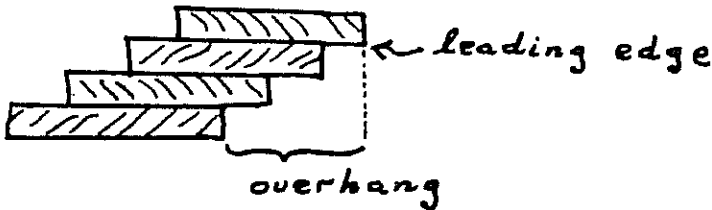# More on the Overhang Problem

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Editor's Note: In the May, 1977 issue of the Bulletin, Ian McGee described the "overhang" problem. Specifically, how much overhang is possible with a pile of blocks, cards, bricks, etc. sitting as in the figure below.

He showed that, in general, if the blocks are arranged so that the combined centre of gravity of a pile of $n-1$ blocks is directly above the edge of the bottom (n' th) block, then an overhang of $\frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{(n-1)}\right]$ is possible. Brailey Sims extends this example, and, in doing so, introduces Euler's constant and solves one of the questions posed in the 1977 article.

To answer the question of how large an overhang is possible, we need to find how large the sum

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2(n-1)} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{(n-1)}\right)$$

can be.

This is just $\frac{1}{2}$ the sum of the harmonic series, which is known to diverge. However, we can illustrate this divergence by considering $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ as an area, as follows,
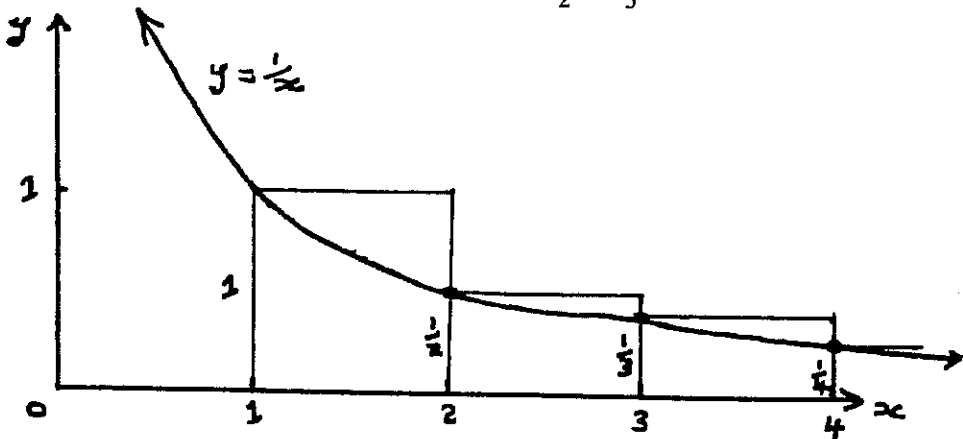
and note that the sum is greater than the area under the curve $y = \dfrac{1}{x}$ between $x = 1$ and $x = n$.

We then have

$$\frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) > \frac{1}{2}\int_1^n \frac{1}{x}dx = \frac{1}{2}[\ln x \;]_1^n = \frac{1}{2}\ln n.$$

Since $\ln n \to \infty$ as $n \to \infty$ we conclude that *the overhang can be as large as we like.*

This fact has been employed in the construction of certain bridges (not always successfully - witness the tragic collapse of a partly completed bridge in Victoria, Australia a few years ago). Stacks are built up from each side to meet in the middle. (Usually the higher blocks in the stack are lighter, allowing a larger overhang to be achieved with fewer "blocks". (See the article by Ian McGee in the *Bulletin, May, 1977 - Problem 4*) To keep *our problem simpler we will continue to assume all our blocks are identical.)*

In the bridge problem the desired overhang is known. Since we start building from the bottom, the number of blocks n needed to achieve this overhang must be determined before construction can commence (the second block must be placed with an overhang of $\dfrac{1}{2(n-1)}$ on the top of the first, etc.). *Our problem is therefore to determine the minimum number of blocks n which need to be stacked to achieve an overhang of at least m.* (Here m is a given number which we will assume is integral.)

That is, we wish to determine the smallest whole number n for which

$$\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) \geqslant m.$$

It is convenient to rephrase this as: *Find the smallest whole number n for which*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \geqslant M$$

(then our answer is the value of n corresponding to $M = 2m$). As a little trial calculation will demonstrate, any attempt to answer this by summing $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots$ until the sum first exceeds M is extremely tedious even for moderate values of M and impossible for larger values. [It took 1 hour 33 minutes on a Tandy TRS-80 using a reasonably efficient Basic II programme to determine the number of blocks for an overhang of 6 ($M = 12$) - the problem of determining n for an overhang of 6 was recently posed in *Parabola* with the answer being given for overhangs of 1 to 5.]

Clearly some other approach is needed. It is at this point that the constant

$$\gamma = 0.5772156649015328606065120900082....$$

known as *Euler's Constant,* comes to the rescue.

The Swiss mathematician, Leonard Euler (1707-1783) discovered in 1744 that

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) - \ln n$$

tends to a limit as $n \rightarrow \infty$. Euler determined the value of this limit to 16 decimal places and christened the limiting value γ. (Much of our modern mathematical notation is due to Euler, for example; he consolidated the use of the symbol π and introduced e to stand for the base of the natural logarithms. His written works (approximately 886 in number) were prolific and far exceed those of any other mathematician.)

Later the value of γ was determined by Mascheroni to 32 decimal places with an error in the 20th. The error was corrected by Gauss and Nicolai. Adams determined the value to 260 places in 1878 and today, using computers, its value is known to over 7000 places (Beyer and Waterman, 1974).†

γ as an Area

From what has been said

$$\gamma = \lim_{n \to \infty} \gamma_n$$

where $\gamma_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} - \ln n$. Now, we also have

$$\gamma_n = \left[1 + \ln \frac{1}{2} \right] + \left[\frac{1}{2} + \ln \frac{2}{3} \right] + \left[\frac{1}{3} + \ln \frac{3}{4} \right] + \cdots + \left[\frac{1}{n-1} + \ln \frac{n-1}{n} \right]$$

as,

$$\ln \frac{1}{2} + \ln \frac{2}{3} + \cdots + \ln \frac{n-1}{n} = \ln \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{n-1}{n}$$

$$= \ln \frac{1}{n} = - \ln \frac{1}{n} \, n$$

Further we can note that

$$\ln \frac{k}{k+1} = \ln k - \ln (k+1) = - \int_k^{k+1} \frac{1}{x} dx$$

while

$$\frac{1}{k} = (k+1) \left[\frac{1}{k} - \frac{1}{k+1} \right] = (k+1) \int_k^{k+1} \frac{1}{x^2} dx.$$

So, the k'th term in the last expression for γ_n, $\dfrac{1}{k} + \ln \dfrac{k}{k+1}$, may be expressed as

$$\int_k^{k+1} \frac{(k+1) - x}{x^2} dx$$

corresponding to the following area, which we readily see is less than $\dfrac{1}{2k^2}$.

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† The constant γ has intrigued mathematicians even since the time of Euler. Other famous constants such as π and e have been shown to be transcendental. That is, numbers which are not roots of any polynomial with integer coefficients. Consequently they are certainly not rational numbers. Any rational number $\dfrac{p}{q}$ is the root of $qx - p = 0$. (Proof of the transcendence of π answered, in the negative, the long standing question of whether it was possible to "square the circle".) γ is conjectured to be transcendental, but as yet, even its irrationality remains unproved.

triangle of area $\frac{1}{2}k^2$
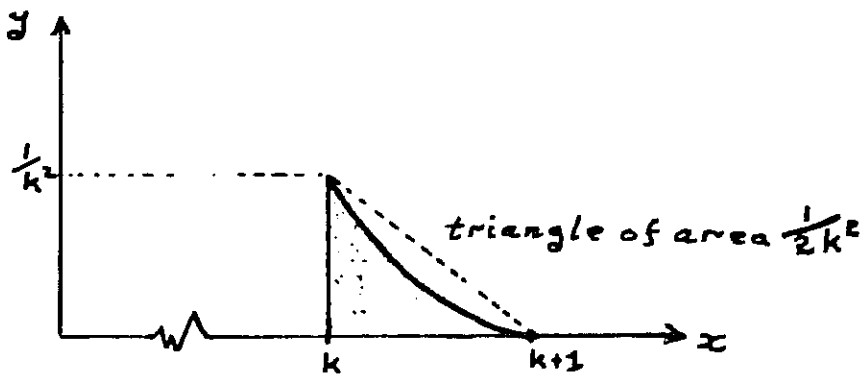
Now,

$$\gamma_n = \int_1^2 \frac{2-x}{x^2}\,dx + \int_2^3 \frac{3-x}{x^2}\,dx + \cdots + \int_{n-1}^n \frac{n-x}{x^2}\,dx$$

is the following area, and γ corresponds to the area of the "saw-toothed" region obtained
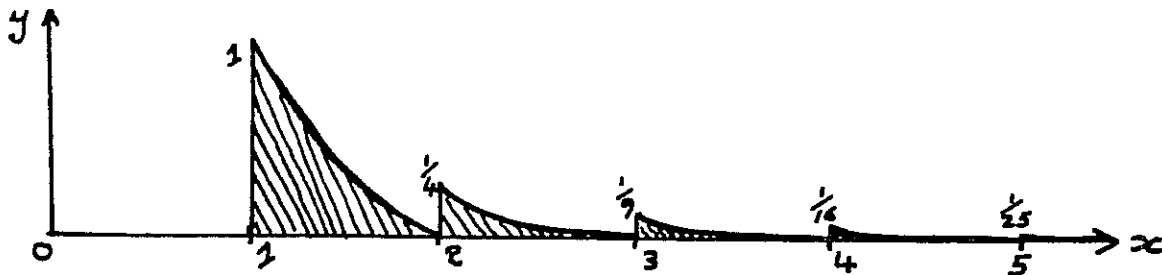
by extending the above picture indefinitely to the right. From this we readily see that the "error" between γ and γ_n is the area of the "saw-toothed" region from n onwards. That is

$$\gamma - \gamma_n = \int_n^{n+1} \frac{n+1-x}{x^2}\,dx + \int_{n+1}^{n+2} \frac{n+2-x}{x^2}\,dx + \cdots$$

and so

$$0 < \gamma - \gamma_n < \frac{1}{2n^2} + \frac{1}{2(n+1)^2} + \cdots$$
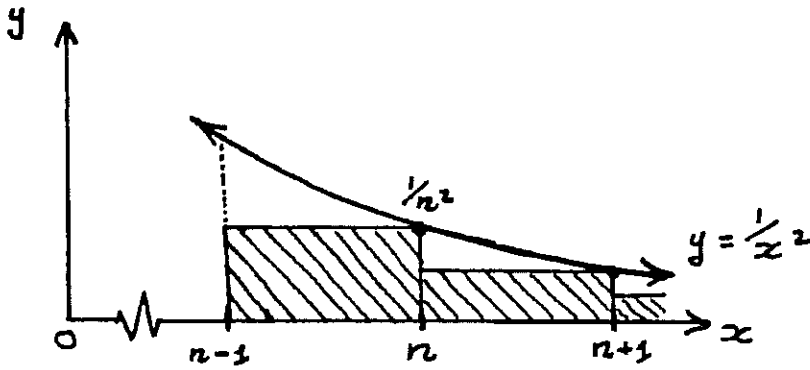
or, substituting for γ_n,

$$0 < \gamma - \left[1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right] + \ln n < \frac{1}{2}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right].$$

or, substituting for γ_n,

$$0 < \gamma - \left[1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right] + \ln n < \frac{1}{2}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right]$$

Using a by now familiar trick, we see that

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots < \int_{n-1}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{n-1}^{\infty} = \frac{1}{n-1}$$

jnd so we have

$$0 < \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + \ln n < \frac{1}{2(n-1)}$$

or

$$\gamma + \ln n - \frac{1}{2(n-1)} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} < \gamma + \ln n .$$

Using this we may readily find the value of n for $1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$ to have any given value.

For example: to find n for an overhang of 6, we need the smallest n for which

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

exceeds 12. Since n will be large, we start by neglecting $\frac{1}{2(n-1)}$ and use $1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \approx \gamma + \ln n$ to determine a value of n for which $\gamma + \ln n = 12$, that is

$$n = e^{12-\gamma} \approx 91380.227 \qquad \text{(by calculator)}.$$

Since for this value of n, $\gamma + \ln n = 12$ and $\gamma + \ln n$ is greater than the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

we see that $n = 91380$ is too small, so we try $n = 91381$; for this value we have

$$\gamma + \ln n - \frac{1}{2(n-1)} \approx 12.000003 < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

and so we conclude the number of blocks necessary to achieve an overhang of 6 is 91,381!

A Problem with Blocks - Exercises

1. Determine the minimum number of blocks which need to be stacked to achieve an overhang of at least *one*. [Do the calculations by hand.] Draw an accurate picture of the stack.

2. Write a computer programme to determine the minimum number of blocks which need to be stacked to achieve an overhang of at least m. That is, a programme which will determine the smallest value of n for which

$$\frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right] \geq m.$$

If possible run your programme to determine the answers for $m = 2, 3, 4$ and 5. Take note of the time needed for these calculations.]

3. Using a calculator, and the inequalities

$$\gamma + \ln n - \frac{1}{2(n-1)} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} < \gamma + \ln n$$

follow the method outlined to determine the number of blocks for which an overhang of 7 can be achieved. [Use $\gamma \approx 0.577215665$]

4. If n_m is the minimum number of blocks necessary to achieve an overhang of m, use the approximation

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \approx \gamma + \ln n$$

to show that

$$n_m \approx e^{2m - \gamma}.$$

Hence conclude that n_{m+1}, is related to n_m by

$$n_{m+1} \approx (e^2) n_m.$$

That is, we need approximately 7.39 more blocks to increase the overhang by the length of one block.

Computer Corner
The Pet Shop

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In the last issue of the *Bulletin* we referred to an article by Richard Clausi and Brian Hahn which discussed the use of micro computers to calculate mortgages. Several readers discovered that no such article could be found in that issue of the *Bulletin*. Unfortunately, there was a slip-up on our part as the *Bulletin* was being prepared for the presses, and that article was overlooked. Please accept our apologies, and do look at that article which is included in this present issue. Would any of our readers like to rewrite that program, and enhance it, to run on the PET?