $y = x^{\alpha}$ for $\alpha = \frac{1}{2}$, $\alpha = 2$, $\alpha = -2$.

(2) If α is increased from 2 to 10, how does the graph of $y = x^{\alpha}$ change?

POLYNOMIALS

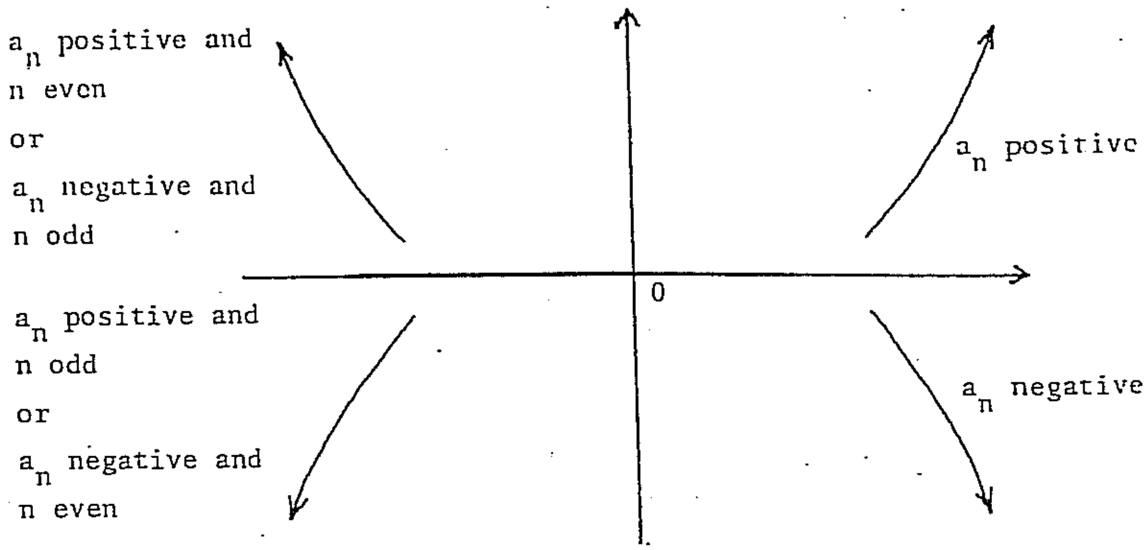
Linear functions, quadratic functions and power functions with whole number exponents are all special cases of the general *polynomial function*

$$y = p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \ .$$

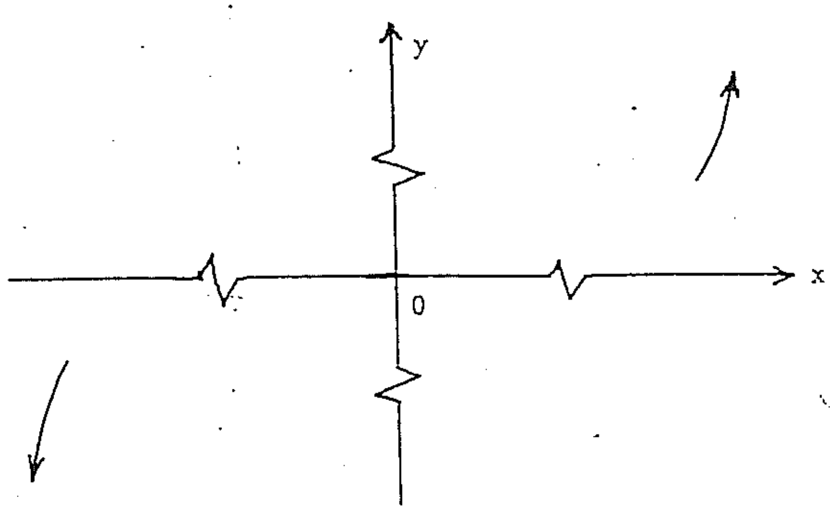
We consider some general features which help in sketching the graph of a polynomial.

Asymptotic Behaviour: When the magnitude of x is large, the value of $p(x)$ is dominated by the highest degree term a_nx^n. Thus the larger the magnitude of x, the more nearly the graph of $y = p(x)$ looks like that of the power function $y = a_nx^n$.

The following diagram shows the different kinds of behaviour that can arise.



For example; when $p(x) = x^3 - 3x^2 + 2x$, we have the following behaviour:



Roots: A *root* of the polynomial $y = p(x)$ is a value x_0 of x for which $p(x) = 0$. Thus x_0 is a root of the polynomial precisely when the graph of $y = p(x)$ intersects the x-axis at

$x = x_0$.

A polynomial of degree n can have up to n roots. In the case of quadratics the roots are given by the *quadratic formula*. More complicated formulas exist for the roots of cubics and quartics (polynomials of degree 4). For polynomials of degree 5 and more it has been proved that no general formula for the roots exists. In general finding roots can be quite difficult.

When the polynomial has integer coefficients it is sometimes possible to locate a root by trying values of x such as 0, 1, -1, 2, or simple fractions such as $1/2$, $-2/3$, etc. Here it is important to remember that when expressed with lowest denominator, *the numerator of any rational root of $p(x)$ must be a factor of the constant term a_0, while the denominator of such a root must be a factor of the highest order coefficient a_n.*

For example. When $p(x) = x^3 - 3x^2 + 2x$:

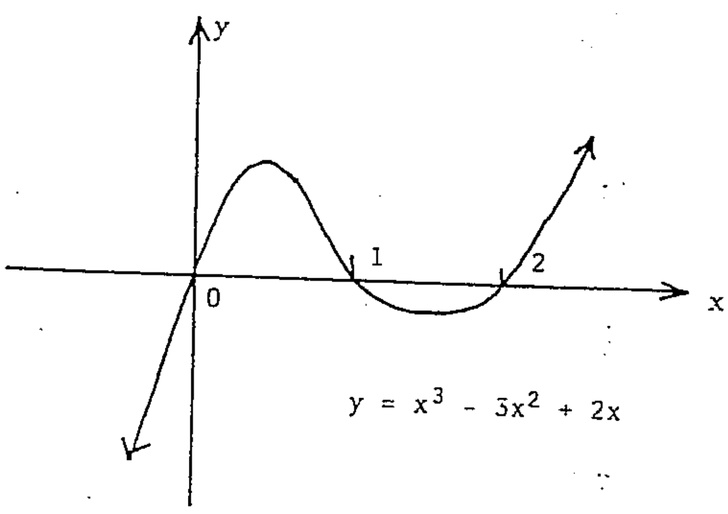
for $x = 0$, we have $p(0) = 0$, so 0 is a root,

for $x = 1$, we have $p(1) = 0$, so 1 is a root,

for $x = -1$, we have $p(-1) = -2$, so -1 is not a root,

for $x = 2$, we have $p(2) = 0$, so 2 is a root.

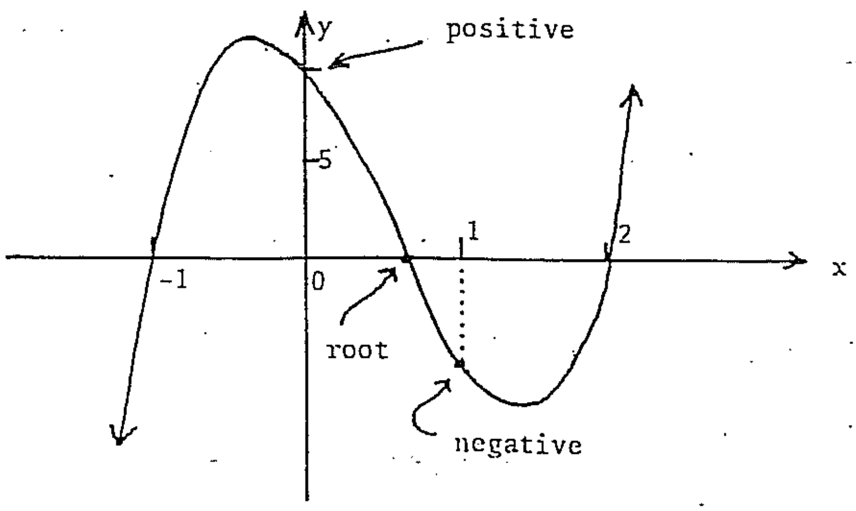
Since $p(x)$ is a cubic and so can have at most 3 roots, we conclude that the roots are 0, 1 and 2. Combining this with the asymptotic behaviour noted above, we see that the graph of $y = x^3 - 3x^2 + 2x$ looks like



As this example illustrates, *between any two consecutive roots of a polynomial, we have either at least one local maximum or at least one local minimum.* Another useful observation is that if the values of $p(x)$ at two points x_1 and x_2 are of opposite sign, then $p(x)$ has a root between x_1 and x_2.

For example. If $y = 8x^3 - 13x^2 - 11x + 10$, we have $p(0) = 10$ and $p(1) = -6$ which are of opposite sign, so there is a root between 0 and 1; $p(-1) = 0$, so -1 is a root; $p(2) = 0$, so 2 is a root.

The graph will therefore look like



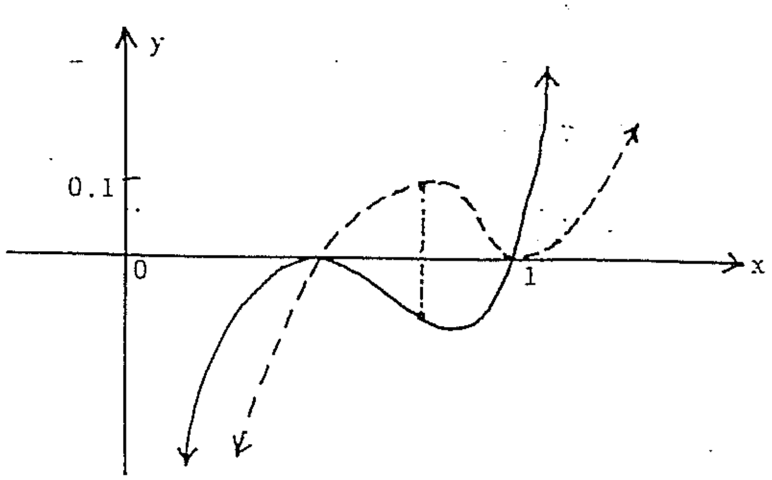
Graph of $y = 8x^3 - 13x^2 - 11x + 10$

Provided we have enough information about the roots, it is usually possible to sketch the graph of a polynomial.

For example. If $y = p(x) = 4x^3 - 8x^2 + 5x - 1$, we find by substituting values for x that $x = 1$ is a root. Factorizing gives

$$\begin{aligned} y &= (x-1)(4x^2 - 4x + 1) \\ &= (x-1)4(x^2 - x + 1/4) \\ &= 4(x-1)(x-1/2)^2 \end{aligned}$$

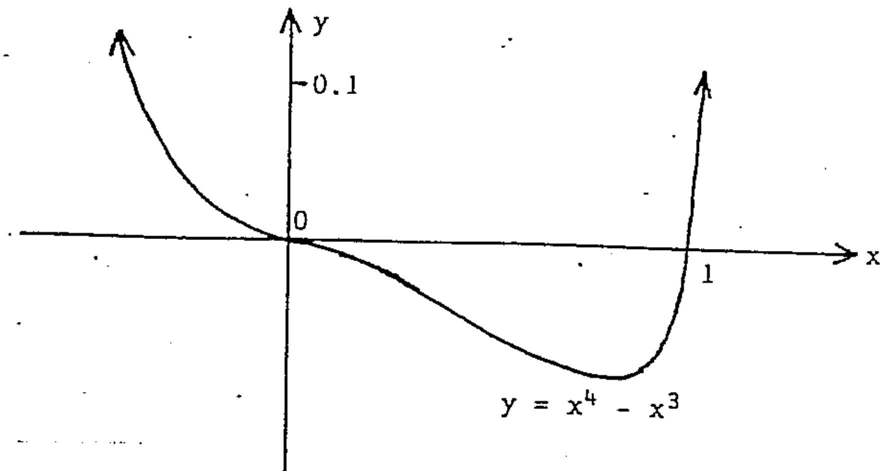
and so in this case $p(x)$ has roots at $x = 1$ and $x = 1/2$, with the root at $x = 1/2$ being *repeated*. Since these are the only points where the graph can cut the x-axis, we are able to infer from the asymptotic behaviour that the graph must look like one of the following possibilities:



That it is indeed the heavy curve and not the broken one is easily confirmed by noting that $p(3/4) = -1/16$ is negative.

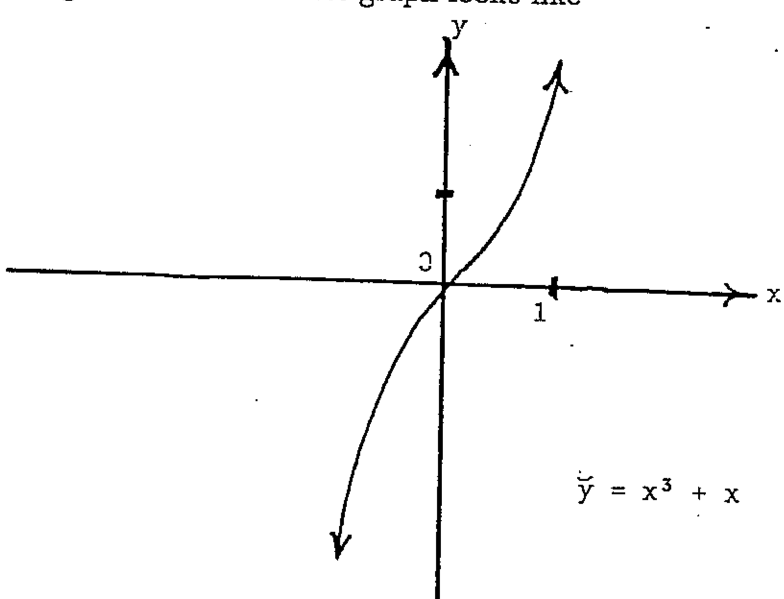
From the observation that a repeated root is also a root of the derivative we are able to see that at a root which is repeated an even number of times, a polynomial will have a local maximum or minimum and the graph will touch the x-axis without crossing it, as illustrated in the last example. At a root which is repeated an odd number of times, the graph crosses the x- axis, but has horizontal slope at the root.

For example. $y = x^4 - x^3 = x^3(x-1)$ has roots at 0 and 1, with the root at 0 being repeated 3 times. The graph looks like



When the roots are difficult to find, or provide us with insufficient information, we must seek alternative methods, such as locating the local maxima and minima.

For example. The polynomial $y = x^3 + x$ is easily seen to have only one real root, at $x = 0$. Its graph cannot be sketched with certainty using the above methods, but noting that $dy/dx = 3x^2 + 1$ is strictly positive allows us to conclude that the polynomial is a strictly increasing function and so its graph looks like



Exercise For each of the following polynomials $p(x)$, sketch the graph of $y = p(x)$:

(a) $p(x) = (x-1)(x-2)(x-3)$

(b) $p(x) = (x+1)(x-2)^2$

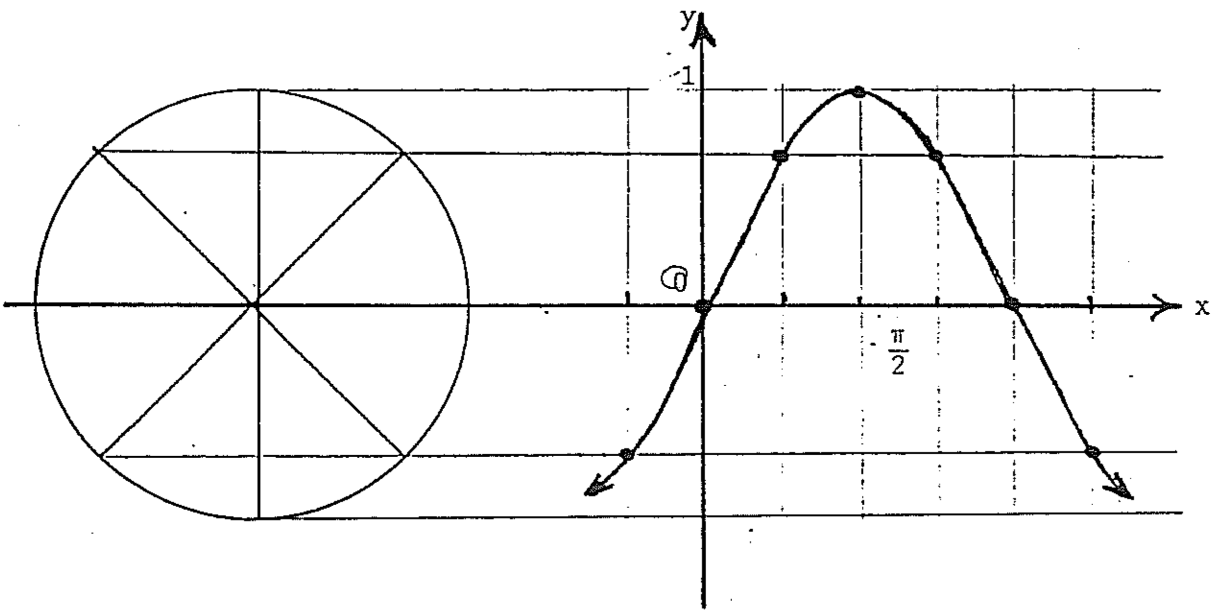
(c) $p(x) = x^2 - 2x - 3$

(d) $x^3 - x^2 - 3x - 1$

(e) $x^4 - 2x^3 + 2x - 1$

TRIGONOMETRIC FUNCTIONS

In order to sketch the functions *sine* (and cosine) it is enough that you understand the following construction.



Oher trigonometric functions can be sketched using identities like

$$\cos(x) = \sin(x + \pi/2)$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

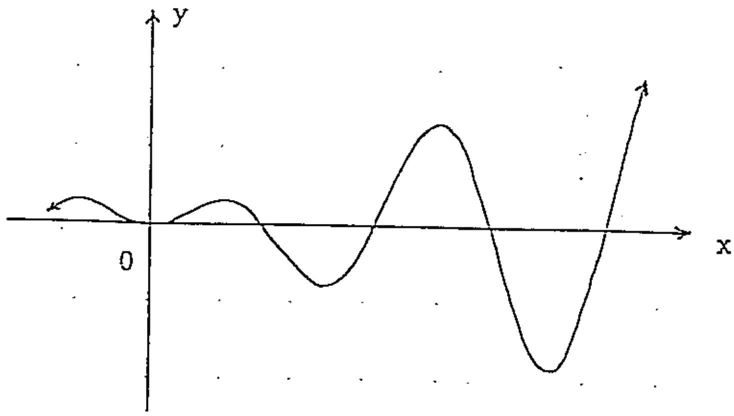
$$a\sin(x) + b\cos(x) = A\sin(x + \phi)$$

where $A = \sqrt{a^2 + b^2}$, and ϕ is such that $\cos(\phi) = a/A$, and $\sin(\phi) = b/A$

and the techniques we are about to describe.

SIMPLE COMPOSITES

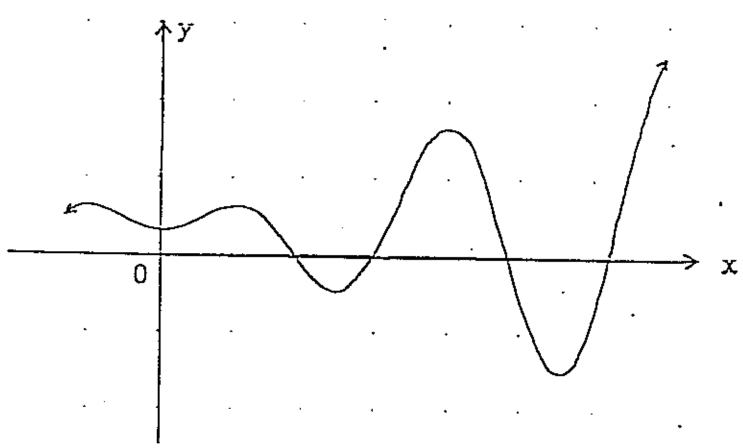
Suppose we know the graph of a function $y = f(x)$, for example;



The graph of $y = f(x) = x\sin(x)$

then:

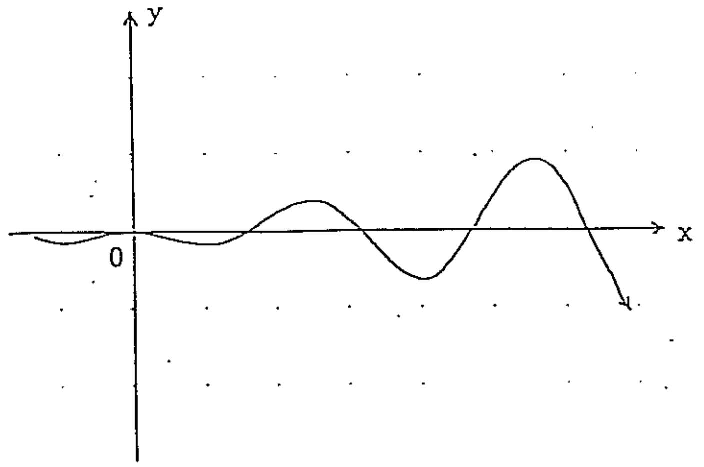
For c a constant the graph of $y = f(x) + c$ is that of $f(x)$, but shifted vertically upward a distance c. Thus for $f(x) = x\sin(x)$ and $c = 2$ we have



Graph of $y = f(x) + 2 = x\sin(x)\ +\ 2$

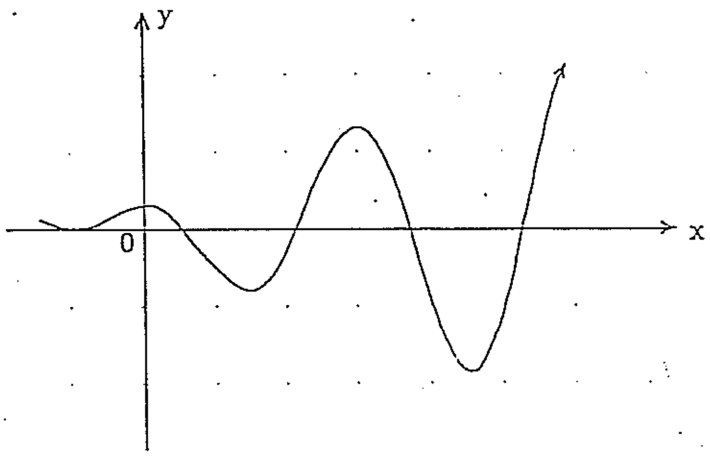
Of course, when $c < 0$ shifting vertically upward a distance c amounts to a downward shift of $|c|$.

For c a constant the graph of $y = cf(x)$ is that of $f(x)$, but with vertical distances scaled by c. When $c < 0$ this involves a scaling of verticle distances by $|c|$ and a reflection in the x-axis. Thus for $f(x)$ given above and $c = -1/2$ we have



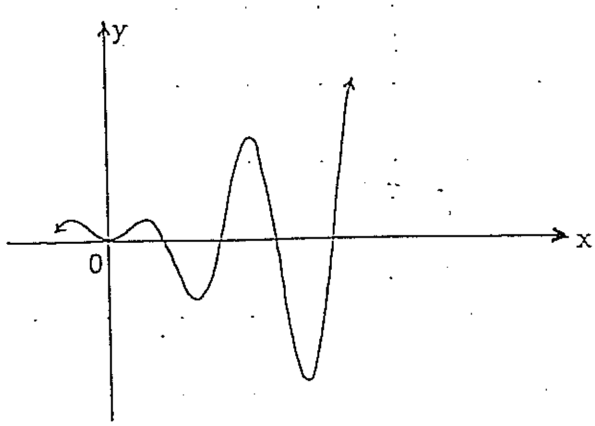
Graph of $y = -\frac{1}{2}f(x) = -\frac{x}{2}\sin(x)$

For c a constant the graph of $y = f(x + c)$ is that of $f(x)$, but shifted horizontally a distance c to the left. When $c < 0$ this means a horizontal shift of $|c|$ to the right. Thus for $f(x)$ given above and $c = 2$ we have



Graph of $y = f(x + 2) = (x + 2)\sin(x + 2)$

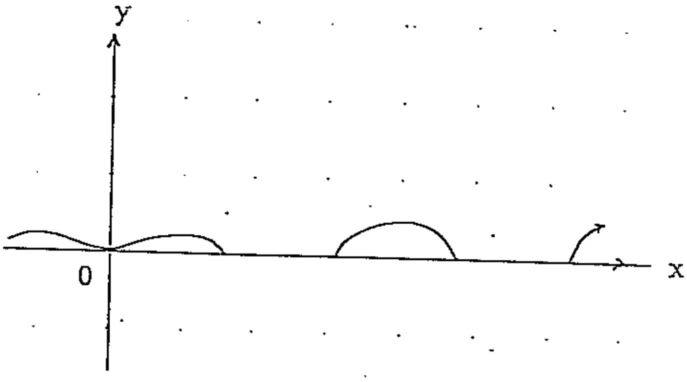
For c a constant in the graph of $y = f(cx)$ as we move along the x-axis 'things' happen c times as quickly as they did for $y = f(x)$. Thus for $f(x)$ given above and $c = 2$ we have



Graph of $y = f(2x) = 2x\sin(2x)$

When $c < 0$ we obtain the graph of $y = f(|c|x)$ reflected about the y-axis.

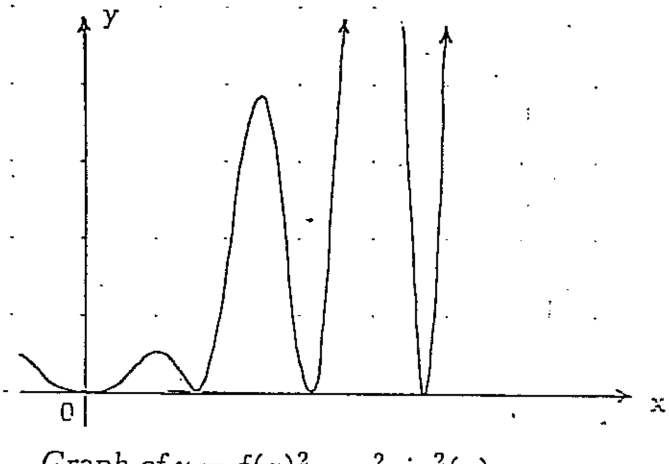
The graph of $y = \sqrt{f(x)}$ is only defined at those x for which $f(x) \geq 0$ and at such points may be obtained from the graph of $y = f(x)$ by plotting an approximate square root of the ordinate. Here it is useful to remember the shape of the square root graph (see the section on power functions), in particular recall that for $0 < a < 1$ we have $a < \sqrt{a} < 1$, while for $a > 1$ we have $1 < \sqrt{a} < a$. Thus for $f(x)$ given above we have



Graph of $y = \sqrt{f(x)} = \sqrt{x\sin(x)}$

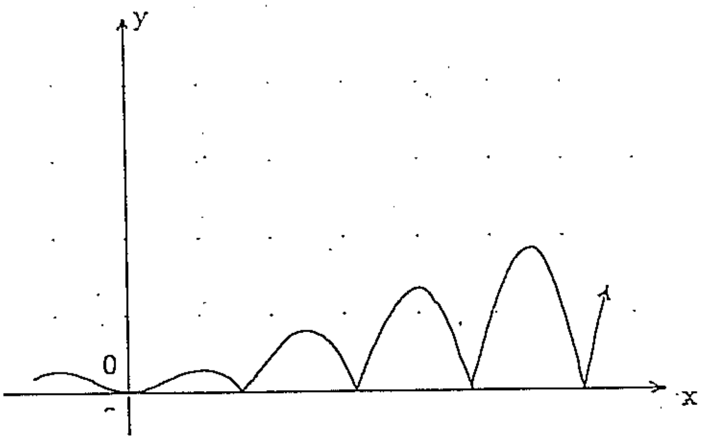
NOTE; since $\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$ we see that the graph of $y = \sqrt{f(x)}$ meets the x-axis vertically at points where $f(x) = 0$, and can only have turning points where $f(x)$ does.

The graph of $y = f(x)^2$ may be sketched from that of $f(x)$ by recalling the square function. Since $dy/dx = 2f(x)f'(x)$ we see that $f(x)^2$ has a turning point wherever $f(x)$ and also has local minima at the roots of $f(x)$. Thus for $f(x) = x\sin(x)$ we have



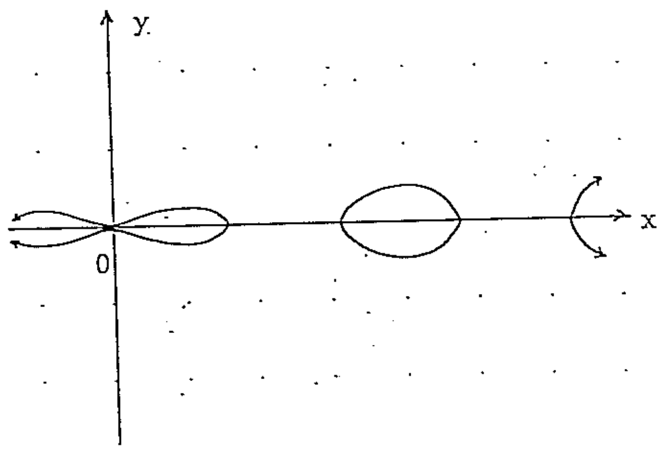
Graph of $y = f(x)^2 = x^2\sin^2(x)$

The graph of $y = |f(x)|$ is identical to that for $f(x)$ where $f(x) \geq 0$, and is the reflection in the x-axis of $y = f(x)$ where $f(x) < 0$. Thus the graph lies entirely above the x-axis. For $f(x)$ given above we have



Graph of $y = |f(x)| = |x\sin(x)|$

Since $y^2 = f(x)$ implies that $y = \pm\sqrt{f(x)}$ the graph of $y^2 = f(x)$ is that of $y = \sqrt{f(x)}$ together with its reflection in the x-axis. Thus for $f(x)$ given above we have



Graph of $y^2 = f(x) = x\sin(x)$

Remark; Functions such as $2\sqrt{f(x)+1}$ which are built up from a sequence of simple operations on $f(x)$ may be graphed by successively constructing the appropriate sequence of graphs, in this case the graph of $y = f(x)$ then $y = f(x) + 1$ from which we can sketch $y = \sqrt{f(x)+1}$ and lastly $y = 2\sqrt{f(x)+1}$.

Exercise Sketch $y = 2\sqrt{f(x)+1}$ when $f(x) = x\sin(x)$.

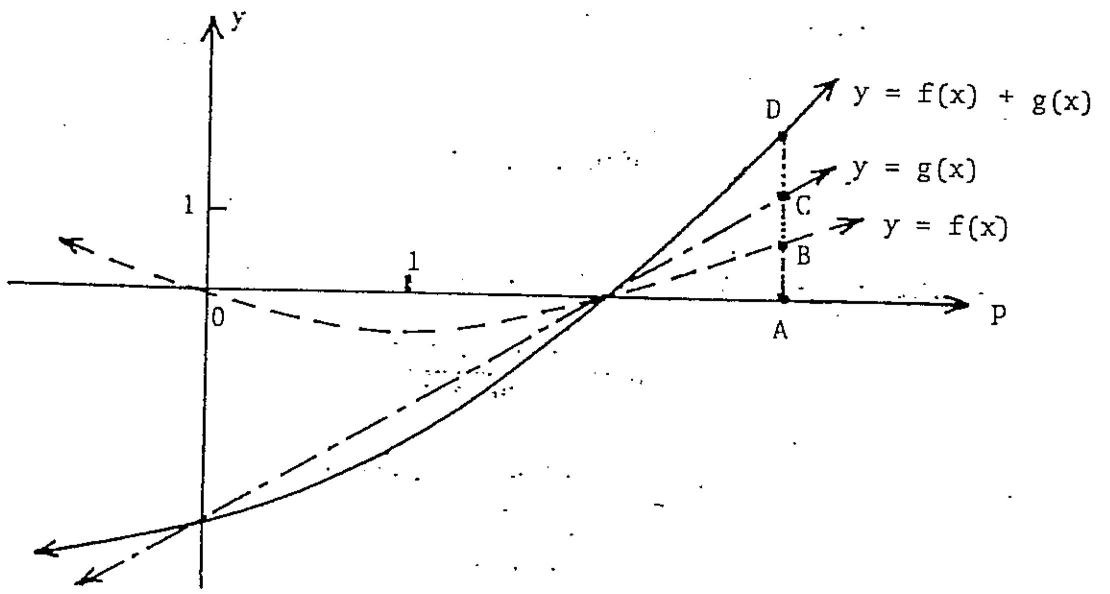
RATIONAL COMBINATIONS

We will see that the task of graphing a complicated function is often made easier if we recognise the function as a combination of simpler functions whose graphs are known or readily found. This extends the process already started in the last section.

Sums of Functions: Given two functions f and g, a graph of their sum $y = f(x) + g(x)$ may be constructed from graphs of f and g (preferably drawn on the same axes) by noting that the vertical distance (*ordinate*) from a point x on the x-axis to the graph of

$y = f(x) + g(x)$ is the sum of the ordinate at x to the graph $y = f(x)$ and the ordinate at x to the graph of $y = g(x)$. Thus, in the illustration below the distance AD = AB + AC. The addition of these distances can be done in many ways, for example; by "eye", by marking them on the edge of a piece of paper, by using a pair of dividers (or your fingers as dividers), or by using a ruler. You should aim to become practised in doing this. Particular care should be taken at points where the graph of one or other of the functions $y = f(x)$ and $y = g(x)$ lies below the x-axis and so the corresponding vertical distance is negative.

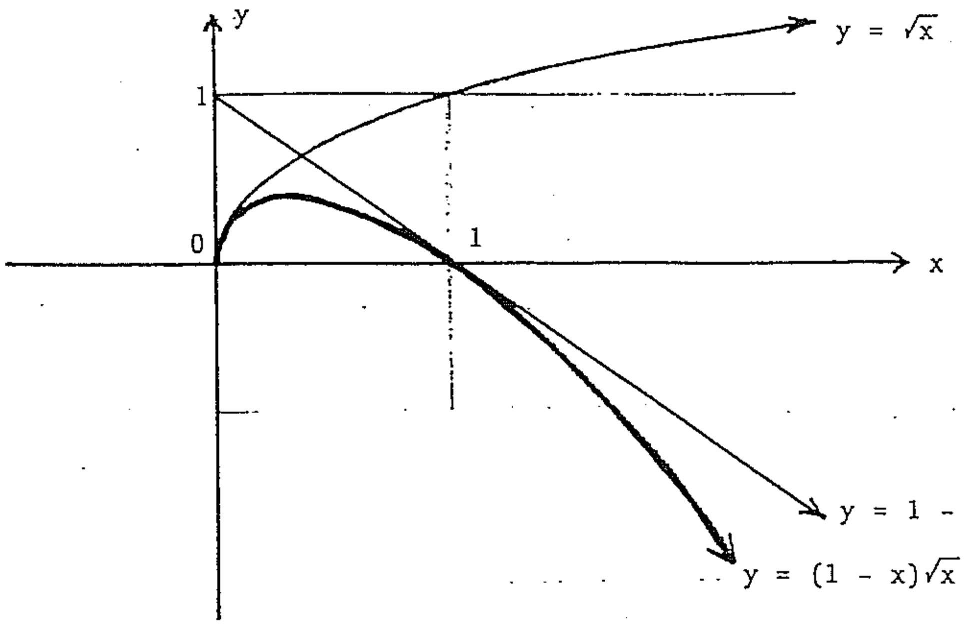
For example. Graps of the functions $y = f(x) = 5x(x-2)$ and $y = g(x) = 15x - 30$ together with the graph of $y = f(x) + g(x)$ constructed from them are sketched below.



The *difference* $f(x) - g(x)$ can be constructed similarly, by taking away rather than adding the appropriate vertical distances. Alternatively, it may be obtained by first sketch $y = -1g(x)$ and then adding this to the graph of $y = f(x)$. Again, special care should be taken when one or other of the graphs is below the x-axis.

Products: From graphs of the two functions f and g we can sketch a graph of their product $y = f(x)g(x)$ by performing 'approximate' multiplications (mentally, or otherwise) of $f(x)$ and $g(x)$. In doing this use should be made of simple observations such as: the product of two numbers larger than 1 is bigger than either of them; the product of a number larger than 1 and a positive number less than 1 lies between them; the product of two positive numbers smaller than 1 is less than either of them; the product of a positive number and a negative number is negative.

For example. From graphs of $y = 1 - x$ and $y = \sqrt{x}$, we see that $y = (1-x)\sqrt{x}$ has the graph shown below:



Quotients: From graphs of the two functions f and g we can prepare a graph of the *quotient* $y = g(x)/f(x)$ by approximately dividing the value of g at x by the value of f at x.

Alternatively the quotient $g(x)/f(x)$ can be regarded as the product of $g(x)$ and the reciprocal $1/f(x)$. Thus it is often convenient to draw, as an intermediate step, the graph of $y = 1/f(x)$. To do this, it is necessary (mentally, or otherwise) to divide 1 by $f(x)$. Remember: taking the reciprocal of a smaller number gives a larger number, and vice versa. In particular, if the values of $f(x)$ are approaching 0 as x approaches x_0, then, as x approaches x_0, the values of $f(x)$ will tend to $+\infty$ or $-\infty$ depending on whether $f(x)$ is positive or negative.

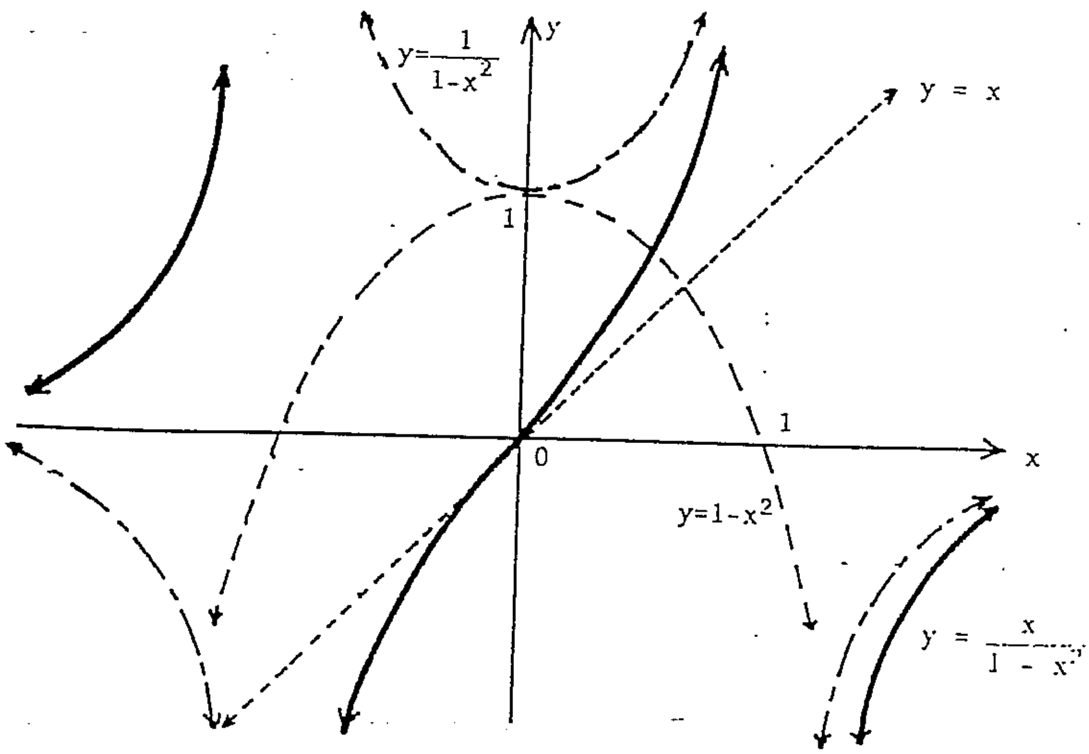
For example. To graph

$$y = \frac{x}{1-x^2}$$

we may first graph $y = 1 - x^2 = (1-x)(1+x)$ and construct from this the graph of

$$y = \frac{1}{1-x^2},$$

and finally multiply this by x to obtain the desired graph. This is illustrated below.



NOTE; for large x, $1 - x^2$ behaves like $-x^2$ and so $x/(1 - x^2)$ is like $x/(-x^2) = -1/x$. Thus, as x becomes large in magnitude, $x/(1 - x^2)$ approaches zero.

Exercises

(1) For the functions $f(x) = x^3 - 4x^2 + 5x - 2$ and $g(x) = 3x - x^2$, sketch graphs of f, g, $f + g$, and fg on a common set of axes.

(2) In each of (a), (b) and (c) graph the functions indicated on a common set of axes.

(a)
- (i) $y = 1 + x^2$
- (ii) $y = 1/(1 + x^2)$
- (iii) $y = x/(1 + x^2)$

(b)
- (i) $y = x^2 - 4x + 3$
- (ii) $y = 1/(x^2 - 4x + 3)$
- (iii) $y = x - 2$
- (iv) $y = \frac{x-2}{x^2-4x+3}$

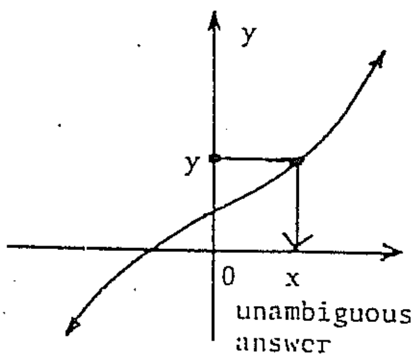
(c) (i) $y = f(x)$, where $f(x) = x^3 - 3x^2 + 2x$

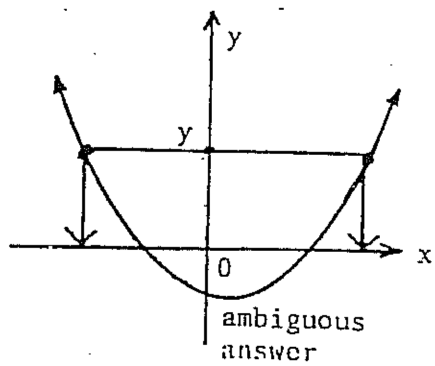
(ii) $y = f(x)^2$

(iii) $y = \sqrt{f(x)}$

INVERSE FUNCTIONS

Given a function $y = f(x)$ we can ask; for what value of x does $f(x)$ have a given value y? In order that this question have an unambiguous answer it is necessary that each value y of the function f come from only one value of x.



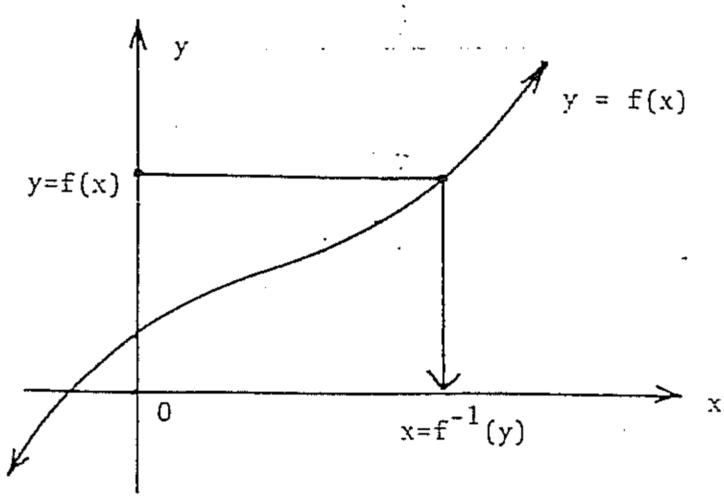


For a function to be unambiguous in this way each horizontal line must cut its graph only once (just as each vertical line must only cut it once in order that f is a function). In any region where the function is continuous this means it must be either strictly increasing or strictly decreasing.

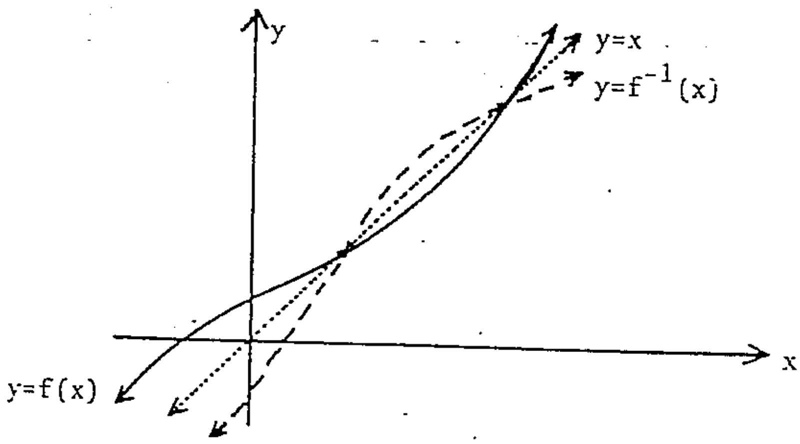
Such an unambiguous f is *invertible* and we can define the *inverse function*

$$x = f^{-1}(y)$$

which assigns to each value y of f the unique x for which $y = f(x)$.



By reflecting this diagram about the line $y = x$ so that the y-axis becomes horizontal and the x-axis vertical we obtain a graph of the inverse function $x = f^{-1}(y)$.



NOTE; when $y = f(x)$ is an invertible function these procedures give the graph for $x = f^{-1}(y)$. If the variables do not have meaningful names which we would wish to preserve, it is conventional to use x and not y for the independent variable. To achieve this it is necessary to swap the roles of x and y in the final answer. Of course the functions given by

$$x = f^{-1}(y)$$

and

$$y = f^{-1}(x)$$

are of course the same function; only the names of the variables have been changed to protect the convention.

Exercise For each of the following functions determine whether or not the function is invertible. When the function is invertible draw graphs of both the function and its inverse on the same set of axes. Find an expression for the inverse function using x to represent the independent variable.

(i) $y = f(x) = 5x - 3$.

(ii) $y = f(x) = x^2 + 1$, where x is positive.

(iii) $y = f(x) = 1/(x + 1)$. $x \neq -1$.

(iv) $y = f(x) = x^3 - x$.

INEQUALITIES

The general inequality $F(x) \leq (\geq, <, >)\, G(x)$ can be converted to the form $f(x) \leq (<)\, 0$ by taking $f(x) = F(x) - G(x)$, or $f(x) = G(x) - F(x)$. Now, the graph of $f(x)$ can only cross the x-axis by either cutting it, or jumping over it; that is, at a zero of $f(x)$, or at a point of discontinuity for $f(x)$. Between consecutive pairs of such points $f(x)$ must remain on one side of the x-axis, and so the inequality is either true, or false, throughout the whole of such a segment. To decide which it is therefore only necessary to test **one** 'conveniently' chosen point in the segment. Doing this for each segment and tesing each division point between segments solves the inequality. It should be noted that if unnecessary division points have been included the method, while requiring additional computation, still leads to a solution.

Zeros of $f(x)$ occur at precisely those points where the original inequality is an equality, and for the type of functions with which we will be dealing discontinuities will always be signalled by the vanishing of a denominator in one of the expressions. Thus we may turn the above observations into an algorithm as follows.

To solve the inequality $F(x) \leq (\geq, <, >)\, G(x)$:

(1) Find those x at which equality occurs; that is, solve $F(x) = G(x)$, and plot these points on the numberline.

(2) Determine those x at which a denominator in one of the expressions involved in either $F(x)$ or $G(x)$ vanishes. Plot all such points on the numberline.

(3) The points identified in (1) and (2) divide the numberline into a number of intervals. From each of these intervals choose a convenient value of x and substitute it into the inequality to test whether or not the inequality is true throughout that interval. Also check whether or not the inequality holds at each of the points from (1) and (2) to obtain a complete solution.

For example. To solve the inequality

$$\frac{1}{2-x} \leq x$$

(1) Equality occurs when $1/(2-x) = x$, that is when $1 = x(2-x)$ or equivalently when

$$0 = x^2 - 2x + 1 = (x-1)^2.$$

Thus we have equality when $x = 1$.

(2) The denominator of the LHS vanishes when $x = 2$.

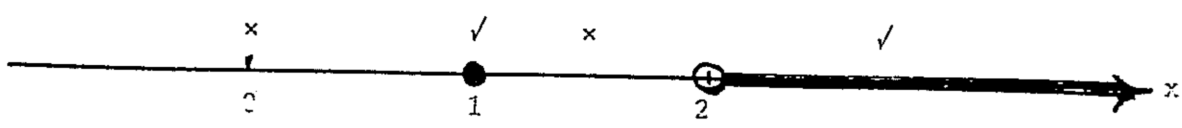
(3) From (1) and (2) the numberline is divided into the intervals:

$$-\infty < x < 1,$$

$$1 < x < 2,$$

and

$$2 < x < \infty.$$



Choosing $x = 0$ from the first of these intervals and substituting the inequality reads $\frac{1}{2} \leq 0$, which is false. Thus the inequality fails to hold for $-\infty < x < 1$.

At $x = 1\frac{1}{2}$ the inequality becomes $1/\frac{1}{2} \leq 1\frac{1}{2}$, so the inequality fails to hold for $1 < x < 2$.

At $x = 3$ the inequality becomes $-1 \leq 3$, so the inequality is true for $2 < x < \infty$.

At $x = 1$ the inequality is true, while at $x = 2$ it is undefined.

Thus, we conclude that the inequality is true when $x = 1$, or $x > 2$, and false otherwise.

Exercise (3/4 Unit, 1990) Solve $\frac{x^2-4}{x} > 0$.

What follows makes particular use of the observation that any unnecessary division points introduced by the first two steps are eliminated when the intervals and division points are tested.

Expressions involving absolute values: When the expressions involved contains absolute values; for example,

$$|x-2| \geq \frac{1}{x}$$

it is often more economical to modify the above procedure in the way illustrated below.

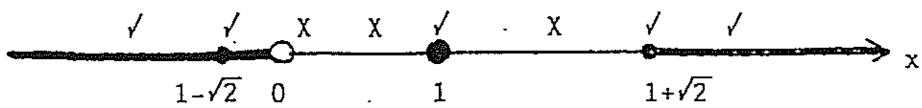
For any value of x the expression $|x-2|$ is equal to $x-2$ or $-(x-2)$. Consequently the values of x for which equality occurs; that is, for which

$$|x-2| = \frac{1}{x}$$

will be included among those x which satisfy either $x - 2 = \frac{1}{x}$ or $-(x-2) = \frac{1}{x}$; namely $x = 1$ (the root of $x^2-2x+1 = 0$) and $x = 1-\sqrt{2}, x = 1+\sqrt{2}$ (the roots of $x^2-2x-1 = 0$).

If we use these x, together with any values of x at which a denominator might vanish (that is, a discontinuity might occur), in this case $x = 0$, to divide the line into intervals, then after testing each of the points and intervals we must have the solution.

In our example the line is divided as follows.



After testing, the results of which are represented by $\surd$'s or $\times$'s in the above diagram, we see that the solution is

$$x < 0, x = 1, \qquad \text{or} \qquad x \geq 1 + \sqrt{2}.$$

Problems involving the absolute value of more than one expression, or absolute valus in the expression for a denominator, may be treated similarly, except that we solve for equality, or the vanishing of the denominator, with every possible choice of signs in front of the expressions appearing inside absolute values.

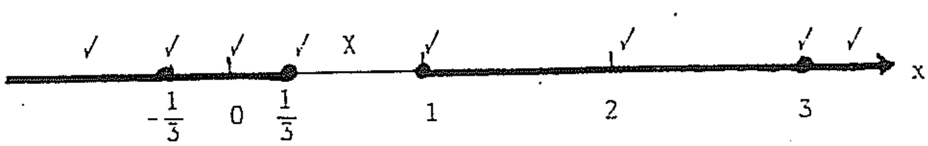
For example. To find those values of x for which

$$|x+1| - |2x-1| \geq 1$$

We find those x for which

$$(x+1) - (2x-1) = 1, \quad \text{that is} \quad x = 1$$
$$(x+1) + (2x-1) = 1, \quad \text{that is} \quad x = \frac{1}{3}$$
$$-(x+1) - (2x-1) = 1, \quad \text{that is} \quad x = -\frac{1}{3}$$
$$-(x+1) + (2x-1) = 1, \quad \text{that is} \quad x = 3.$$

These points divide the line as follows



Testing the points and regions we have the solution

$$\frac{1}{3} \leq x \leq 1.$$

Constrained regions in the plane: A procedure similar to that used above may be used to find the region in the plane determined by a set of inequality constraints. Each inequality is investigated for equality. When absolute values occur in an inequality we replace it by several equalities, one for each choice of signs in front of the terms inside absolute values. The resulting equations divide the plane into regions. Each region is then tested against the corresponding inequality using a convenient point in it and shaded out if the inequality fails to hold.

For example. To determine the region consisting of those points (x, y) which satisfy the constraints

$$\begin{aligned} |x-y| &\leq 1 \\ y &\leq 2(1-x^2) \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

we proceed as follows.

From the first constraint, $|x - y| \leq 1$, we obtain the two equalities

$$x - y = 1$$

and

$$-(x - y) = 1,$$

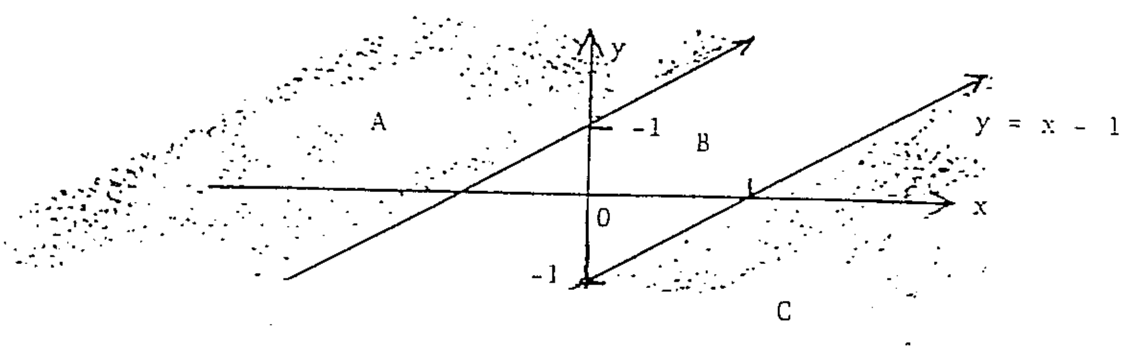
or equivalently

$$y = x - 1$$

and

$$y = x + 1.$$

These are equations for the two lines illustrated below, which divide the plane into three regions labelled A, B and C.



To test each of the regions we may note that:

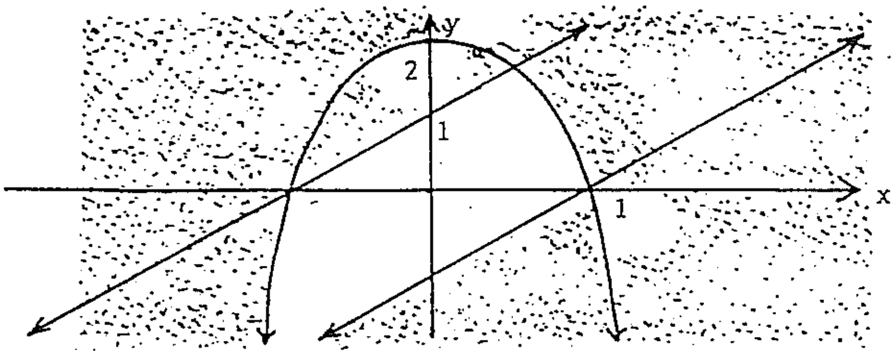
$(0, 2)$ is in A and $|0 - 2| = 2 \not\leq 1$ so points in A do not satisfy the inequality and should be shaded out.

The origin $(0,0)$ is in B and $|0-0| = 0 \leq 1$, so points in B satisfy the inequality.

$(0,2)$ is in C and $|0-(-2)| = 2 \not\leq 1$, so C should be shaded out.

Thus the first inequality leads to the diagram and shading illustrated above.

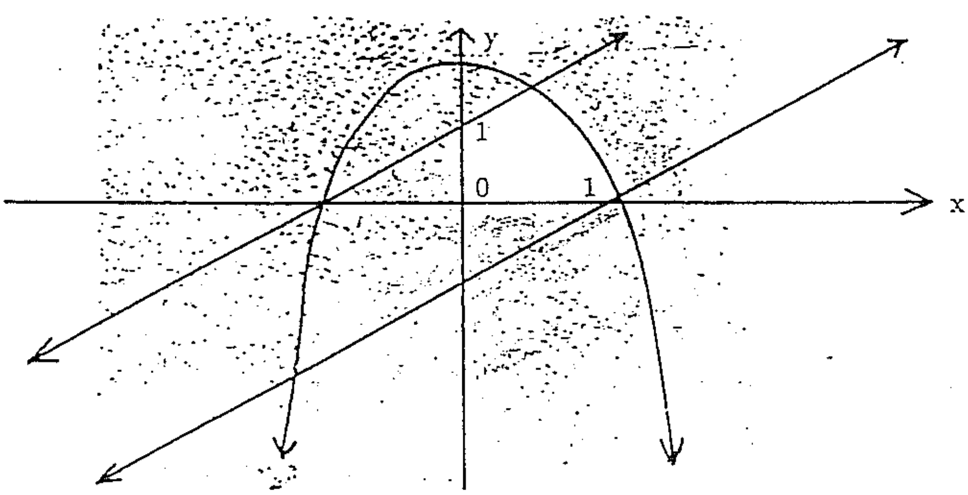
Replacing the second constraint, $y \leq 2(1-x^2)$ with an equality leads to the parabola $y = 2(1-x^2)$, illustrated below superimposed on our previous picture.



The parabola divides the plane into two regions.

Testing the region above the parabola using the point $(0,3)$ we see that $3 \not\leq 2(1-0^2) = 2$ and so this region must be shaded out, as has been done in the diagram above. On the other hand $(0,0)$ is in the region inside the parabola and $0 \leq 2(1-0^2) = 2$ so this region receives no further shading.

Similarly, we may complete the picture by including the two constraints $x \geq 0, y \geq 0$, to obtain the situation depicted below in which the solution is the region left unshaded together with its boundary along which the non-strict inequalities of the constraints are satisfied.



Exercises

(1) For each of the following indicate on a number line those points x which satisfy the inequality, also describe the set of such x by means of simple inequalities.

(a) $x + 1 \leq 3x - 2$.

(b) $|2x - 5| \leq 4$.

(c) $|2x - 1| \leq x + 2$.

(d) $|2x - 1| - |x + 1| \geq 1$.

(e) $\dfrac{2}{(x-3)} \leq x - 2, \quad x \neq 3$.

(2) For each of (a) and (b) indicate those points (x, y) in the plane which satisfy the given inequalities.

(a)

$$\begin{aligned} x + y &\leq 4 \\ y &\leq 5x \\ 2x &\leq y \\ 3 &\leq 3y + x. \end{aligned}$$

(b)

$$\begin{aligned} |y - 2x| &\leq 1 \\ x^2 &\leq y + 2 \\ 0 &\leq x. \end{aligned}$$